Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 792431, 9 pages
http://dx.doi.org/10.1155/2013/792431

Research Article

Relaxation Problems Involving Second-Order Differential Inclusions

Adel Mahmoud Gomaa[1,2]

[1] *Taibah University, Faculty of Applied Science, Department of Applied Mathematics, Al-Madinah, Saudi Arabia*
[2] *Mathematics Department, Faculty of Science, Helwan University, Cairo, Egypt*

Correspondence should be addressed to Adel Mahmoud Gomaa; gomaa_5@hotmail.com

Received 12 November 2012; Accepted 12 March 2013

Academic Editor: Malisa R. Zizovic

We present relaxation problems in control theory for the second-order differential inclusions, with four boundary conditions, $\ddot{u}(t) \in F(t, u(t), \dot{u}(t))$ a.e. on $[0,1]$; $u(0) = 0$, $u(\eta) = u(\theta) = u(1)$ and, with $m \geq 3$ boundary conditions, $\ddot{u}(t) \in F(t, u(t), \dot{u}(t))$ a.e. on $[0,1]$; $\dot{u}(0) = 0$, $u(1) = \sum_{i=1}^{m-2} a_i u(\xi_i)$, where $0 < \eta < \theta < 1$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$ and F is a multifunction from $[0,1] \times \mathbb{R}^n \times \mathbb{R}^n$ to the nonempty compact convex subsets of $\mathbb{R}^n$. We have results that improve earlier theorems.

1. Introduction

Second-order differential inclusions of three boundary conditions were studied by many authors [1–6], using Hartman-type functions. Such a function was first introduced by [7] for two boundary conditions. Moreover, in [8] we consider second-order differential inclusions with four boundary conditions,

$$\begin{gathered} \ddot{u}(t) \in F(t, u(t), \dot{u}(t)), \quad \text{a.e. on } [0,T], \\ u(0) = x_0, \quad u(\eta) = u(\theta) = u(T), \end{gathered} \tag{1}$$

where $0 < \eta < \theta < T$ and F is a multifunction from $[0,T] \times \mathbb{R}^n \times \mathbb{R}^n$ to the nonempty compact subsets of $\mathbb{R}^n$, while in [9] we study four-point boundary value problems for differential inclusions and differential equations with and without multivalued moving constraints.

In the present paper, we study relaxation results for the second-order differential inclusions, with four boundary conditions,

$$\begin{gathered} \ddot{u}(t) \in F(t, u(t), \dot{u}(t)), \quad \text{a.e. on } [0,1], \\ u(0) = 0, \quad u(\eta) = u(\theta) = u(1) \end{gathered} \tag{P}$$

and, with $m \geq 3$ boundary conditions,

$$\begin{gathered} \ddot{u}(t) \in F(t, u(t), \dot{u}(t)), \quad \text{a.e. on } [0,1], \\ \dot{u}(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} a_i u(\xi_i), \end{gathered} \tag{Q}$$

where $0 < \eta < \theta < 1$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$, and F is a multifunction from $[0,1] \times \mathbb{R}^n \times \mathbb{R}^n$ to the non-empty compact subsets of $\mathbb{R}^n$.

In conjunction with Problem (P) and Problem (Q) we also consider the following problems:

$$\begin{gathered} \ddot{u}(t) \in \operatorname{ext} F(t, u(t), \dot{u}(t)), \quad \text{a.e. on } [0,1], \\ u(0) = 0, \quad u(\eta) = u(\theta) = u(1), \end{gathered} \tag{P_e}$$

$$\begin{gathered} \ddot{u}(t) \in \operatorname{ext} F(t, u(t), \dot{u}(t)), \quad \text{a.e. on } [0,1], \\ \dot{u}(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} a_i u(\xi_i). \end{gathered} \tag{Q_e}$$

By $\operatorname{ext} F(t, u(t), \dot{u}(t))$, we denote the set of extreme points of $F(t, u(t), \dot{u}(t))$.

2. Notations and Preliminaries

Throughout this paper we let $I = [0,1]$ and $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$. We will use the following definitions, notations, and summarize some results.

(i) A multifunction F from a metric space (X,d) to the set $P_f(Y)$ of all closed subsets of another metric space Y is lower semicontinuous *(l. s. c.)* at $x_0 \in X$ if for every open subset V in Y with $F(x_0) \cap V \neq \emptyset$ there exists an open subset U in X such that $x_0 \in U$ and $F(x) \cap V \neq \emptyset$ for all $x \in U$. F is *l. s. c.* if it is *l. s. c.* at each $x_0 \in X$.

(ii) F is upper semicontinuous *(u. s. c.)* at $x_0 \in X$ if for every open subset V in Y and containing $F(x_0)$ there exists an open subset U in X such that $x_0 \in U$ and $F(x) \subseteq V$, for all $x \in U$. F is *u. s. c.* if it is *u. s. c.* at each $x_0 \in X$.

(iii) A multifunction F from I into the set $P_f(X)$ of all closed subsets of X is measurable if for all $x \in X$ the function $t \to d(x, F(t)) = \inf\{\| x - y \|: y \in F(t)\}$ is measurable [10–13].

(iv) Let (Ω, Σ) be a measurable space and X a separable Banach space. We say that $F : \Omega \to P_f(X)$ is graph measurable if

$$gr(F) = \{(z,x) \in \Omega \times X : x \in F(z)\} \in \Sigma \times \mathscr{B}(X), \quad (2)$$

where $\mathscr{B}(X)$ is the Borel σ-field of X. For further details we refer to [14–16].

(v) F is continuous if it is lower and upper semicontinuous.

(vi) For each $A, B \in P_f(X)$, the Hausdorff metric is defined by

$$d_H(A,B) = \max\left[\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\right]. \quad (3)$$

It is known that the space $(P_f(X), d_H)$ is a generalized metric space, if the sets are not bounded (see, for instance, [14, 15]).

(vii) A multifunction F is Hausdorff continuous (d_H-continuous) if it is continuous from X into the metric space $(P_f(Y), d_H)$.

(viii) If F has compact values in Y, then F is d_H-continuous if and only if it is continuous [14, 17].

(ix) We denote by $P_{kc}(\mathbb{R}^n)$ the nonempty compact convex subsets of $\mathbb{R}^n$.

(x) The Banach spaces $C(I,\mathbb{R}^n)$, $C^1(I,\mathbb{R}^n)$, and $C^2(I,\mathbb{R}^n)$ endowed with the norms

$$\|u\|_C = \max_{t\in I} |u(t)|, \qquad \|u\|_{C^1} = \max\{\|u\|_C, \|\dot{u}\|_C\},$$
$$\|u\|_{C^2} = \max\{\|u\|_C, \|\dot{u}\|_C, \|\ddot{u}\|_C\}, \quad (4)$$

respectively.

(xi) $L^1_w(I,\mathbb{R}^n)$ denotes the space $L^1(I,\mathbb{R}^n)$ equipped with weak norm $\| \cdot\|_w$ which is defined by

$$\|h\|_w = \sup\left\{\left\|\int_a^b h(t)\,dt\right\| : 0 \le a \le b \le 1\right\}. \quad (5)$$

(xii) $W^{2,1}(I,\mathbb{R}^n)$ is the Sobolev space of functions $u : I \to \mathbb{R}^n$, u and $\dot{u}$ are both absolutely continuous functions so $\ddot{u}(t) \in L^1(I,\mathbb{R}^n)$ and it is equipped with the norm $\|u\|_{W^{2,1}(I,\mathbb{R}^n)} = \|u\|_{L^1(I,\mathbb{R}^n)} + \|\dot{u}\|_{L^1(I,\mathbb{R}^n)} + \|\ddot{u}\|_{L^1(I,\mathbb{R}^n)}$.

(xiii) Let $R : I \to 2^{\mathbb{R}^n}$ be a multifunction and $\delta^1_R = \{h \in L^1(I,\mathbb{R}^n) : h(t) \in R(t)\}$.

(xiv) By a solution of (P) (resp., of (P_e)) we mean a function $u \in W^{2,1}(I,\mathbb{R}^n)$ such that $\ddot{u}(t) = h(t)$ a.e. on I with $h \in \delta^1_{F(\cdot,u(\cdot),\dot{u}(\cdot))}$ (resp., $h \in \delta^1_{\text{ext}F(\cdot,u(\cdot),\dot{u}(\cdot))}$) and $u(0) = 0$, $u(\eta) = u(\theta) = u(1)$.

(xv) By a solution of (Q) (resp., of (Q_e)) we mean a function $u \in W^{2,1}(I,\mathbb{R}^n)$ such that $\ddot{u}(t) = h(t)$ a.e. on I with $h \in \delta^1_{F(\cdot,u(\cdot),\dot{u}(\cdot))}$ (resp., $h \in \delta^1_{\text{ext}F(\cdot,u(\cdot),\dot{u}(\cdot))}$) and $\dot{u}(0) = 0$, $u(1) = \sum_{i=1}^{m-2} a_i u(\xi_i)$.

(xvi) In the sequel by Δ_P (resp., Δ_{P_e}) we denote the solution set of Problem (P) (resp., of Problem (P_e)). Moreover, by Δ_Q (resp., Δ_{Q_e}) we denote the solution set of Problem (Q) (resp., of Problem (Q_e)).

Definition 1. Let E be a Banach space and let Y be a metric space. A multifunction $G : I \times Y \to P_{ck}(E)$ has the Scorza-Dragoni property (the SD-property) if for every $\varepsilon > 0$ there exists a closed set $A \subset I$ such that the Lebesgue measure $\mu(I \setminus A)$ is less than ε and $G|_{A\times Y}$ is continuous. The multifunction G is called integrably bounded on compacta in Y if, for any compact subset $Q \subset Y$, we can find an integrable function $\mu_Q : I \to \mathbb{R}^+$ such that $\sup\{\| y \|: y \in G(t,z)\} \le \mu_Q(t)$, for almost every $z \in Q$.

Theorem 2 (see [18]). *Let Y be a complete metric space, E a separable Banach space, E_σ the Banach space E endowed with the weak topology, $M : I \times Y \to P_{ck}(E_\sigma)$, and K a compact subset of $C(I,Y)$. Furthermore, let $R : K \to 2^{L^1(I,E)}$ be a multifunction defined by*

$$R(y) = \left\{g \in L^1(I,E) : g(t) \in M(t, y(t)) \text{ a.e. on } I\right\}. \quad (6)$$

If M has the SD-property and is integrably bounded on compacta in Y, then the set

$$A_K = \left\{f \in C\left(K, L^1_w(I,E)\right) : f(y) \in R(y) \ \forall y \in K\right\} \quad (7)$$

is nonempty complete subset of the space $C(K, L^1_w(I,E))$. Moreover, $A_K = \overline{A_{\text{ext}\,K}}$ where $L^1_w(I,E)$ is the space of equivalence classes of Bochner-integrable functions $v : I \to E$ with the norm $\| v\|_w = \sup_{t\in T} \| \int_0^t v(s)ds \|$ and

$$A_{\text{ext}\,K} = \left\{f \in C\left(K, L^1_w(I,E)\right) : f(y) \in \text{ext}R(y) \ \forall y \in K\right\}. \quad (8)$$

Lemma 3 (see [19]). *For p such that $1 < p < \infty$ let $\{u_n, u\}_{n\in\mathbb{N}} \subseteq L^p(I,\mathbb{R}^n)$, $\sup_{n\in\mathbb{N}} \|u_n\|_p < \infty$ and $u_n \to u$ with respect to the weak norm $\|\cdot\|_w$. Then $u_n \to u$ weakly in $L^p(I,\mathbb{R}^n)$.*

Next we state a preliminary lemma, for $0 < \eta < \theta < 1$, which is useful in the study of four boundary problems for the differential equations and the differential inclusions, and moreover we summarize some properties of a Hartman-type function.

Lemma 4 (see [8]). *Let $G : I \times I \to \mathbb{R}$ be the function defined as follows:*

as $0 \le t < \eta$,

$$G(t,\tau) = \begin{cases} -\tau & \text{if } 0 \le \tau \le t \\ -t & \text{if } t < \tau \le \eta \\ \dfrac{t(\tau-\theta)+(\tau-\eta)}{\theta-\eta} & \text{if } \eta < \tau \le \theta \\ \dfrac{1-\tau}{1-\theta} & \text{if } \theta < \tau \le 1, \end{cases} \tag{9}$$

when $\eta \le t < \theta$,

$$G(t,\tau) = \begin{cases} -\tau & \text{if } 0 \le \tau \le \eta \\ \dfrac{\tau(t-\theta+1)+\eta(\tau-t-1)}{\theta-\eta} & \text{if } \eta < \tau \le t \\ \dfrac{t(\tau-\theta)+(\tau-\eta)}{\theta-\eta} & \text{if } t < \tau \le \theta \\ \dfrac{1-\tau}{1-\theta} & \text{if } \theta < \tau \le 1, \end{cases} \tag{10}$$

lastly if $\theta \le t \le 1$,

$$G(t,\tau) = \begin{cases} -\tau & \text{if } 0 \le \tau \le \eta \\ \dfrac{\eta(\tau-t-1)+\tau(t-\theta+1)}{\theta-\eta} & \text{if } \eta < \tau \le \theta \\ \dfrac{1-\tau}{1-\theta}+(t-\tau) & \text{if } \theta < \tau \le t \\ \dfrac{1-\tau}{1-\theta} & \text{if } t < \tau \le 1. \end{cases} \tag{11}$$

Then the following hold.

(i) *If $u \in W^{2,1}(I,\mathbb{R}^n)$ with $u(0)=x_0, u(1)=u(\theta)=u(\eta)$, then*

$$u(t) = x_0 + \int_0^1 G(t,\tau)\,\ddot{u}(\tau)\,d\tau, \quad \forall t \in I; \tag{12}$$

(ii) *if $w \in L^1(I,\mathbb{R}^n)$, then for all $t \in I$,*

$$\begin{aligned} \int_0^1 G(t,\tau)\,w(\tau)\,d\tau &= \int_0^t (t-\tau)\,w(\tau)\,d\tau \\ &\quad - \int_0^\eta \frac{t(\tau-\eta)(t+1)}{\theta-\eta} w(\tau)\,d\tau \\ &\quad + \int_0^\theta \frac{t(\tau-\theta)+(\tau-\eta)}{\theta-\eta} w(\tau)\,d\tau \\ &\quad + \int_\theta^1 \frac{1-\tau}{1-\theta} w(\tau)\,d\tau; \end{aligned} \tag{13}$$

(iii) $\sup_{t,\tau\in I}|G(t,\tau)| \le 2$, $\sup_{t,\tau\in I}|\partial G(t,\tau)/\partial t| \le 1$.

Let $c_1, c_2, a \in L^p(I,\mathbb{R}^+)$, $1 < p < \infty$, and let L be a linear operator from $C(I,\mathbb{R}) \times C(I,\mathbb{R})$ to $C(I,\mathbb{R}) \times C(I,\mathbb{R})$ defined by $L(f,g) = (\underline{f}, \underline{g})$ such that, for all $t \in I$,

$$\begin{aligned} \underline{f}(t) &= \int_0^T |G(t,\tau)| \left(c_1(\tau) f(\tau) + c_2(\tau) g(\tau)\right) d\tau, \\ \underline{g}(t) &= \int_0^T \left|\frac{\partial G(t,\tau)}{\partial t}\right| \left(c_1(\tau) f(\tau) + c_2(\tau) g(\tau)\right) d\tau. \end{aligned} \tag{14}$$

If $c_1 = c_2 = 0$, then clearly $L = 0$. We note that if $\mathcal{K} = \{(h_1,h_2) \in C(I,\mathbb{R}) \times C(I,\mathbb{R}) : h_1(t), h_2(t) \ge 0, \ \forall t \in I\}$, then $L(\mathcal{K}) \subseteq \mathcal{K}$. Moreover, the spectral radius $r(L) = \lim \| L^n \|^{1/n}$ is an eigenvalue of L with an eigenvector in $\mathcal{K}$ [20].

3. Relaxation Theorems

In this section, both Theorems 5 and 7 improve [19, Theorem 4.1] with [21, Theorem 6]. Indeed in [19] Papageorgiou considered (P) and (P_e) with the two boundary conditions $u(0) = u(1) = 0$ and in [21] Ibrahim and Gomaa study the same problems with three boundary conditions $u(0) = x_0$, $u(\eta) = u(1)$.

Theorem 5. *Let $F : I \times \mathbb{R}^n \times \mathbb{R}^n \to P_{kc}(\mathbb{R}^n)$ be a multifunction such that*

(i) *for each $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$, the multifunction $F(\cdot,x,y)$ is measurable,*

(ii) *$d_H(F(t,x,y), F(t,x',y')) \le \alpha_1(t) \| x - x' \| + \alpha_2(t) \| y - y' \|$ a.e. with $\alpha_1, \alpha_2 \in L^1(I,\mathbb{R}^+)$ and $\| \alpha_1 + \alpha_2 \| < 1/2$,*

(iii) *for each $(t,x,y) \in I \times \mathbb{R}^n \times \mathbb{R}^n$,*

$$\begin{aligned} \|F(t,x,y)\| &= \sup\{\|v\| : v \in F(t,x,y)\} \\ &\le a(t) + c_1(t)\|x\| + c_2(t)\|y\| \end{aligned} \tag{15}$$

with $a, c_1, c_2 \in L^p(I,\mathbb{R}^+)$ $1 < p < \infty$,

(iv) *the spectral radius, $r(L)$, is less than 1.*

Then for each solution $u \in \Delta_{P_e}$, there is a sequence $(u_m(\cdot))_{m\in\mathbb{N}} \subset \Delta_P$ converging to $u(\cdot)$ in $(C^1(I,\mathbb{R}^n), \|\cdot\|_{C^1})$.

Proof. From [9, Theorem 2.1], we obtain $\Delta_{P_e} \ne \emptyset$. Moreover, we can say that $\| F(t,x,y) \| \le a_1(t)$ a.e. on I for some $a_1 \in L^p(I,\mathbb{R}^+)$. Let $u \in \Delta_P$. Then

$$\begin{aligned} \ddot{u}(t) &= h(t), \quad \text{a.e. on } I, \\ u(0) = 0, \ u(\eta) &= u(\theta) = u(1), \end{aligned} \tag{16}$$

where $h(t) \in F(t,u(t),\dot{u}(t))$ a.e. on I. Assume that $f : L^1(I,\mathbb{R}^n) \to C^1(I,\mathbb{R}^n)$ is a function such that, for each $h \in L^1(I,\mathbb{R}^n)$, $f(h) \in W^{1,2}(I,\mathbb{R}^n)$ is the unique solution of the second-order differential equation

$$\begin{aligned} \ddot{u}(t) &= h(t), \quad \text{a.e. on } I, \\ u(0) = 0, \ u(\eta) &= u(\theta) = u(1). \end{aligned} \tag{P_h}$$

Let $\mathcal{S} = \{u \in L^1(I, \mathbb{R}^n) : \| u(t) \| \leq a_1(t)$ a.e. on $I\}$. It is easy to see that $f(\mathcal{S})$ is convex. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $f(\mathcal{S})$. Hence, $u_n \in W^{2,1}(I, \mathbb{R}^n)$ with $u_n(0) = x_0$, $u_n(\eta) = u_n(\theta) = u_n$ (17) and

$$\begin{aligned} u_n(t) = x_0 &+ \int_0^t (t-\tau)\ddot{u}_n(\tau)\,d\tau \\ &- \int_0^\eta \frac{t(\tau-\eta)(t+1)}{\theta-\eta}\ddot{u}_n(\tau)\,d\tau \\ &+ \int_0^\theta \frac{t(\tau-\theta)+(\tau-\eta)}{\theta-\eta}\ddot{u}_n(\tau)\,d\tau \\ &+ \int_\theta^1 \frac{1-\tau}{1-\theta}\ddot{u}_n(\tau)\,d\tau. \end{aligned} \tag{17}$$

Then,

$$\lim_{n\to\infty} u_n(t) = \int_0^1 G(t,\tau)\ddot{u}(\tau)\,d\tau = u(t), \tag{18}$$

which means that $f(\mathcal{S})$ is a compact subset of $C^1(I, \mathbb{R}^n)$. Set

$$\begin{aligned} \mathcal{P}_\varepsilon(t) = \{x \in F(t, v(t), \dot{v}(t)) : \|h(t)-x\| &< \varepsilon \\ &+ d(h(t), F(t, v(t), \dot{v}(t)))\}, \end{aligned} \tag{19}$$

where $\varepsilon > 0$ and $v \in f(\mathcal{S})$. Hence, for each $t \in I$, $\mathcal{P}_\varepsilon(t) \neq \emptyset$. Assume that $\mathcal{B}(I)$ and $\mathcal{B}(\mathbb{R}^n)$ are the Borel σ-fields of I and $\mathbb{R}^n$, respectively. From condition, (i) the function $t \to F(t, v(t), \dot{v}(t))$ is measurable. Hence, $grF(\cdot, v(\cdot), \dot{v}(\cdot)) \in \mathcal{B}(I) \times \mathcal{B}(\mathbb{R}^n)$ and $(t,x) \to \varepsilon d(h(t), F(t, v(t), \dot{v}(t))) - \| h(t) - x \|$ is measurable in t and continuous in x that is jointly measurable. Thus, by Aumann's selection theorem, there exists a measurable selection s_ε of $\mathcal{P}_\varepsilon$ such that $s_\varepsilon(t) \in \mathcal{P}_\varepsilon(t)$ for each $t \in I$. Now we define a multifunction $\mathcal{Q}_\varepsilon : f(\mathcal{S}) \to 2^{L^1(I,\mathbb{R}^n)}$ by the following:

$$\begin{aligned} &\mathcal{Q}_\varepsilon(v) \\ &= \{x \in \delta^1_{F(\cdot,v(\cdot),\dot{v}(\cdot))} : \\ &\qquad \|h(t)-x\| < \varepsilon + d(h(t), F(t, v(t), \dot{v}(t))) \text{ a.e. on } I\}, \end{aligned} \tag{20}$$

with $\mathcal{Q}_\varepsilon(v)(t) \neq \emptyset$ for each $v \in f(\mathcal{S})$. From [22, Proposition 4], $\mathcal{Q}_\varepsilon$ is $l.\ s.\ c.$ and clearly has decomposable values. Applying [22, Theorem 3], we have a continuous selection S_ε of $\overline{\mathcal{Q}_\varepsilon}$. Therefore,

$$\begin{aligned} \|h(t) - S_\varepsilon(v)(t)\| &\leq \varepsilon + d(h(t), F(t, v(t), \dot{v}(t))) \\ &\leq \varepsilon + \alpha_1(t)\|u(t)-v(t)\| \\ &\quad + \alpha_2(t)\|\dot{u}(t)-\dot{v}(t)\| \quad \text{a.e. on } I. \end{aligned} \tag{21}$$

From Theorem 2, we find a continuous function $\xi_\varepsilon : f(\mathcal{S}) \to L^1_w(I, \mathbb{R}^n)$ such that $\xi_\varepsilon(v) \in \text{ext}\delta^1_{F(\cdot,v(\cdot),\dot{v}(\cdot))}$ and $\| S_\varepsilon(v) - \xi_\varepsilon(v) \| < \varepsilon$ for each $v \in f(\mathcal{S})$. Define a multifunction $R : f(\mathcal{S}) \to 2^{L^1(I,\mathbb{R}^n)}$ by

$$R(u) = \{g \in L^1(I, \mathbb{R}^n) : g(t) \in F(t, u(t), \dot{u}(t)) \text{ a.e. on } I\}. \tag{22}$$

Assume that $Y = \mathbb{R}^n \times \mathbb{R}^n$ and set a multifunction $M : I \times Y \to 2^{\mathbb{R}^n}$ such that $M(t,(x,y)) = F(t,x,y)$. From Theorem 3.1 in [23], M has SD-property. R has nonempty convex values. Let $(g_n)_{n\in\mathbb{N}}$ be a sequence in $R(u)$ for some $u \in f(\mathcal{S})$. So, for each $t \in I$,

$$\lim_{n\to\infty} g_n(t) = g(t) \in F(t, u(t), \dot{u}(t)) \tag{23}$$

because F has closed values in $\mathbb{R}^n$. Therefore, $g \in \delta^1_{F(\cdot,u(\cdot),\dot{u}(\cdot))}$ which implies that $R(\cdot)$ has compact values in $\mathbb{R}^n$. We can apply Theorem 2 to find a continuous function $\theta : f(\mathcal{S}) \to L^1_w(I, \mathbb{R}^n)$ such that $\theta(u) \in \text{ext}(R(u))$, for all $u \in f(\mathcal{S})$. We see that $\theta(u)(t) \in \text{ext}(M(t,(u(t),\dot{u}(t))))$ [24], hence $\theta(u)(t) \in \text{ext}F(t, u(t), \dot{u}(t))$ a.e. on I. Assume that $\eta : f(\mathcal{S}) \to W^{1,2}(I, \mathbb{R}^n)$ is the function which for each $u \in f(\mathcal{S})$, $\eta(u) = g(\theta(u))$. For each $u \in f(\mathcal{S})$, we have $\| \theta(u)(t) \| \leq a_1$ and so $\theta(u) \in \mathcal{S}$. Then, η is a function from $f(\mathcal{S})$ into $f(\mathcal{S})$ and also we see that η is continuous [19]. Now let $\varepsilon_n \to 0$, $S_{\varepsilon_n} = S_n$ and $\xi_n = \xi_{\varepsilon_n}$. Then, for each $n \in \mathbb{N}$, the function $fo\xi_n$ is a continuous function from the compact set $f(\mathcal{S})$ into itself. From Schauder's fixed point theorem, $fo\xi_n$ has a fixed point u_n, but $\text{ext}\delta^1_{F(\cdot,v(\cdot),\dot{v}(\cdot))} = \delta^1_{\text{ext}F(\cdot,v(\cdot),\dot{v}(\cdot))}$ [24] so $u_n \in \Delta_{P_e}$. By passing to a subsequence if necessary, we may assume that $u_n \to \hat{u}$ in $C^1(I, \mathbb{R}^n)$. Then, we obtain

$$\begin{aligned} &\|u_n(t) - u(t)\| \\ &\leq \int_0^1 \Bigg\| \Bigg[\int_0^t (t-\tau)(\xi_n(\tau) - h(\tau))\,d\tau \\ &\qquad - \int_0^\eta \frac{t(\tau-\eta)(t+1)}{\theta-\eta}(\xi_n(\tau)-h(\tau))\,d\tau \\ &\qquad + \int_0^\theta \frac{t(\tau-\theta)+(\tau-\eta)}{\theta-\eta}(\xi_n(\tau)-h(\tau))\,d\tau \\ &\qquad + \int_\theta^1 \frac{1-\tau}{1-\theta}(\xi_n(\tau)-h(\tau))\,d\tau \Bigg] \Bigg\| ds \\ &\leq \int_0^1 \Bigg[\int_0^t (t-\tau)\|\xi_n(\tau) - S_n(\tau)\|\,d\tau \\ &\qquad + \int_0^t (t-\tau)\|h(\tau) - S_n(\tau)\|\,d\tau \\ &\qquad + \int_0^\eta \frac{t(\tau-\eta)(t+1)}{\theta-\eta}(\xi_n(\tau)-S_n(\tau))\,d\tau \\ &\qquad + \int_0^\eta \frac{t(\tau-\eta)(t+1)}{\theta-\eta}\|h(\tau)-S_n(\tau)\|\,d\tau \end{aligned}$$

$$+\int_{\theta}^{1}\frac{1-\tau}{1-\theta}\left\|\xi_n(\tau)-S_n(\tau)\right\|d\tau$$
$$\left.+\int_{\theta}^{1}\frac{1-\tau}{1-\theta}\left\|h(\tau)-S_n(\tau)\right\|d\tau\right]ds. \tag{24}$$

But $\xi_n-S_n \to 0$ with respect to the norm $\|\cdot\|_w$ from Lemma 3 we get $\xi_n-S_n \to 0$ weakly in $L^1(I,\mathbb{R}^n)$. So we have

$$\int_0^t (t-\tau)\left\|\xi_n(\tau)-S_n(\tau)\right\|d\tau + \int_0^{\eta}\frac{t(\tau-\eta)(t+1)}{\theta-\eta}\left\|\xi_n(\tau)-S_n(\tau)\right\|d\tau + \int_{\theta}^{1}\frac{1-\tau}{1-\theta}\left\|\xi_n(\tau)-S_n(\tau)\right\|d\tau \longrightarrow 0. \tag{25}$$

Moreover,

$$\begin{aligned}
&\int_0^1\left[\int_0^t (t-\tau)\left\|h(\tau)-S_n(\tau)\right\|d\tau + \int_0^{\eta}\frac{t(\tau-\eta)(t+1)}{\theta-\eta}\left(\xi_n(\tau)-S_n(\tau)\right)d\tau + \int_{\theta}^{1}\frac{1-\tau}{1-\theta}\left\|h(\tau)-S_n(\tau)\right\|d\tau\right]ds\\
&\leq \int_0^1\left[\int_0^t (t-\tau)\left(\varepsilon_n+\alpha_1(\tau)\left\|u(\tau)-u_n(\tau)\right\| +\alpha_2(\tau)\left\|\dot{u}(\tau)-\dot{u}_n(\tau)\right\|\right)\right.\\
&\quad + \int_0^{\eta}\frac{t(\tau-\eta)(t+1)}{\theta-\eta}\left(\varepsilon_n+\alpha_1(\tau)\left\|u(\tau)-u_n(\tau)\right\| +\alpha_2(\tau)\left\|\dot{u}(\tau)-\dot{u}_n(\tau)\right\|\right)\\
&\quad \left.+ \int_{\theta}^{1}\frac{1-\tau}{1-\theta}\left(\varepsilon_n+\alpha_1(\tau)\left\|u(\tau)-u_n(\tau)\right\| +\alpha_2(\tau)\left\|\dot{u}(\tau)-\dot{u}_n(\tau)\right\|\right)\right]ds.
\end{aligned} \tag{26}$$

As $n \to \infty$, we have

$$\begin{aligned}
&\left\|\widehat{u}(t)-u(t)\right\|\\
&\leq \int_0^1\left[\int_0^t (t-\tau)\left(\alpha_1(\tau)\left\|u(\tau)-\widehat{u}(\tau)\right\| +\alpha_2(\tau)\left\|\dot{u}(\tau)-\dot{\widehat{u}}(\tau)\right\|\right)d\tau\right.\\
&\quad + \int_0^{\eta}\frac{t(\tau-\eta)(t+1)}{\theta-\eta}\left(\alpha_1(\tau)\left\|u(\tau)-\widehat{u}(\tau)\right\| +\alpha_2(\tau)\left\|\dot{u}(\tau)-\dot{\widehat{u}}(\tau)\right\|\right)d\tau\\
&\quad \left.+ \int_{\theta}^{1}\frac{1-\tau}{1-\theta}\left(\alpha_1(\tau)\left\|u(\tau)-\widehat{u}(\tau)\right\| +\alpha_2(\tau)\left\|\dot{u}(\tau)-\dot{\widehat{u}}(\tau)\right\|\right)d\tau\right]ds\\
&\leq \left\|u-\widehat{u}\right\|_{C^1(I,\mathbb{R}^n)}\left(\int_0^t (t-\tau)\left(\alpha_1(\tau)+\alpha_2(\tau)\right)d\tau + \int_0^{\eta}\frac{t(\tau-\eta)(t+1)}{\theta-\eta}\left(\alpha_1(\tau)+\alpha_2(\tau)\right)d\tau + \int_{\theta}^{1}\frac{1-\tau}{1-\theta}\left(\alpha_1(\tau)+\alpha_2(\tau)\right)d\tau\right)\\
&= \left\|u-\widehat{u}\right\|_{C^1(I,\mathbb{R}^n)}\int_0^1\left|G(t,\tau)\right|\left(\alpha_1(\tau)+\alpha_2(\tau)\right)d\tau\\
&\leq 2\left\|u-\widehat{u}\right\|_{C^1(I,\mathbb{R}^n)}\left\|\alpha_1(\tau)+\alpha_2(\tau)\right\|.
\end{aligned} \tag{27}$$

Since by assumption (ii), $\| \alpha_1+\alpha_2 \| < 1/2$ we get $u=\widehat{u}$. So $u_n \to u$ in $C^1(I,\mathbb{R}^n)$ and $u \in \overline{\Delta_{P_e}}$ where the closure is taken in $C^1(I,\mathbb{R}^n)$ which means that $\Delta_P \subseteq \overline{\Delta_{P_e}}$. Therefore, the proof is complete if we show that Δ_P is closed. Indeed if $v_n \in \Delta_P$ and $v_n \to v$ in $C^1(I,\mathbb{R}^n)$, then $v_n = f(y_n)$ for $y_n \in \delta^1_{F(\cdot,v(\cdot),\dot{v}(\cdot))}$. From assumption (iii) and the Dunford-Pettis theorem, $\{y_n\}_{n\in\mathbb{N}}$ is weakly sequentially compact in $L^1(I,\mathbb{R}^n)$. So we can say that $\{y_n\}_{n\in\mathbb{N}}$ in $L^1(I,\mathbb{R}^n)$. By [25, Theorem 3.1], we get

$$\begin{aligned}
y(t) &\in \overline{\operatorname{conv}}\ \overline{\lim}\{y_n(t)\}_{n\in\mathbb{N}} \subseteq \overline{\operatorname{conv}}\ \overline{\lim}\, F\left(t,v_n(t),\dot{v}_n(t)\right)\\
&= F\left(t,v(t),\dot{v}(t)\right) \quad \text{a.e. on } I.
\end{aligned} \tag{28}$$

Moreover, $f(y_n) \to f(y)$ in $L^1(I,\mathbb{R}^n)$ for $y \in L^1(I,\mathbb{R}^n)$ and $y(t) \in F(t,v(t),\dot{v}(t))$ a.e. on I. Hence, $v \in \Delta_P$; that is Δ_P is closed in $C^1(I,\mathbb{R}^n)$. □

Now we consider the following assumptions:

(A_1) $\beta \in (0,\pi/2)$, $a_i > 0$ and $\sum_{i=1}^{m-2} a_i < 1$;

(A_2) $\sum_{i=1}^{m-2} a_i \cos\beta\xi_i - \cos\beta > 0$ and $K_m = 1/\sum_{i=1}^{m-2} a_i \cos\beta\xi_i - \cos\beta$;

(A_3) $C_0 = (\sin\beta/\beta)(1+K_m)$ and $C_1 = \min\{K_m+1, K_m\sin^2\beta\}$;

(A_4) $S = \{u \in C^2(I, \mathbb{R}^n) : \dot{u}(0) = 0,\ u(1) = \sum_{i=1}^{m-2} a_i u(\xi_i)\}$;

(A_5) $\mathcal{G} : I \times I \rightarrow \mathbb{R}$ is defined by

$$\mathcal{G}(t,s) = \begin{cases} \frac{1}{\beta}\sin\beta(t-s) & \text{if } 0 \le s \le t \le 1 \\ 0 & \text{if } 0 \le t \le s \le 1 \end{cases} + \frac{K_m}{\beta}\cos\beta t \begin{cases} \sin\beta(1-s) - \sum_{i=1}^{m-2} a_i \sin\beta(\xi_i - s), & \text{if } 0 \le s \le \xi_1, \\ \sin\beta(1-s) - \sum_{i=2}^{m-2} a_i \sin\beta(\xi_i - s), & \text{if } \xi_1 < s \le \xi_2, \\ \sin\beta(1-s) - \sum_{i=3}^{m-2} a_i \sin\beta(\xi_i - s), & \text{if } \xi_2 < s \le \xi_3, \\ \vdots \\ \sin\beta(1-s) - \sum_{i=k}^{m-2} a_i \sin\beta(\xi_i - s), & \text{if } \xi_{k-1} < s \le \xi_k, \\ \vdots \\ \sin\beta(1-s), & \text{if } \xi_{m-2} < s \le 1. \end{cases} \quad (29)$$

Lemma 6 (see [26]). *If the assumptions (A_1)–(A_5) hold, then*

(i) $0 \le \mathcal{G}(t,s) \le C_0$ *for all* $(t,s) \in I \times I$,

(ii) $\sup_{t,s\in I} |\partial\mathcal{G}(t,s)/\partial t| \le C_1$,

(iii) *for each* $x \in C^1(I, \mathbb{R}^n)$ *there exists a unique function* $u_x \in S$ *such that*

$$u_x(t) = \int_0^1 \mathcal{G}(t,s)\, x(s)\, ds, \quad (30)$$

(iv) $(\int_0^1 |\mathcal{G}(t,s)|^k ds)^{1/k} \le C_0$ *and* $(\int_0^1 |(\partial\mathcal{G}/\partial t)(t,s)|^k ds)^{1/k} \le C_1$.

Proof. (ii) Since

$$\frac{\partial\mathcal{G}(t,s)}{\partial t} = \begin{cases} \cos\beta(t-s) & \text{if } 0 \le s \le t \le 1 \\ 0 & \text{if } 0 \le t \le s \le 1 \end{cases} - K_m \sin\beta t \begin{cases} \sin\beta(1-s) - \sum_{i=1}^{m-2} a_i \sin\beta(\xi_i - s), & \text{if } 0 \le s \le \xi_1, \\ \sin\beta(1-s) - \sum_{i=2}^{m-2} a_i \sin\beta(\xi_i - s), & \text{if } \xi_1 < s \le \xi_2, \\ \sin\beta(1-s) - \sum_{i=3}^{m-2} a_i \sin\beta(\xi_i - s), & \text{if } \xi_2 < s \le \xi_3, \\ \vdots \\ \sin\beta(1-s) - \sum_{i=k}^{m-2} a_i \sin\beta(\xi_i - s), & \text{if } \xi_{k-1} < s \le \xi_k, \\ \vdots \\ \sin\beta(1-s), & \text{if } \xi_{m-2} < s \le 1, \end{cases} \quad (31)$$

then $\sup_{t,s\in I} \partial\mathcal{G}(t,s)/\partial t \le 1 + K_m$. Furthermore,

$$\frac{\partial\mathcal{G}(t,s)}{\partial t} \ge K_m \sin\beta t \left[\sum_{i=1}^{m-2} a_i \sin(\xi_i - s) - \sin\beta(1-\beta)\right] \ge -K_m \sin^2\beta \quad (32)$$

and thus $\sup_{t,s\in I} |\partial\mathcal{G}(t,s)/\partial t| \le C_1$. □

Theorem 7. *Assume that the assumptions (A_1) and (A_2) hold. Let F be a multifunction from $I \times \mathbb{R}^n \times \mathbb{R}^n$ to $P_{kc}(\mathbb{R}^n)$ satisfying the following conditions:*

(a) *for each $(x, y) \in \mathbb{R} \times \mathbb{R}$, the multifunction $F(\cdot, x, y)$ is measurable;*

(b) *for each $t \in I$, the function $(x, y) \rightarrow F(t, x, y)$ is continuous with respect to the Hausdorff metric d_H;*

(c) *for each $(t, x, y) \in I \times \mathbb{R}^n \times \mathbb{R}^n$*

$$\|F(t,x,y)\| \le \sup\{\|v\| : v \in F(t,x,y)\} \le a(t) + c_1(t)\|x\| + c_2(t)\|y\|; \quad (33)$$

(d) *the spectral radius $r(L)$ of L is less than one.*

Then Problem (Q_e) admits a solution in S.

Proof. We can say that $\| F(t,x,y) \| \le a_1(t)$ a.e. on I for some $a_1 \in L^p(I,\mathbb{R}^+)$ [9]. Let $x \in C^1(I,\mathbb{R}^n)$ and let $u \in C^2(I,\mathbb{R}^n)$ be the unique solution of the problem

$$\ddot{u}(t) = x(t), \quad \text{a.e. on } I,$$
$$\dot{u}(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} a_i u(\xi_i). \qquad (*)$$

From Lemma 6, we have $u(t) = \int_0^1 \mathcal{G}(t,s)x(s)ds$, $\forall t \in I$. Thus, we define a function $f : C^1(I,\mathbb{R}^n) \to C^2(I,\mathbb{R}^n)$ such that $f(x)$ is the unique solution of $(*)$. Let

$$\mathcal{V} = \left\{x \in C^1(I,\mathbb{R}^n) : \|x(t)\| \le a_1(t) \text{ a.e. on } I\right\}. \qquad (34)$$

From the Dunford-Pettis theorem, $\mathcal{V}$ is weakly compact and then $f(\mathcal{V})$ is convex and compact subset of $C^2(I,\mathbb{R}^n)$. Let $\mathcal{Y} = \mathbb{R}^n \times \mathbb{R}^n$. If $\mathcal{K} = f(\mathcal{V})$, $\mathcal{R} : \mathcal{K} \to 2^{L^1(I,\mathbb{R}^n)}$ and $\mathcal{M} : I \times \mathcal{Y} \to 2^{\mathbb{R}^n}$, where $\mathcal{R}(u) = \{g \in L^1(I,\mathbb{R}^n) : g(t) \in F(t,u(t),\dot{u}(t))$ a.e. on $I\}$ and $\mathcal{M}(t,(x,y)) = F(t,x,y)$, then $\mathcal{M}$ has SD-property [23]. It is easy to show that $\mathcal{R}$ is nonempty and convex subset of $L^1(I,\mathbb{R}^n)$. If f_n is a sequence in $\mathcal{R}(u)$ for some $u \in \mathcal{K}$, then $\lim_{n\to\infty} f_n(t) = f(t) \in F(t,u(t),\dot{u}(t))$, where the values of F are closed. Therefore, the values of $\mathcal{R}$ are weakly compact. According to Theorem 5 there exists a continuous function $r : \mathcal{K} \to L^1_w(I,\mathbb{R}^n)$ with $r(u) \in \text{ext}(\mathcal{R}(u))$, for all $u \in \mathcal{K}$. Thus, $r(u)(t) \in \text{ext}(\mathcal{M}(t,u(t),\dot{u}(t)))$ a.e. on I [24] which implies $r(u)(t) \in \text{ext}(F(t,u(t),\dot{u}(t)))$ a.e. on I. If $u \in f(\mathcal{V})$, then $\| r(u)(t) \| \le a_1$ and so $r(u) \in \mathcal{V}$. Put $\theta : f(\mathcal{V}) \to W^{2,1}(I,\mathbb{R}^n)$ such that $\theta(u) = f(r(u))$, thus θ is a continuous function from $f(\mathcal{V})$ into $f(\mathcal{V})$ [19]. From Schauder's fixed point theorem, there exists $x \in f(\mathcal{V})$ such that $x = \theta(x) = f(r(x))$ which means that there is $x \in S \subseteq C^2(I,\mathbb{R}^n)$ such that $\ddot{x}(t) \in \text{ext}(F(t,x(t),\dot{x}(t)))$. □

Theorem 8. *In the setting of Theorem 7, if one replaces condition (b) by the following condition:*

(b)′ $d_H(F(t,x,y), F(t,x',y')) \le k_1 \| x - x' \| + k_2 \| y - y' \|$ a.e. with $k_1 \ge 0$, $k_2 \ge 0$ and $|k_1 + k_2| < 1/2C_0$.

Then Δ_{Q_e} is nonempty and $\overline{\Delta_{Q_e}} = \Delta_Q$ where the closure taken in $C^2(I,\mathbb{R}^n)$.

Proof. From Theorem 7, we have $\Delta_{Q_e} \neq \emptyset$. Moreover, $\| F(t,x,y) \| \le b_1(t)$ a.e. on I for some $b_1 \in L^p(I,\mathbb{R}^+)$. Let $u \in \Delta_Q$. Then

$$\ddot{u}(t) = h(t), \quad \text{a.e. on } I,$$
$$\dot{u}(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} a_i u(\xi_i), \qquad (35)$$

where $h(t) \in F(t,u(t),\dot{u}(t))$ a.e. on I. Assume that $f' : C^1(I,\mathbb{R}^n) \to C^2(I,\mathbb{R}^n)$ is a function such that, for each $h \in C^1(I,\mathbb{R}^n)$, $f'(h) \in C^2(I,\mathbb{R}^n)$ is the unique solution of the second-order differential equation

$$\ddot{u}(t) = h(t), \quad \text{a.e. on } I,$$
$$\dot{u}(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} a_i u(\xi_i). \qquad (Q_h)$$

Let $S = \{u \in C^1(I,\mathbb{R}^n) : \| u(t) \| \le b_1(t)$ a.e. on $I\}$. So $f'(S)$ is convex. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $f'(S)$. Hence, $u_n \in C^2(I,\mathbb{R}^n)$ with $u_n(0) = 0$, $\dot{u}_n(0) = 0$, $u_n(1) = \sum_{i=1}^{m-2} a_i u_n(\xi_i)$. Then from Lemma 6,

$$\lim_{n\to\infty} u_n(t) = \int_0^1 \mathcal{G}(t,\tau)\ddot{u}(\tau)\,d\tau = u(t), \qquad (36)$$

hence, $f'(S)$ is a compact subset of $C^2(I,\mathbb{R}^n)$. Set

$$\mathcal{Q}_\varepsilon(t) = \{x \in F(t,v(t),\dot{v}(t)) : \|h(t) - x\| < \varepsilon + d(h(t),F(t,v(t),\dot{v}(t)))\}, \qquad (37)$$

where $\varepsilon > 0$ and $v \in f'(S)$. Hence, for each $t \in I$, $\mathcal{Q}_\varepsilon(t) \neq \emptyset$. Assume that $\mathcal{B}(I)$ and $\mathcal{B}(\mathbb{R}^n)$ are the Borel σ-fields of I and $\mathbb{R}^n$, respectively. From condition (i), the function $t \to F(t,v(t),\dot{v}(t))$ is measurable. Hence, $grF(\cdot,v(\cdot),\dot{v}(\cdot)) \in \mathcal{B}(I) \times \mathcal{B}(\mathbb{R}^n)$ and $(t,x) \to \varepsilon d(h(t),F(t,v(t),\dot{v}(t))) - \| h(t) - x \|$ is measurable in t and continuous in x that is jointly measurable. Thus, by Aumann's selection theorem, there exists a measurable selection s_ε of $\mathcal{Q}_\varepsilon$ such that $s_\varepsilon(t) \in \mathcal{Q}_\varepsilon(t)$ for each $t \in I$. Now we define a multifunction $\mathcal{Q}_\varepsilon : f'(S) \to 2^{C^1(I,\mathbb{R}^n)}$ by the following:

$$\mathcal{Q}_\varepsilon(v) = \left\{x \in \delta^1_{F(\cdot,v(\cdot),\dot{v}(\cdot))} : \|h(t) - x\| < \varepsilon + d(h(t),F(t,v(t),\dot{v}(t))) \text{ a.e. on } I\right\}, \qquad (38)$$

with $\mathcal{Q}_\varepsilon(v)(t) \neq \emptyset$ for each $v \in f'(S)$. From [22, Proposition 4], $\mathcal{Q}_\varepsilon$ is *l. s. c.* and clearly has decomposable values. Applying [22, Theorem 3], we have a continuous selection S_ε of $\overline{\mathcal{Q}_\varepsilon}$. Therefore,

$$\begin{aligned}\left\|h(t) - S_\varepsilon(v)(t)\right\| &\le \varepsilon + d(h(t),F(t,v(t),\dot{v}(t))) \\ &\le \varepsilon + k_1(t)\|u(t) - v(t)\| \\ &\quad + k_2(t)\|\dot{u}(t) - \dot{v}(t)\| \quad \text{a.e. on } I.\end{aligned} \qquad (39)$$

From Theorem 2, we find a continuous function $\xi'_\varepsilon : f'(S) \to L^1_w(I,\mathbb{R}^n)$ such that $\xi'_\varepsilon(v) \in \text{ext}\,\delta^1_{F(\cdot,v(\cdot),\dot{v}(\cdot))}$ and $\| S_\varepsilon(v) - \xi'_\varepsilon(v) \| < \varepsilon$ for each $v \in f'(S)$. Define a multifunction $R' : f'(S) \to 2^{C^1(I,\mathbb{R}^n)}$ by

$$R'(u) = \left\{g \in C^1(I,\mathbb{R}^n) : g(t) \in F(t,u(t),\dot{u}(t)) \text{ a.e. on } I\right\}. \qquad (40)$$

As in Theorem 5, let $Y = \mathbb{R}^n \times \mathbb{R}^n$ and set a multifunction $M : I \times Y \to 2^{\mathbb{R}^n}$ such that $M(t,(x,y)) = F(t,x,y)$. From [23, Theorem 3.1], M has SD-property. R' has nonempty convex values. Let $(g_n)_{n\in\mathbb{N}}$ be a sequence in $R'(u)$ for some $u \in f'(S)$. So, for each $t \in I$,

$$\lim_{n\to\infty} g_n(t) = g(t) \in F(t, u(t), \dot{u}(t)) \tag{41}$$

because F has closed values in $\mathbb{R}^n$. Therefore, $g \in \delta^1_{F(\cdot,u(\cdot),\dot{u}(\cdot))}$ which implies $R'(\cdot)$ has compact values in $\mathbb{R}^n$. We can apply Theorem 2 to find a continuous function $\theta' : f'(S) \to L^1_w(I,\mathbb{R}^n)$ such that $\theta'(u) \in \operatorname{ext}(R'(u))$, for all $u \in f'(S)$. We see that $\theta'(u)(t) \in \operatorname{ext}(M(t,(u(t),\dot{u}(t))))$ [24], hence $\theta'(u)(t) \in \operatorname{ext}F(t,u(t),\dot{u}(t))$ a.e. on I. Assume that $\eta' : f'(S) \to C^2(I,\mathbb{R}^n)$ is the function which for each $u \in f'(S)$, $\eta'(u) = g(\theta'(u))$. For each $u \in f'(S)$, we have $\| \theta'(u)(t) \| \le b_1$ and so $\theta'(u) \in S$. Then, η' is a function from $f'(S)$ into $f'(S)$ and also we see that η' is continuous [19]. Now let $\varepsilon_n \to 0$, $S_{\varepsilon_n} = S_n$ and $\xi'_n = \xi'_{\varepsilon_n}$. Then, for each $n \in \mathbb{N}$, the function $f' o \xi'_n$ is a continuous function from the compact set $f'(S)$ into itself. From Schauder's fixed point theorem, $f o \xi'_n$ has a fixed point u_n, but $\operatorname{ext}\delta^1_{F(\cdot,v(\cdot),\dot{v}(\cdot))} = \delta^1_{\operatorname{ext}F(\cdot,v(\cdot),\dot{v}(\cdot))}$ [24] so $u_n \in \Delta_{P_e}$. Assume that $u_n \to \widehat{u}$ in $C^2(I,\mathbb{R}^n)$. From Lemma 6, we obtain

$$\|u_n(t) - u(t)\| \le \int_0^1 \left[\int_0^1 |\mathcal{G}(t,\tau)| \left\| \xi'{}_n(\tau) - S_n(\tau) \right\| d\tau + \int_0^1 |\mathcal{G}(t,\tau)| \left\| (S_n(\tau) - h(\tau)) \right\| d\tau \right] ds. \tag{42}$$

But $\xi'_n - S_n \to 0$ with respect to the norm $\| \cdot \|_w$ and from Lemma 3 we get $\xi'_n - S_n \to 0$ weakly in $C^1(I,\mathbb{R}^n)$. So we have

$$\int_0^1 |\mathcal{G}(t,\tau)| \left\| \xi'_n(\tau) - S_n(\tau) \right\| \, d\tau \longrightarrow 0. \tag{43}$$

Moreover, as $n \to \infty$ we have

$$\begin{aligned} \|\widehat{u}(t) - u(t)\| &\le \|u - \widehat{u}\|_{C^1(I,\mathbb{R}^n)} \int_0^1 |\mathcal{G}(t,\tau)| \left(k_1(\tau) + k_2(\tau)\right) d\tau \\ &\le \|u - \widehat{u}\|_{C^1(I,\mathbb{R}^n)} \left\| k_1(\tau) + k_2(\tau) \right\| C_0. \end{aligned} \tag{44}$$

Since by assumption (ii), $\| k_1 + k_2 \| < 1/2C_0$, thus from Lemma 6, we get $u = \widehat{u}$. So $u_n \to u$ in $C^2(I,\mathbb{R}^n)$ and $u \in \overline{\Delta}_{Q_e}$ where the closure is taken in $C^2(I,\mathbb{R}^n)$ which means that $\Delta_P \subseteq \overline{\Delta}_{P_e}$. If $v_n \in \Delta_Q$ and $v_n \to v$ in $C^2(I,\mathbb{R}^n)$, then $v_n = f'(y_n)$ for $y_n \in \delta^1_{F'(\cdot,v(\cdot),\dot{v}(\cdot))}$. From assumption (iii) and the Dunford-Pettis theorem, $\{y_n\}_{n\in\mathbb{N}}$ is weakly sequentially compact in $C^2(I,\mathbb{R}^n)$. By [25, Theorem 3.1], we get

$$\begin{aligned} y(t) \in \overline{\operatorname{conv}}\ \overline{\lim}\{y_n(t)\}_{n\in\mathbb{N}} &\subseteq \overline{\operatorname{conv}}\ \overline{\lim}\, F(t, v_n(t), \dot{v}_n(t)) \\ &= F(t, v(t), \dot{v}(t)) \quad \text{a.e. on } I. \end{aligned} \tag{45}$$

Moreover, $f'(y_n) \to f'(y)$ in $C^2(I,\mathbb{R}^n)$ for $y \in C^2(I,\mathbb{R}^n)$ and $y(t) \in F(t,v(t),\dot{v}(t))$ a.e. on I. Hence, $v \in \Delta_Q$; that is, Δ_Q is closed in $C^2(I,\mathbb{R}^n)$. □

Acknowledgments

The author is deeply indebted and thankful to the deanship of the scientific research and his helpful and distinct team of employees at Taibah University, Al-Madinah Al-Munawarah, Saudia Arabia. This research work was supported by a Grant no. 3029/1434.

References

[1] A. M. Gomaa, *Set-valued functions and set-valued differential equations [Ph.D. thesis]*, Faculty of Science, Cairo University, 1999.

[2] A. G. Ibrahim and A. M. Gomaa, "Existence theorems for a functional multivalued three-point boundary value problem of second order," *Journal of the Egyptian Mathematical Society*, vol. 8, no. 2, pp. 155–168, 2000.

[3] A. G. Ibrahim and A. G. Gomaa, "Extremal solutions of classes of multivalued differential equations," *Applied Mathematics and Computation*, vol. 136, no. 2-3, pp. 297–314, 2003.

[4] A. M. Gomaa, "On the solution sets of three-point boundary value problems for nonconvex differential inclusions," *Journal of the Egyptian Mathematical Society*, vol. 12, no. 2, pp. 97–107, 2004.

[5] D. L. Azzam, C. Castaing, and L. Thibault, "Three boundary value problems for second order differential inclusions in Banach spaces," *Control and Cybernetics*, vol. 31, no. 3, pp. 659–693, 2002.

[6] B. Satco, "Second order three boundary value problem in Banach spaces via Henstock and Henstock-Kurzweil-Pettis integral," *Journal of Mathematical Analysis and Applications*, vol. 332, no. 2, pp. 919–933, 2007.

[7] P. Hartman, *Ordinary Differential Equations*, John Wiley and Sons, New York, NY, USA, 1964.

[8] A. M. Gomaa, "On the solution sets of four-point boundary value problems for nonconvex differential inclusions," *International Journal of Geometric Methods in Modern Physics*, vol. 8, no. 1, pp. 23–37, 2011.

[9] A. M. Gomaa, "On four-point boundary value problems for differential inclusions and differential equations with and without multivalued moving constraints," *Czechoslovak Mathematical Journal*, vol. 62, no. 137, pp. 139–154, 2012.

[10] C. J. Himmelberg and F. S. Van Vleck, "Selection and implicit function theorems for multifunctions with Souslin graph," *Bulletin de l'Académie Polonaise des Sciences*, vol. 19, pp. 911–916, 1971.

[11] C. J. Himmelberg and F. S. Van Vleck, "Some selection theorems for measurable functions," *Canadian Journal of Mathematics*, vol. 21, pp. 394–399, 1969.

[12] C. J. Himmelberg and F. S. Van Vleck, "Extreme points of multifunctions," *Indiana University Mathematics Journal*, vol. 22, pp. 719–729, 1973.

[13] K. Kuratowski and C. Ryll-Nardzewski, "A general theorem on selectors," *Bulletin de l'Académie Polonaise des Sciences*, vol. 13, pp. 397–403, 1965.

[14] E. Klein and A. C. Thompson, *Theory of Correspondences*, John Wiley and Sons, New York, NY, USA, 1984.

[15] C. Castaing and M. Valadier, *Convex Analysis and Measurable Multifunctions*, vol. 580 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 1977.

[16] J.-P. Aubin and A. Cellina, *Differential Inclusions Set-Valued Maps and Viability Theory*, vol. 264, Springer, Berlin, Germany, 1984.

[17] F. S. De Blasi and J. Myjak, "On continuous approximations for multifunctions," *Pacific Journal of Mathematics*, vol. 123, no. 1, pp. 9–31, 1986.

[18] A. A. Tolstonogov, "Extremal selectors of multivalued mappings and the "bang-bang" principle for evolution inclusions," *Bulletin de l'Académie Polonaise des Sciences, Série des Sciences Mathématiques*, vol. 317, no. 3, pp. 589–593, 1991.

[19] D. Kravvaritis and N. S. Papageorgiou, "Boundary value problems for nonconvex differential inclusions," *Journal of Mathematical Analysis and Applications*, vol. 185, no. 1, pp. 146–160, 1994.

[20] S. R. Bernfeld and V. Lakshmikantham, *An Introduction to Nonlinear Boundary Value*, Academic Press, New York, NY, USA, 1974.

[21] A. G. Ibrahim and A. M. Gomaa, "Topological properties of the solution sets of some differential inclusions," *Pure Mathematics and Applications*, vol. 10, no. 2, pp. 197–223, 1999.

[22] A. Bressan and G. Colombo, "Extensions and selections of maps with decomposable values," *Studia Mathematica*, vol. 90, no. 1, pp. 69–86, 1988.

[23] N. S. Papageorgiou, "On measurable multifunctions with applications to random multivalued equations," *Mathematica Japonica*, vol. 32, no. 3, pp. 437–464, 1987.

[24] M. Benamara, *Point Extrémaux, Multi-applications et Fonctionelles Intégrales [M.S. thesis]*, Université de Grenoble, 1975.

[25] N. S. Papageorgiou, "Convergence theorems for Banach space valued integrable multifunctions," *International Journal of Mathematics and Mathematical Sciences*, vol. 10, no. 3, pp. 433–442, 1987.

[26] L. X. Truong, L. T. P. Ngoc, and N. T. Long, "Positive solutions for an m-point boundary-value problem," *Electronic Journal of Differential Equations*, vol. 2008, no. 111, 10 pages, 2008.