Research Article
Best Polynomial Approximation in L^p-Norm and (p,q)-Growth of Entire Functions

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The classical growth has been characterized in terms of approximation errors for a continuous function on $[-1, 1]$ by Reddy (1970), and a compact K of positive capacity by Nguyen (1982) and Winiarski (1970) with respect to the maximum norm. The aim of this paper is to give the general growth $((p,q)$-growth) of entire functions in $\mathbb{C}^n$ by means of the best polynomial approximation in terms of L^p-norm, with respect to the set $\Omega_r = \{z \in \mathbf{C}^n; \exp V_K(z) \le r\}$, where $V_K = \sup\{(1/d) \log |P_d|, P_d$ polynomial of degree $\le d, \|P_d\|_K \le 1\}$ is the Siciak's extremal function on an L-regular nonpluripolar compact K is not pluripolar.

1. Introduction

Let $f(z) = \sum_{k=0}^{+\infty} a_k z^{\lambda_k}$ be a nonconstant entire function and $M(f, r) = \max_{|z|=r} |f(z)|$. It is well known that the function $r \mapsto \log(M(f, r))$ is indefinitely increasing convex function of $\log(r)$. To estimate the growth of f precisely, Boas (see [1]) has introduced the concept of order, defined by the number ρ ($0 \le \rho \le +\infty$):

$$\rho = \limsup_{r \to +\infty} \frac{\log \log (M(f, r))}{\log (r)}. \tag{1}$$

The concept of type has been introduced to determine the relative growth of two functions of the same nonzero finite order. An entire function, of order ρ, $0 < \rho < +\infty$, is said to be of type σ, $0 \le \sigma \le +\infty$, if

$$\sigma = \limsup_{r \to +\infty} \frac{\log (M(f, r))}{r^\rho}. \tag{2}$$

If f is an entire function of infinite or zero order, the definition of type is not valid and the growth of such function cannot be precisely measured by the above concept. Bajpai et al. (see [2]) have introduced the concept of index-pair of an

entire function. Thus, for $p \ge q \ge 1$, they have defined the number

$$\rho (p, q) = \limsup_{r \to +\infty} \frac{\log^{[p]} (M (f, r))}{\log^{[q]} (r)}, \tag{3}$$

$b \le \rho(p, q) \le +\infty$, where $b = 0$ if $p > q$ and $b = 1$ if $p = q$, where $\log^{[0]}(x) = x$, and $\log^{[p]}(x) = \log(\log^{[p-1]}(x))$, for $p \ge 1$.

The function f is said to be of index-pair (p, q) if $\rho(p-1, q-1)$ is nonzero finite number. The number $\rho(p, q)$ is called the (p, q)-order of f.

Bajpai et al. have also defined the concept of the (p, q)-type $\sigma(p, q)$, for $b < \rho(p, q) < +\infty$, by

$$\sigma (p, q) = \limsup_{r \to +\infty} \frac{\log^{[p-1]} ((M (f, r)))}{\left(\log^{[q-1]} (r)\right)^{\rho(p,q)}}. \tag{4}$$

In their works, the authors established the relationship of (p, q)-growth of f with respect to the coefficients a_k in the Maclaurin series of f.

We have also many results in terms of polynomial approximation in classical case. Let K be a compact subset

of the complex plane $\mathbb{C}$ of positive logarithmic capacity and f a complex function defined and bounded on K. For $k \in \mathbb{N}$, put

$$E_k(K, f) = \|f - T_k\|_K, \tag{5}$$

where the norm $\|\cdot\|_K$ is the maximum on K and T_k is the kth Chebytchev polynomial of the best approximation to f on K.

Bernstein showed (see [3, page 14]), for $K = [-1, 1]$, that there exists a constant $\rho > 0$ such that

$$\lim_{k \to +\infty} k^{1/\rho} \sqrt[k]{E_k(K, f)} \tag{6}$$

is finite, if and only if f is the restriction to K of an entire function of order ρ and some finite type.

This result has been generalized by Reddy (see [4, 5]) as follows:

$$\lim_{k \to +\infty} \sqrt[k]{E_k(K, f)} = (\rho e \sigma) \, 2^{-\rho} \tag{7}$$

if and only if f is the restriction to K of an entire function g of order ρ and type σ for $K = [-1, 1]$.

In the same way Winiarski (see [6]) generalized this result to a compact K of the complex plane $\mathbb{C}$ of positive logarithmic capacity, denoted $c = \text{cap}(K)$ as follows.

If K is a compact subset of the complex plane $\mathbb{C}$, of positive logarithmic capacity, then

$$\lim_{k \to +\infty} k(E_k(K, f))^{\rho/k} = c^\rho e \rho \sigma \tag{8}$$

if and only if f is the restriction to K of an entire function of order ρ ($0 < \rho < +\infty$) and type σ.

Recall that the capacity of $[-1, 1]$ is $\text{cap}([-1, 1]) = 1/2$ and the capacity of a unit disc is $\text{cap}(D(0, 1)) = 1$.

The authors considered, respectively, the Taylor development of f with respect to the sequence $(z_n)_n$ and the development of f with respect to the sequence $(W_n)_n$ defined by

$$W_n(z) = \prod_{j=1}^{j=n} (z - \eta_{nj}), \quad n = 1, 2, \ldots, \tag{9}$$

where $\eta^{(n)} = (\eta_{n0}, \eta_{n1}, \ldots, \eta_{nn})$ is the nth extremal points system of K (see [6, page 260]).

We remark that the above results suggest that the rate at which the sequence $(\sqrt[k]{E_k(K, f)})_k$ tends to zero depends on the growth of the entire function (order and type).

Harfaoui (see [7]) obtained a result of generalized order in terms of approximation in L^p-norm for a compact of $\mathbb{C}^n$.

The aim of this paper is to generalize the growth $((p, q)$-order and (p, q)-type), studied by Reddy (see [4, 5]) and Winiarski (see [6]), in terms of approximation in L^p-norm for a compact of $\mathbb{C}^n$ satisfying some properties which will be defined later.

We also obtain a general result of Harfaoui (see [7]) in term of (p, q)-order and (p, q)-type for the functions

$$\alpha(x) = \log^{p-1}(x), \quad \beta(x) = \log^{q-1}(x) \quad \text{for } (p, q) \in \mathbb{N}^2. \tag{10}$$

So we establish relationship between the rate at which $(\pi_k^p(K, f))^{1/k}$, for $k \in \mathbb{N}$, tends to zero in terms of best approximation in L^p-norm, and the generalized growth of entire functions of several complex variables for a compact subset K of $\mathbb{C}^n$, where K is a compact well selected and

$$\pi_k^p(K, f) = \inf \left\{ \|f - P\|_{L^p(K, \mu)}; P \in \mathscr{P}_k(\mathbb{C}^n) \right\}, \tag{11}$$

where $\mathscr{P}_k(\mathbb{C}^n)$ is the family of all polynomials of degree $\leq k$ and μ is the well selected measure (the equilibrium measure $\mu = (dd^c V_K)^n$ associated to a L-regular compact K) (see [8]) and $L^p(K, \mu)$, $p \geq 1$, is the class of all functions such that

$$\|f\|_{L^p(K, \mu)} = \left(\int_K |f|^p \, d\mu \right)^{1/p} < \infty. \tag{12}$$

In this work we give the generalization of these results in $\mathbb{C}^n$, replacing the circle $\{z \in \mathbb{C}; |z| = r\}$ by the set $\{z \in \mathbb{C}^n; \exp(V_K(z)) < r\}$, where V_K is the Siciak's extremal function of K, a compact of $\mathbb{C}^n$ satisfying some properties (see [9, 10]), and using the development of f with respect to the sequence $(A_k)_{k \in \mathbb{N}}$ constructed by Zeriahi (see [11]).

Recall that in the paper of Winiarski (see [6]) the author used the Cauchy inequality. In our work we replace this inequality by an inequality given by Zeriahi (see [11]).

2. Definitions and Notations

Before we give some definitions and results which will be frequently used in this paper, let K be a compact of $\mathbb{C}^n$ and let $\|\cdot\|_K$ denote the maximum norm on K.

Multivariate polynomial inequalities are closely related to the Siciak extremal function associated with a compact subset K of $\mathbb{C}^n$,

$$V_K = \sup \left\{ \frac{1}{d} \log |P_d|, P_d \text{ polynomial in } \mathbb{C}^n \text{ of degree} \right.$$

$$\left. \leq d, \|P_d\|_K \leq 1 \right\}. \tag{13}$$

Siciak's function establishes an important link between polynomial approximation in several variables and pluripotential theory.

It is known (see [10]) that

$$V_K(z) = \sup \left\{ u(z) : u \in \mathscr{L}(\mathbb{C}^n); u \leq 0 \text{ on } K \right\}, \tag{14}$$

where

$$\mathscr{L}(\mathbb{C}^n) = \{ u \in \text{PSH}(\mathbb{C}^n) : u(z) - \log(|z|) $$

$$\leq O(1) \text{ as } |z| \longrightarrow \infty \} \tag{15}$$

is the Lelong class of plurisubharmonic functions with logarithmic growth at infinity. If K is nonpluripolar (i.e., there is no plurisubharmonic function u such that $K \subset \{u(z) = -\infty\}$), then the plurisubharmonic function $V_K^*(z) = \limsup_{w \to z} V_K(w)$ is the unique function in the class $\mathscr{L}(\mathbb{C}^n)$

which vanishes on K except perhaps for a pluripolar subset and satisfies the complex Monge-Ampère equation (see [12]):

$$\left(dd^c V_K\right)^n = 0 \quad \text{on } \mathbb{C}^n \setminus K. \tag{16}$$

If $n = 1$, the Monge-Ampère equation reduces to the classical Laplace equation.

For this reason, the function V_K^* is considered as a natural counterpart of the classical Green function with logarithmic pole at infinity and it is called the pluricomplex Green function associated with K.

Definition 1 (Siciak [10]). The function

$$V_K = \sup \left\{ \frac{1}{d} \log |P_d|, P_d \text{ polynomial of degree} \leq d, \right. \tag{17}$$

$$\left. \|P_d\|_K \leq 1 \right\}$$

is called the Siciak's extremal function of the compact K.

Definition 2. A compact K in $\mathbb{C}^n$ is said to be L-regular if the extremal function, V_K, associated to K is continuous on $\mathbb{C}^n$.

Regularity is equivalent to the following Bernstein-Markov inequality (see [9]).

For any $\epsilon > 0$, there exists an open $U \supset K$ such that for any polynomial P

$$\|P\|_U \leq e^{\epsilon \cdot \deg(P)} \|P\|_K. \tag{18}$$

In this case we take $U = \{z \in \mathbb{C}^n; V_K(z) < \epsilon\}$.

Regularity also arises in polynomial approximation. For $f \in \mathscr{C}(K)$, we let

$$\epsilon_k(K, f) = \inf \left\{ \|f - P\|_K, P \in \mathscr{P}_k(\mathbb{C}^n) \right\}, \tag{19}$$

where $\mathscr{P}_k(\mathbb{C}^n)$ is the set of polynomials of degree at most d. Siciak showed that (see [10]).

If K is L-regular, then

$$\limsup_{k \to +\infty} (\epsilon_k(K, f))^{1/k} = \frac{1}{R} < 1 \tag{20}$$

if and only if f has an analytic continuation to

$$\left\{ z \in \mathbf{C}^n; V_K(z) < \log\left(\frac{1}{R}\right) \right\}. \tag{21}$$

It is known that if K is a compact L-regular of $\mathbb{C}^n$, there exists a measure μ, called extremal measure, having interesting properties (see [9, 10]), in particular, we have the following properties.

(P_1) Bernstein-Markov inequality: for all $\epsilon > 0$, there exists a constant $C = C_\epsilon$ such that

$$(\text{BM}): \|P_d\|_K = C(1 + \epsilon)^{s_k} \|P_d\|_{L^2(K,\mu)}, \tag{22}$$

for every polynomial of n complex variables of degree at most d.

(P_2) Bernstein-Walsh (BW) inequality: for every set L-regular K and every real $r > 1$ we have

$$\|f\|_K \leq M r^{\deg(f)} \left(\int_K |f|^p \, d\mu \right)^{1/p}. \tag{23}$$

Note that the regularity is equivalent to the Bernstein-Markov inequality.

Let $\alpha : \mathbb{N} \to \mathbb{N}^n, k \mapsto \alpha(k) = (\alpha_1(k), \ldots, \alpha_n(k))$ be a bijection such that

$$|\alpha(k+1)| \geq |\alpha(k)|, \quad \text{where } |\alpha(k)| = \alpha_1(k) + \cdots + \alpha_n(k). \tag{24}$$

Zeriahi (see [11]) has constructed according to the Hilbert-Schmidt method a sequence of monic orthogonal polynomials according to an extremal measure (see [9]), $(A_k)_k$, called extremal polynomial, defined by

$$A_k(z) = z^{\alpha(k)} + \sum_{j=1}^{k-1} a_j z^{\alpha(j)} \tag{25}$$

such that

$$\|A_k\|_{L^p(K,\mu)}$$

$$= \left[\inf \left\{ \left\| z^{\alpha(k)} + \sum_{j=1}^{k-1} a_j z^{\alpha(j)} \right\|_{L^p_{(K,\mu)}}, \right. \right. \tag{26}$$

$$\left. \left. (a_1, a_2, \ldots, a_n) \in \mathbb{C}^n \right\} \right]^{1/\alpha_k}.$$

We need the following notations and lemma which will be used in the sequel (see [2]):

(N_1) $v_k = v_k(K) = \|A_k\|_{L^2(K,\mu)}$,

(N_2) $a_k = a_k(K) = \|A_k\|_K = \max_{z \in K} |A_k(z)|$ and $\tau_k = (a_k)^{1/s_k}$, where $s_k = \deg(A_k)$.

For $p \in \mathbb{N}$, put, for $p \geq 1$ and $x > 0$,

$$\log^{[p]}(x) = \log\left(\log^{[p-1]}(x)\right),$$

$$\exp^{[p]}(x) = \exp\left(\exp^{[p-1]}(x)\right),$$

$$\Lambda_{[p]} = \prod_{k=1}^{p} \log^{[k]}(x), \qquad E_{[p]}(x) = \prod_{k=0}^{p} \exp^k(x), \tag{27}$$

$$\log^{[0]}(x) = x, \qquad \exp^{[0]}(x) = x.$$

Lemma 3 (see [2]). *With the above notations one has the following results:*

(RR1) $E_{[-p]}(x) = x/\Lambda_{[p-1]}(x)$ and $\Lambda_{[-p]}(x) = x/E_{[p-1]}(x)$,

(RR2) $(d/dx)\exp^{[p]}(x) = E_{[p]}(x)/x = 1/\Lambda_{[-p-1]}(x)$,

(RR3) $(d/dx)\log^{[p]}(x) = E_{[-p]}(x)/x = 1/\wedge_{[p-1]}(x)$,

(RR4)

$$E_{[p]}^{-1}(x) = \begin{cases} x, & \text{if } p = 0, \\ \log^{[p-1]}\left\{\log(x) - \log^{[2]}(x) \right. \\ \left. \quad + o\left(\log_{[3]}(x)\right)\right\}, & \text{if } p = 1, 2, \dots, \end{cases}$$
(28)

(RR5)

$$\lim_{x \to +\infty} \exp\left(E_{[p-2]}(x)\right) = \begin{cases} e, & \text{if } p = 2, \\ 1, & \text{if } p \geq 3, \end{cases}$$
(29)

(RR6)

$$\lim_{x \to +\infty} \left[\exp^{[p-1]}\left(E_{[p-2]}^{-1}(x)\right)\right]^{1/x} = \begin{cases} e, & \text{if } p = 2, \\ 1, & \text{if } p \geq 3. \end{cases}$$
(30)

For more details of these results, see [2].

Definition 4. Let K be a compact L-regular and put

$$\Omega_r = \left\{z \in \mathbb{C}^n; \exp V_K(z) \leq r\right\}.$$
(31)

An entire function f is said to be of (K, p, q)-order $\rho_K(p, q)$ if it is of index-pair (p, q) such that

$$\rho = \rho_K(p, q) = \limsup_{r \to +\infty} \frac{\log^{[p]}\left(\|f\|_{\overline{\Omega}_r}\right)}{\log^{[q]} r}.$$
(32)

If $\rho \in [\beta, +\infty[$, the (K, p, q)-type is defined by

$$\sigma = \sigma_K(p, q) = \limsup_{r \to +\infty} \frac{\log^{[p-1]}\left(\|f\|_{\overline{\Omega}_r}\right)}{\left(\log^{[q-1]} r\right)^\rho},$$
(33)

with $\beta = 1$ if $p = q$ and $\beta = 0$ and $p > q$.

3. (p, q)-Growth in terms of the Coefficients of the Development with respect to Extremal Polynomials

The object of this section is to establish the relationship of (p, q)-growth of an entire function with respect to the set

$$\Omega_r = \left\{\exp\left(V_K\right) < r\right\}$$
(34)

and the coefficients of entire function f on $\mathbb{C}^n$ of the development with respect to the sequence of extremal polynomials.

The (p, q)-growth of an entire function is defined by (K, p, q)-order and (K, p, q)-type of f.

Let $(A_k)_k$ be the basis of extremal polynomials associated to the set K defined by (25). Recall that $(A_k)_k$ is a basis of the vector space of entire functions, hence if f is an entire function, then

$$f = \sum_{k \geq 1} f_k A_k.$$
(35)

To prove the aim result of this section we need Brernstein-Walsh inequality and the following lemmas which have been proved by Zeriahi (see [11]).

Lemma 5. *Let K be a compact L-regular subset of $\mathbb{C}^n$ and let f be an entire function such that $f = \sum_{k=0}^{+\infty} f_k A_k$. Then for every $\theta > 1$, there exists an integer $N_\theta \geq 1$ and a constant C_θ such that*

$$\pi_k^p(K, f) \leq C_\theta \frac{(r+1)^{N_\theta}}{(r-1)^{2N-1}} \frac{\|f\|_{\overline{\Omega}_{r\theta}}}{r^k},$$
(36)

$$|f_k| \, v_k \leq C_\theta \frac{(r+1)^{N_\theta}}{(r-1)^{2N-1}} \frac{\|f\|_{\overline{\Omega}_{r\theta}}}{r^{s_k}},$$
(37)

where $N_\theta \in \mathbb{N}$ and $C_\theta > 0$ are constant not depending on (r, k, f).

Lemma 6. *If K is an L-regular, then the sequence of extremal polynomials $(A_k)_k$ satisfies*

$$\lim_{k \to +\infty} \left(\frac{|A_k(z)|}{v_k}\right)^{1/s_k} = \exp\left(V_K(z)\right)$$
(38)

for every $z \in \mathbb{C}^n$, and

$$\lim_{k \to +\infty} \left(\frac{\|A_k\|_K^{s_k}}{v_k}\right)^{1/s_k} = 1.$$
(39)

Recall that the second assertion (37) of Lemma 5 replaces the Cauchy inequality for complex function defined on the complex plane $\mathbb{C}$.

Theorem 7. *Let $f = \sum_{k \geq 1} f_k A_k$ be an entire function. Then f is said of a finite (K, p, q)-order $\rho_K(p, q)$ if and only if*

$$L(p, q) = \limsup_{k \to +\infty} \frac{\log^{[p-1]}(s_k)}{\log^{[q-1]}\left[(-1/s_k)\log\left(|f_k| \cdot \tau_k^{s_k}\right)\right]} < +\infty$$
(40)

and $\rho_K(p, q) = P_1(L(p, q))$, where

$$P_1(L(p, q)) = \begin{cases} L(p, q), & \text{if } p > q, \\ 1 + L(2, 2), & \text{if } p = q = 2, \\ \max(1, L(p, q)), & \text{if } 3 \leq p = q < +\infty, \\ +\infty, & \text{if } p = q = +\infty, \end{cases}$$
(41)

for $(p, q) \in \mathbb{N}^2$ with $p \geq q$.

Proof. Put $\rho = \rho_K(p, q)$. Let us prove that $\varrho \geq P_1(L(p, q))$. If f is of finite (p, q)-order ρ, then we have

$$\rho = \limsup_{r \to +\infty} \frac{\log^{[p]}\|f\|_{\overline{\Omega}_r}}{\log^{[q]} r} = \limsup_{r \to +\infty} \frac{\log^{[p]}\|f\|_{\overline{\Omega}_{r\theta}}}{\log^{[q]}(r\theta)}.$$
(42)

Thus for every $\varepsilon > 0$ there exists $r(\varepsilon)$ such that for every $r > r(\varepsilon)$

$$\log\left(\|f\|_{\overline{\Omega}_{r\theta}}\right) \leq \exp^{[p-2]}\left(\log^{[q-1]}(r\theta)\right)^{\rho+\varepsilon}.$$
(43)

Using the inequalities (37) of **Lemma 5** and (39) of **Lemma 6**, one has, for every $\varepsilon > 0$, there exist $r(\varepsilon)$ and $k(\varepsilon)$ such that for every $r > r(\varepsilon)$ and $k > k(\varepsilon)$

$$\log\left(\left|f_k\right| \tau_k^{s_k}\right) \le s_k \log\left(1 + \varepsilon\right) + \log\left(C_\theta\right) + N_\theta \log\left(1 + r\right)$$
$$- (2N - 1)\log\left(r - 1\right)$$
$$- s_k \log\left(r\right) + \log\left(\|f\|_{\overline{\Omega}_{r\theta}}\right)$$
$$(44)$$

for $r > r(\varepsilon)$ and $k > k(\varepsilon)$. But for $r > r(\varepsilon)$ and $k > k(\varepsilon)$ we have

$$-\frac{1}{s_k}\log\left(\left|f_k\right| \tau_k^{s_k}\right) \ge -\frac{1}{s_k} \cdot \log\left(\frac{C_\theta\left(1 + \varepsilon\right) \cdot \log\left(1 + r\right)^{N_\theta}}{(r - 1)^{(2N - 1)}}\right)$$
$$+ \exp^{[p-2]}\left(\log^{[q-1]}\left(r\theta\right)\right)^{\rho + \varepsilon}.$$
$$(45)$$

Then, by proceeding to limits as $k \to \infty$, we get for r sufficiently large

$$\log\left(\left|f_k\right| \tau_k^{s_k}\right) \le (1 + o(1))\log\left(r\right).$$
$$(46)$$

(i) For $(p, q) \ne (2, 2)$ with $p > q$, let

$$r_k = \frac{1}{\theta}\exp^{[q-1]}\left(\log^{[p-2]}\left(\frac{s_k}{\rho + \varepsilon}\right)^{1/\rho+\varepsilon}\right) > r(\varepsilon).$$
$$(47)$$

Then if we replace in the equality (46) r by r_k, we get easily that for k sufficiently large

$$-\frac{1}{s_k}\log\left(\left|f_k\right| \tau_k^{s_k}\right)$$
$$\ge \exp^{[q-2]}\left(\log^{[p-2]}\left(\frac{s_k}{\rho + \varepsilon}\right)^{1/(\rho+\varepsilon)}\right)[1 + o(1)].$$
$$(48)$$

After passing to the upper limit, we get for $p > q$

$$\limsup_{k \to +\infty}\frac{\log^{[p-1]}\left(s_k\right)}{\log^{[q-1]}\left((-1/s_k)\log\left(\left|f_k\right| \cdot \tau_k^{s_k}\right)\right)} \le \rho.$$
$$(49)$$

(ii) For $3 \le p = q < +\infty$, the inequality (46) gives $\rho \ge \max(1, L(p, q))$ (because $\rho \ge 1$ for $p = q$).

(iii) For $(p, q) = (2, 2)$, choose $r_k = (1/\theta)\exp(s_k/(\rho + \varepsilon))^{1/(\rho-1+\varepsilon)} > r(\varepsilon)$ and in the same way we show that

$$-\frac{1}{s_k}\log\left(f_k \mid \tau_k^{s_k}\right) \ge \left(\frac{s_k}{\rho + \varepsilon}\right)^{1/(\rho-1+\varepsilon)} \cdot \frac{\rho - 1 + \varepsilon}{\rho + \varepsilon} \cdot (1 + o(1))$$
$$(50)$$

for k sufficiently large, thus

$$\limsup_{k \to +\infty}\frac{\log\left(s_k\right)}{\log\left((-1/s_k)\log\left(\left|f_k\right| \cdot \tau_k^{s_k}\right)\right)} \le \rho - 1.$$
$$(51)$$

By combining (i), (ii), and (iii) we have $\rho \ge P_1(L(p, q))$. This result holds obviously if $\rho = +\infty$.

We prove now reverse inequality $\rho \le P_1(L(p, q))$. By the definition of $L(p, q)$, for every $\varepsilon > 0$ there exists $k(\varepsilon)$ such that for every $k \ge k(\varepsilon)$

$$\left|f_k\right| \tau_k^{s_k} \le \exp\left(-s_k \exp^{[q-2]}\left(\log^{[p-2]}\left(s_k\right)\right)^{1/(L+\varepsilon)}\right),$$
$$(52)$$

where $L = L(p, q)$, for simplification.

Let $k(r)$ be a positive integer such that, for $k \ge k(r)$,

$$s_{k(r)} \le \exp^{[p-2]}\left(\log^{[q-1]}(2(1 + \varepsilon)r)^{L+\varepsilon}\right) < s_{k(r)} + 1$$
$$(53)$$

and $k(r) > k(\varepsilon)$, by (39), (37), (BM) and (BW) inequalities, there exists $k_0 \in \mathbb{N}$, such that

$$\|f\|_{\overline{\Omega}_r} \le \sum_{k=0}^{k=k(\varepsilon)}\left|f_k\right|\left|A_k\left(z\right)\right| + C_\varepsilon\sum_{k=k(\varepsilon)+1}^{k=k(r)}\left|f_k\right| \tau_k^{s_k} r^{s_k} + C_\varepsilon'$$
$$\cdot \sum_{k=k(r)+1}^{+\infty}\left|f_k\right| \tau_k^{s_k}((1 + \varepsilon)r)^{s_k}.$$
$$(54)$$

Indeed,

$$\|f\|_{\overline{\Omega}_r} \le \sum_{k=0}^{k=+\infty}\left|f_k\right| \cdot \left|A_k\left(z\right)\right|$$
$$(55)$$

(because z satisfies $\exp(V_K(z)) \le r$).

By (38) and (39), for k sufficiently large we have

$$\left|A_k\right| \le (1 + \varepsilon)^{s_k}\left[\exp\left(V_K\left(z\right)\right)\right]^{s_k},$$
$$\tau_k^{s_k} \le (1 + \varepsilon)^{s_k}\nu_k.$$
$$(56)$$

Therefore, for r sufficiently large we have

$$\|f\|_{\overline{\Omega}_r} \le A_{k(\varepsilon)} + \underbrace{\sum_{k=k_\varepsilon+1}^{N(r)}f_k\tau_k^{s_k}}_{1} + \underbrace{\sum_{k=N(r)}^{+\infty}2^{-s_k}}_{2},$$
$$(57)$$

where $A_{k(\varepsilon)}$ is a polynomial of degree not exceeding k_ε. By using (46) we get

$$\|f\|_{\overline{\Omega}_r} \le A_{k(\varepsilon)} + c_\varepsilon(2(1 + \varepsilon)r)^{s_{k(r)}}$$
$$\times \underbrace{\sum_{k=0}^{+\infty}\exp\left\{-s_k\exp^{[q-2]}\left(\ln^{[p-2]}(s_k)^{L+\varepsilon}\right)\right\}}_{1}$$
$$+ \underbrace{\sum_{k=0}^{+\infty}2^{-s_k}}_{2}.$$
$$(58)$$

By (52) the series (1) is convergent, and (2) is obviously convergent, hence we have for r sufficiently large

$$\|f\|_{\overline{\Omega}_r} \le A_{k(\varepsilon)} + A_1(2(1 + \varepsilon)r)^{s_{k(r)}} + 1,$$
$$(59)$$

where A_1 is a constant. Thus, for r sufficiently large we obtain

$$\log\left(\|f\|_{\overline{\Omega}_r}\right) \le s_k \log\left((1 + \varepsilon)r\right)(1 + o(1)).$$
$$(60)$$

Therefore, for r sufficiently large

$$\log^{[2]} \left(\|f\|_{\overline{\Omega}_r} \right) \leq \log(s_k) + \log^{[2]} ((1+\varepsilon) r) + o(1). \quad (61)$$

For r sufficiently large let

$$s_k = E\left[\exp^{[p-2]} \left(\left(\log^{[q-1]} (2r) \right)^{L+\varepsilon} \right) \right], \quad (62)$$

where $E(x)$ is the integer part of x. Replacing s_k in the inequality (61) we get

$$\log^{[2]} \left(\|f\|_{\overline{\Omega}_r} \right) \leq \exp^{[p-3]} \left(\log^{[q-1]}(2r)^{L+\varepsilon} \right) \\ + \log^{[2]} ((1+\varepsilon) r) + o(1). \quad (*)$$

To prove the result we proceed in three steps.

Step 1. For $(p,q) = (2,2)$, we have

$$\log^{[2]} \left(\|f\|_{\overline{\Omega}_r} \right) \leq (L+\varepsilon) \log^{[2]} (2r) + \log^{[2]} ((1+\varepsilon) r) + o(1). \quad (63)$$

Then

$$\frac{\log^{[2]} \left(\|f\|_{\overline{\Omega}_r} \right)}{\log^{[2]} (2r)} \leq (L+\varepsilon) + 1 + o(1). \quad (64)$$

Proceeding to the upper limit we get $\rho \leq 1 + L(2,2) = P_1(L(2,2))$.

Step 2. For $3 \leq p = q$, since $\rho \geq 1$, we get $\rho \leq \max(1(L(p,p)) = P_1(L(p,q))$.

Step 3. For $p < q$, the relation $(*)$ is equivalent to

$$\log^{[2]} \left(\|f\|_{\overline{\Omega}_r} \right) \leq \exp^{[p-3]} \left(\log^{[q-1]}(2r)^{L+\varepsilon} \right) (1 + +o(1)). \quad (65)$$

Then, for r sufficiently large

$$\log^{[p]} \left(\|f\|_{\overline{\Omega}_r} \right) \leq \log \left(\log^{[q-1]}(2r)^{L+\varepsilon} \right) (1+o(1)) \\ = (L+\varepsilon) \left(\log^{[q]} (2r) \right) (1+o(1)). \quad (66)$$

Passing to the upper limit after division by $\log^{[q]}(2r)$ we obtain $\rho \leq L(p,q)$.

By combining (i), (ii), and (iii) we obtain for $p \geq q \geq 1$, $\rho \leq P_1(L(p,q))$. The inequality is obviously true for $L(p,q) = +\infty$. □

Theorem 8. *Let f be an entire function of (K,p,q)-order $\rho_K(p,q) \in\,]\beta, \infty[$. Then f is of finite (K,p,q)-type $\sigma_K(p,q)$ if and only if*

$$\gamma(p,q) = \limsup_{k \to +\infty} \frac{\log^{[p-2]} (s_k)}{\left[\log^{[q-2]} \left((-1/s_k) \log |f_k| \tau_k^{s_k} \right) \right]^{\rho-C}} < +\infty \quad (67)$$

and $\sigma_K(p,q) = \gamma(p,q)M(p,q)$, where $\beta = 1$ if $p = q$, $\beta = 0$ if $p > q$, $C = 1$ if $p = q = 2$, $C = 0$ if $(p,q) \neq (2,2)$, $s_k = \deg(A_k)$, and

$$M(p,q) = \begin{cases} 1, & \text{if } p \geq 3, \\ \dfrac{1}{e \cdot \rho(2,1)}, & \text{if } (p,q) = (2,1), \\ \dfrac{(\rho(2,2)-1)^{\rho(2,2)-1}}{(\rho(2,2))^{\rho(2,2)}}, & (p,q) = (2,2). \end{cases} \quad (68)$$

Proof. Let us first prove that $\sigma_K(p,q) \leq M(p,q) \cdot \gamma(p,q)$. By the definition of $\gamma = \gamma(p,q)$, for every $\varepsilon > 0$ there exists $k(\varepsilon)$ such that for every $k \geq k(\varepsilon)$,

$$|f_k| \tau_k^{s_k} \leq \exp \left(-s_k \exp^{[q-2]} \left(\frac{\log^{[p-2]} (s_k)}{\gamma+\varepsilon} \right)^{1/(\rho-C)} \right). \quad (69)$$

Let $k(r)$ be a positive integer such that

$$k(r) \leq \exp^{[p-2]} \left((\gamma+\varepsilon) \left(\log^{[q-1]} (2(1+\varepsilon) r) \right)^{\rho-C} \right) \\ < k(r) + 1 \quad (70)$$

and $k \geq k(r)$. By the estimate (39) from **Lemma 6** and the (BM) and (BW) inequalities, there exists $k_0 \in \mathbb{N}$ such that

$$\|f\|_{\overline{\Omega}_r} \leq k(r) + \max_{k \geq 0} \left\{ f_k \cdot \tau_k^{s_k} ((1+\varepsilon) r)^{s_k} \right\} + \sum_{k=0}^{\infty} 2^{-s_k}. \quad (71)$$

Put

$$H(r) = \max_{x \geq 0} \{ \omega(r,x) \}, \quad (72)$$

where

$$\omega(r,x) = -x \exp^{[q-2]} \left(\left[\frac{\log^{[p-2]} (x)}{\gamma+\varepsilon} \right]^{1/\rho-C} \right) \\ + x \log((1+\varepsilon) r). \quad (73)$$

By repeating the argument used in the proof of **Theorem 7** one may easily check that

$$\sigma_K(p,1) \leq M(p,1) \gamma(p,1). \quad (74)$$

For example we will show that $\sigma_K(p,1) \leq M(p,1)\gamma(p,1)$. By the relation (69) we have

$$|f_k| \tau_k^{s_k} \leq \left(\frac{\gamma+\varepsilon}{\log^{[p-2]} (s_k)} \right)^{s_k/\rho}. \quad (75)$$

The maximum of the function $x \mapsto \omega(x,r)$ is reached for $x = x_r$, where x_r is the solution of the equation

$$-\log \left(\frac{\log^{[p-2]} (x)}{\gamma+\varepsilon} \right)^{1/\rho} + \log((1+\varepsilon) r) = \frac{E_{[1-p]} (x)}{\rho}. \quad (76)$$

For $p = 2$ the relation (75) becomes

$$-\log\left(\frac{x}{\gamma+\varepsilon}\right)^{1/\rho} + \log\left((1+\varepsilon)r\right) = \frac{1}{\rho}. \tag{77}$$

Thus

$$x_r = \frac{1}{e}\left(\gamma+\varepsilon\right)\left((1+\varepsilon)r\right)^\rho. \tag{78}$$

Therefore $H(r) = \omega(r, x_r) = x_r/\rho$ and $\omega(r, x_r) = (1/e\rho)(\gamma + \varepsilon)((1+\varepsilon)r)^\rho$ and by (69) we get

$$\|f\|_{\overline{\Omega}_r} \le C_0 + (\gamma+\varepsilon)\left((1+\varepsilon)r\right)^\rho \exp\left(H(r)\right) \tag{79}$$

which gives for r sufficiently large

$$\log\left(\|f\|_{\overline{\Omega}_r}\right) \le H(r) + \rho \log\left((1+\varepsilon)r\right)(1 + o(1)). \tag{80}$$

Passing to the upper limit when $r \to +\infty$ we get

$$\sigma_K(2,1) \le M(2,1)\gamma(2,1). \tag{81}$$

For $p \ge 3$ we have

$$\omega(r, x_r) = \frac{x_r}{\rho} K_{[1-p](x)} = \frac{x_r^2}{\rho \bigwedge_{[p-2]}(x_r)}. \tag{82}$$

Therefore

$$\log\left(\omega(r, x_r)\right) = 2\log(x_r) - \log(\rho) - \log\left(\bigwedge_{[p-2]}(x_r)\right)$$
$$= 2\log(x_r) - \sum_{k=1}^{p-1}\log^k(x_r). \tag{83}$$

Hence

$$\frac{\log\left(\omega(r, x_r)\right)}{\log(x_r)} = 1 - \sum_{k=1}^{p-1}\frac{\log^k(x_r)}{\log(x_r)}. \tag{84}$$

Then, for r sufficiently large,

$$\log\left(\omega(r, x_r)\right) \sim \log(x_r) \sim \exp^{[p-2]}\left[(\gamma+\varepsilon)((1+\varepsilon)r)^\rho\right],$$

$$\|f\|_{\overline{\Omega}_r} \le \exp^{[p-2]}\left[(\gamma+\varepsilon)((1+\varepsilon)r)^\rho\right]. \tag{85}$$

We obtain the result after passing to the upper limit. □

Remark 9. If $n = 1$ and $(p, q) = (2, 1)$, we know that

$$\lim_{k \to +\infty}(\tau_k)^{1/s_k} = \tau(K) = \operatorname{cap}(K). \tag{86}$$

Then by using Theorems 7 and 8 we get

$$\rho_K(2,1) = \lim_{k \to +\infty} - \frac{s_k \log(s_k)}{\log(|f_k|)} = \rho(2,1),$$

$$\sigma_K(2,1) = \limsup_{r \to +\infty} s_k \cdot \left(\left[|f_k| \cdot \tau_k\right]^{1/s_k}\right)^{\rho_K(2,1)} \tag{87}$$

$$= \sigma(2,1)\left(\operatorname{cap}(K)\right)^{\rho_K(2,1)}$$

which gives the result of Winiarski.

Remark 10. The notion of the type associated to a compact in $\mathbb{C}$ was considered by Nguyen (see [13]). In this work the concept of the general type seems to be a new result for a compact in $\mathbb{C}^n$, $(n \ge 2)$, which is not Cartesian product. Also the generalized order is independent of the norm but not the generalized type.

4. Best Polynomial Approximation in terms of L^p-Norm

Let f be a bounded function defined on a L-regular compact K of $\mathbb{C}^n$.

The object of this section is to study the relationship between the rate of the best polynomial approximation of f in L^p-norm and the (p, q)-growth of an entire function g such that $g_{|K} = f$.

To our knowledge, no similar result is known according to polynomial approximation in L^p-norm ($1 \le p \le \infty$) with respect to a measure μ on K in $\mathbb{C}^n$. To prove the aim results we use the results obtained in the second section to give relationship between the general growth of f and the sequence

$$\pi_k^p(K, f) = \inf\left\{\|f - P\|_{L^p(K,\mu)}, P \in \mathscr{P}_k(\mathbb{C}^n)\right\} \tag{88}$$

which extend the classical results of Reddy and Winiarsk in $\mathbb{C}^n$. We need the following lemmas.

Lemma 11. *If K is compact L-regular in $\mathbb{C}^n$, then every function $f \in L_\mathscr{P}^2(K, \mu)$ can be written in the form*

$$f = \sum_{k=0}^{+\infty} f_k \cdot A_k, \tag{89}$$

where $L_\mathscr{P}^2(K, \mu)$ is the closed subspace of $L^2(K, \mu)$ generated by the restrictions to E of polynomials $\mathbb{C}^n$ and (A_k) is the sequence defined by (25).

Lemma 12. *Let (A_k) be the sequence defined by (25) and $f = \sum_{k \ge 0} f_k A_k$ an element of $L^p(K, \mu)$, for $p \ge 1$, then*

$$\limsup_{k \to +\infty}\frac{\log^{[p-1]}(k)}{\log^{[q-1]}\left((-1/k)\log\left(\pi_k^p(K, f)\right)\right)}$$
$$= \limsup_{k \to +\infty}\frac{\log^{[p-1]}(s_k)}{\log^{[q-1]}\left((-1/s_k)\log\left(|f_k| \cdot \tau_k^{s_k}\right)\right)}, \tag{90}$$

$$\limsup_{k \to +\infty}\frac{\log^{[p-2]}(k)}{\left(\log^{[q-2]}\left((-1/k)\log\left(\pi_k^p(K, f)\right)\right)\right)^{\rho-C}}$$
$$= \limsup_{k \to +\infty}\frac{\log^{[p-2]}(s_k)}{\left(\log^{[q-2]}\left((-1/s_k)\log\left(|f_k| \cdot \tau_k^{s_k}\right)\right)\right)^{\rho-C}}. \tag{91}$$

Proof of Lemma 12. The proof is done in two steps ($p \ge 2$ and $1 < p < 2$). Let $f = \sum_{k \ge 0} f_k \cdot A_k$ be an element of $L^p(K, \mu)$.

Step 1. If $f \in L^p(K, \mu)$ with $p \ge 2$, then $f = \sum_{k=0}^{+\infty} f_k \cdot A_k$ with convergence in $L^2(K, \mu)$, where $f_k = 1/v_k^2 \int_K f \overline{A}_k d\mu$,

$k \geq 0$ and therefore $f_k = (1/v_k^2) \int_K (f - P_{s_k-1}) \cdot \overline{A}_k d\mu$ (because $\deg(A_k) = s_k$). Since $|f_k| \leq (1/v_k^2) \int_K |f - P_{s_k-1}| \cdot |\overline{A}_k| d\mu$, we obtain easily, using Bernstein-Walsh inequality and de Hölder inequality, that we have for any $\varepsilon > 0$

$$|f_k| \cdot v_k \leq C_\varepsilon \cdot (1+\varepsilon)^{s_k} \pi_{s_k-1}^p (K, f) \tag{92}$$

for every $k \geq 0$.

Step 2. If $1 \leq p < 2$, let p' such $1/p + 1/p' = 1$, then $p' \geq 2$. By the Hölder inequality we have

$$|f_k| v_k^2 \leq \|f - P_{k-1}\|_{L^p(K,\mu)} \|A_k\|_{L^{p'}(K,\mu)}. \tag{93}$$

But $\|A_k\|_{L^{p'}(E,\mu)} \leq C\|A_k\|_K = Ca_k(K)$, therefore, by the (BM) inequality, we have

$$|f_k| v_k^2 \leq CC_\varepsilon(1+\varepsilon)^{s_k} \|f - P_{s_k-1}\|_{L^p(K,\mu)}. \tag{94}$$

Hence

$$|f_k| v_k^2 \leq C_\varepsilon'(1+\varepsilon)^{s_k} \pi_{s_k}^p (K, f). \tag{95}$$

Thus in both cases we have

$$|f_k| v_k \leq A_\varepsilon(1+\varepsilon)^{s_k} \pi_{s_k}^p (K, f), \tag{96}$$

where A_ε is a constant which depends only on ε. After passing to the upper limit (96) gives

$$\begin{aligned} &\limsup_{k \to +\infty} \frac{\log^{[p-1]}(k)}{\log^{[q-1]}\left((-1/k)\log\left(\pi_k^p (K, f)\right)\right)} \\ &\geq \limsup_{k \to +\infty} \frac{\log^{[p-1]}(s_k)}{\log^{[q-1]}\left((-1/s_k)\log\left(|f_k| \cdot \tau_k^{s_k}\right)\right)}. \end{aligned} \tag{97}$$

To prove the other inequality we consider the polynomial of degree s_k

$$P_{s_k}(z) = \sum_{j=0}^{j=k} f_j A_j, \tag{98}$$

then

$$\pi_{s_k-1}^p (K, f) \leq \sum_{s_j=s_k}^{+\infty} |f_j| \|A_j\|_{L^p(K,\mu)} \leq C_0 \sum_{s_j=s_k+1}^{+\infty} |f_j| \|A_j\|_K. \tag{99}$$

By the Bernstein-Walsh inequality we have

$$\pi_k^p (K, f) \leq C_\varepsilon \sum_{s_j=s_k}^{+\infty} (1+\varepsilon)^{s_j} |f_j| v_j \tag{100}$$

for $k \geq 0$ and $p \geq 1$. If we take as a common factor $(1+\varepsilon)^{s_k} v_k$, the other factor is convergent, thus we have

$$\pi_k^p (K, f) \leq C(1+\varepsilon)^{s_k} \cdot v_k \tag{101}$$

and by (39) of Lemma 6 we have then

$$\pi_k^p (K, f) \leq C(1+\varepsilon)^{s_k} \cdot |f_k| \cdot \tau_k^{s_k}. \tag{102}$$

We then deduce that

$$\begin{aligned} &\limsup_{k \to +\infty} \frac{\log^{[p-1]}(k)}{\log^{[q-1]}\left((-1/k)\log\left(\pi_k^p (K, f)\right)\right)} \\ &\leq \limsup_{k \to +\infty} \frac{\log^{[p-1]}(s_k)}{\log^{[q-1]}\left((-1/s_k)\log\left(|f_k| \cdot \tau_k^{s_k}\right)\right)}. \end{aligned} \tag{103}$$

This inequality is a direct consequence of (102) and the inequality on coefficients $|f_k|$ given by

$$|f_k| \cdot \tau_k^{s_k} \leq \exp\left(-s_k \exp^{[q-2]}\left(\ln^{[p-2]}(s_k)^{1/(\rho+\varepsilon)}\right)\right). \tag{104}$$

$\square$

Applying the above lemma we get the following main result.

Theorem 13. *Let f be an element of $L^p(K, \mu)$, then*

(1) *f is $\mu - a \cdot s$ the restriction to K of an entire function in $\mathbb{C}^n$ of finite (K, p, q)-order ρ if and only if*

$$\rho_1(p, q) = \limsup_{k \to +\infty} \frac{\log^{[p-1]}(k)}{\log^{[q-1]}\left((-1/k)\log\left(\pi_k^p (K, f)\right)\right)} < +\infty \tag{105}$$

and $\rho = L(\rho_1(p, q))$.

(2) *f is $\mu - a \cdot s$ the restriction to K of an entire function of (K, p, q)-order ρ $(\beta < \rho < +\infty)$ and of (K, p, q)-type σ $(0 < \sigma < +\infty)$ if and only if*

$$\sigma_1(p, q) = \limsup_{k \to +\infty} \frac{\log^{[p-2]}(k)}{\left(\log^{[q-2]}\left((-1/k)\log\left(\pi_k^p (K, f)\right)\right)\right)^{\rho-C}}$$

$$< +\infty \tag{106}$$

and $\sigma = M(p, q)(\sigma_1(p, q))^{\rho-C}$, where $C = 0$ if $(p, q) = (2, 2)$ and $C = 1$ if $p = q = 2$.

Proof. Suppose that f is μ-a.s the restriction to K of an entire function g of (K, p, q)-order ρ $(\beta < \rho < +\infty)$ and show that $\rho = L(\rho_1(p, q))$. We have $g \in L^p(K, \mu)$, $p \geq 2$ and $g = \sum_{k \geq 0} g_k \cdot A_k$ in $L^2(K, \mu)$, where $g_k = (1/v_k) \int_K f \overline{A}_k d\mu, k \geq 0$. From (40) of Theorem 7 we get $\rho = P_1(L(p, q))$, where

$$L(p, q) = \limsup_{k \to +\infty} \frac{\log^{[p-1]}(s_k)}{\log^{[q-1]}\left((-1/s_k)\log|g_k| \cdot \tau_k^{s_k}(E)\right)}. \tag{107}$$

But $g = f$ on K, thus by Lemma 11 we have $\rho = L(\rho_1(p, q))$.

Conversely, suppose now that f is a function of $L^p(K, \mu)$ such that the relation (105) holds.

(1) Let $p \geq 2$, then we have $f = \sum_{k \geq 0} f_k A_k$ because $f \in L^2(K, \mu)$, $(L^p(K, \mu) \subset L^2(K, \mu))$ and $(A_k)_k$ is a basis of $L^2(K, \mu)$ as in Section 3. Consider in $\mathbb{C}^n$ the series $\sum f_k A_k$. By (90) of Lemma 12 one may easily check that this series converges normally on every compact subset of $\mathbb{C}^n$ to an entire function denoted f_1 (this result is a direct consequence of the inequality (BM) and the inequality on coefficients $|f_k|$). We have obviously $f_1 = f$ μ-a.s on K, and by Theorem 8, the (K, p, q)-order of f_1 is

$$\rho(f_1, p, q) = \limsup_{k \to +\infty} \frac{\log^{[p-1]}(s_k)}{\log^{[q-1]}\left((-1/s_k)\log|f_k| \cdot \tau_k^{s_k}\right)}. \tag{108}$$

By Lemma 12 we check that $\rho(f_1, p, q) = \rho$ so the proof is completed for $p \geq 2$.

(2) Now let $p \in [1, 2[$ and $f \in L^p(K, \mu)$, by (BM) inequality and Hölder inequality we have again the inequality (96) and (102), and by the previous arguments we obtain the result.

The proof of the second assertion follows in a similar way of the proof of the first assertion with the help of Theorem 8 and the arguments discussed above, hence we omit the details. $\qquad \square$

Remark 14. (1) If $n = 1$ and $(p, q) = (2, 1)$, using the results of Theorem 13 we obtain the result of Winiarski (see [6]):

$$\limsup_{k \to +\infty} k\left(\pi_k^p(K, f)\right)^{\rho/k} = (e\rho\sigma)\left(\text{cap}(K)\right)^\rho. \tag{109}$$

(2) If $n = 1$, $p > 2$, and $q = 1$, using the results of Theorem 13 we obtain the result of Nguyen (see [13]):

$$\limsup_{k \to +\infty} \log^{[p-2]}(k)\left(\pi_k^p(K, f)\right)^{\rho/k} = \sigma\left(\text{cap}(K)\right)^\rho. \tag{110}$$

Remark 15. The above result holds for $0 < p < 1$ (see [14]).

Let $0 < p < 1$; of course, for $0 < p < 1$, the L^p-norm does not satisfy the triangle inequality. But our relations (92) and relation (102) are also satisfied for $0 < p < 1$, because by using Holder's inequality we have, for some $M > 0$ and all $r > p$ (p fixed),

$$\|f\|_{L^p(E, \mu)} \leq M\|f\|_{L^r(E, \mu)}. \tag{111}$$

Using the inequality

$$\int_E |f|^p \, d\mu \leq \|f\|_E^{p-r} \int_E |f|^r \, d\mu \tag{112}$$

we get

$$\|f\|_{L^p(E, \mu)} \leq \|f\|_E^{1-(r/p)} \|f\|_{L^r(E, \mu)}^{r/p}. \tag{113}$$

We deduce that (E, μ) satisfies the Bernstein-Markov inequality. For $\epsilon > 0$ there is a constant $C = C(\epsilon, p) > 0$ such that for all (analytic) polynomials P we have

$$\|P\|_E \leq C(1 + \epsilon)_{\deg(P)}\|P\|_{L^p(E, \mu)}. \tag{114}$$

Thus if (E, μ) satisfies the Bernstein-Markov inequality for one $p > 0$, then (92) and (95) are satisfied for all $p > 0$.

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