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Review Article

Controllability of Impulsive Neutral Functional Differential Inclusions in Banach Spaces

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We investigate the controllability of impulsive neutral functional differential inclusions in Banach spaces. Our main aim is to find an effective method to solve the controllability problem of impulsive neutral functional differential inclusions with multivalued jump sizes in Banach spaces. Based on a fixed point theorem with regard to condensing map, sufficient conditions for the controllability of the impulsive neutral functional differential inclusions in Banach spaces are derived. Moreover, a remark is given to explain less conservative criteria for special cases, and work is improved in the previous literature.

1. Introduction

During the last decade, differential inclusions [1–3] were well known for applications to mechanics, engineering, and so on. Impulsive differential equations [4–9] were important in the study of physical fields. Ahmed [10] first introduced three different models of impulsive differential inclusions and studied the existence of them, respectively. From then on, there have been many focuses on various properties of impulsive differential inclusions, see [11–17] and references therein.

Controllability is one of the primary problems in control theory [11, 13, 14, 17–24]. Study on controllability has always been considered as a hot topic given its numerous applications to mechanics, electrical engineering, medicine, biology, and so forth. Because of their various application backgrounds, there were a number of researches on controllability of differential inclusions, see [11, 13, 14, 17]. Controllability of impulsive functional differential inclusions is an attractive subject, thanks to their outstanding performance in applications. But as far as we are concerned, there were very few results on controllability of the model with multivalued jump sizes [13]. As for the third model initiated by Ahmed [10], we were impressed by the statement that the model of differential inclusions with multi-valued jump sizes may arise under many different situations, for example, in case of a control problem where one wishes to control the jump sizes in order to achieve certain objectives. In this paper, we aim to find an effective method to solve the controllability problem of impulsive neutral functional differential inclusions with multi-valued jump sizes in Banach spaces.

Liu [11] studied impulsive neutral functional differential inclusions in Banach spaces. However, to the best of our knowledge, there has not any result considering the controllability of the impulsive neutral functional differential inclusions with multi-valued jump sizes in Banach spaces. This work is both challenging and interesting, since our systems are more general than those studied ever before. Based on a fixed point theorem with regard to condensing map, we work out the sufficient conditions for the controllability of impulsive neutral functional differential inclusions in Banach spaces. In [11], Liu considered the controllability basing on Martelli's fixed point theorem [25]. He took advantage of the statement that a completely continuous map is a condensing map. However, condensing map may not be completely continuous. We notice this inequality and consider the controllability on the strength of a special property of Kuratowski measure of noncompactness in Banach spaces. Due to the property, we are allowed to prove that a map is condensing according to its definition. When jumps are single-valued maps in our system, the system degenerates into the system (1.1) in [11]. At this time, less conservative criteria can be given for controllability of system (1.1) [11] after appropriate degeneration. Work in [11] is improved.

The content of this paper is organized as follows. In Section 2, some preliminaries are recalled; the impulsive neutral functional differential inclusions is proposed. In Section 3, the results on controllability of impulsive neutral functional differential inclusions in Banach spaces are derived, as well as strictly proof; a remark is given to show our criteria are less conservative. In Section 4, conclusions are given to explain our work in this paper.

2. Preliminaries

Definition 1. Let X be a Banach space, a multi-valued map $\mathfrak{F} : X \to 2^X$ is called convex valued, if $\mathfrak{F}(x)$ is convex for all $x \in X$.

$\mathfrak{F}$ is called closed valued, if $\mathfrak{F}(x)$ is closed for all $x \in X$.

$\mathfrak{F}$ is called bounded on bounded set, if $\mathfrak{F}(E) = \cup_{x\in E}\mathfrak{F}(x)$ is bounded in X for any bounded subset $E \subset X$.

$\mathfrak{F}$ is called upper semicontinuous on X, if for every $\hat{x} \in X$, the set $\mathfrak{F}(\hat{x})$ is a nonempty and closed subset of X, and for every open set E of X containing $\mathfrak{F}(\hat{x})$, there is an open neighborhood B of $\hat{x}$, such that $\mathfrak{F}(B) \subseteq E$.

We make the following notations: $\mathfrak{B}_h = \{\chi : (-\infty, 0] \to X$; for any $\varepsilon > 0$, χ is a bounded and measurable function on $[-\varepsilon, 0]$, and $\int_{-\infty}^{0} h(s)\sup_{[s,0]} |\chi| ds < +\infty\}$, where $h : (-\infty, 0] \to (0, +\infty)$ is a continuous function. Define norm on $\mathfrak{B}_h$, as $\|\chi\|_{\mathfrak{B}_h} = \int_{-\infty}^{0} h(s)\sup_{[s,0]} |\chi| ds$. $(\mathfrak{B}_h, \|\cdot\|_{\mathfrak{B}_h})$ is a Banach space [11].

$H_d(A, B) = \max\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(A, b)\}$, where A, B are subsets of $X, d(a, B) = \inf_{b\in B} d(a, b), d(A, b) = \inf_{a\in A} d(a, b)$,
$P_{bd}(X) = \{E \subset X : E \text{ is bounded in } X\}$,
$P_{cv}(X) = \{E \subset X : E \text{ is convex in } X\}$,
$P_{cl}(X) = \{E \subset X : E \text{ is closed in } X\}$,
$P_{bd,cv,cl}(X) = \{E \subset X : E \text{ is bounded, convex, and closed in } X\}$.

In this paper, we consider the neutral functional differential inclusions in Banach space X as follows:

$$\begin{cases} \dfrac{d}{dt}\left[x(t) - g(t, x_t)\right] \in Ax(t) \\ \qquad +F(t, x_t) + Bu(t), \quad t \in J \setminus \{t_k\}_{k=1}^{m}, \\ \Delta x|_{t=t_k} = x(t_k^+) - x(t_k^-) \in I_k(x(t_k^-)), \\ \qquad\qquad k = 1, 2, \dots, m, \\ x_0 = \phi \in \mathfrak{B}_h, \end{cases} \tag{1}$$

where $x \in X$. For $t \in J, x_t$ represents the $x_t : (-\infty, 0] \to \mathfrak{B}_h$ defined by $x_t(\theta) = x(t + \theta), \theta \in (-\infty, 0]$ which belongs to some abstract phase space $\mathfrak{B}_h$; $g : J \times \mathfrak{B}_h \to X$; $J = [0, b]$, where b is a positive constant; A is the infinitesimal generator of a strongly continuous operator semigroup $(T(t))_{t>0}$ [26]; $F : J \times \mathfrak{B}_h \to 2^X$ is a closed, bounded, and convex valued multivalued map; $B : U \to X$ is a continuous linear operator, where U is a Banach space with $u(\cdot) \in L^2(J, U)$, here $u(\cdot)$ is the control function; $\{I_k : X \to 2^X\}_{k=1}^{m}$ are closed, bounded, and convex valued multi-valued maps, $x(t_k^+)$, and $x(t_k^-)$ represent the left and right limits of $x(\cdot)$ at $t = t_k$, respectively. The histories $x_t : (-\infty, 0] \to X, x_t(\theta) = x(t + \theta)$.

We introduce definitions the following.

Definition 2. A function $x : (-\infty, b] \to X$ is called a mild solution of system (1) if the following holds: $x_0 = \phi \in \mathfrak{B}_h$ on $(-\infty, 0]$ and for each $s \in [0, t)$, the function $AT(t - s)g(s, x_s)$ is integrable, and there exists $\mathcal{I}_k(x(t_k^-)) \in I_k(x(t_k^-))$, such that the integral equation

$$\begin{aligned} x(t) = {} & T(t)\left[\phi(0) - g(0, \phi)\right] \\ & + g(t, x_t) + \int_0^t AT(t - s) g(s, x_s) ds \\ & + \int_0^t T(t - s) f(s) ds \\ & + \int_0^t T(t - s)(Bu)(s) ds \\ & + \sum_{0<t_k<t} T(t - t_k)\mathcal{I}_k(x(t_k^-)), \quad t \in J, \end{aligned} \tag{2}$$

is satisfied, where $f \in S_{F,x} = \{f \in L^1(J, X) : f(t) \in F(t, x_t)$, for a.e. $t \in J\}$.

Definition 3. The system (1) is said to be controllable on the interval J, if for every initial function $\phi \in \mathfrak{B}_h$ and every $x_1 \in X$, there exists a control function $u \in L^2(J, U)$, such that the mild solution x of (1) satisfies $x(b) = x_1$.

Definition 4 (see [27]). A map $N : X \to X$ is called α-condensing, if for any bounded subset S of X, $N(S)$ is bounded and $\alpha(N(S)) < \alpha(S), \alpha(S) > 0$.

Remark 5. The $\alpha(\cdot)$ in the Definition 4 is called the Kuratowski measure of noncompactness, which is defined as $\alpha(S) = \inf\{d > 0 :$ there exist finitely many sets of diameter at most d which cover $S\}$. Measures of noncompactness are useful in the study of infinite-dimensional Banach spaces, where any ball $\mathcal{B}$ of diameter d has $\alpha(\mathcal{B}) = d$.

Lemma 6 (see [25]). *Let E be a Banach space, and $N : E \to P_{bd,cv,cl}(E)$ is a condensing map. If the set*

$$\Omega = \{x \in E : \lambda x \in Nx, \text{ for some } \lambda > 1\} \tag{3}$$

is bounded, then N has a fixed point.

3. Main Results

In order to study system (1), we introduce hypotheses hereinafter:

(A_0) $T(t)$ is bounded, that is to say there are constants M_1, such that $\|T\| \le M_1$.

(A_1) The linear operator $W: L^2(J,U) \to X$ defined by

$$Wu = \int_0^b T(b-s)Bu(s)\,ds \tag{4}$$

has an inverse operator W^{-1}, which takes value in $L^2(J,U)/\ker W$. And W^{-1} is bounded. There exist positive constants M_2 and M_3 satisfying $\|B\| \le M_2$ and $\|W^{-1}\| \le M_3$.

(A_2) For each $k \in \{1,2,\dots,m\}$, there is a positive constant α_k, such that $\|I(x(t_k^-))\| = \sup\{|\mathcal{I}_k(x(t_k^-))| : \mathcal{I}(x(t_k^-)) \in I(x(t_k^-))\} \le \alpha_k$ for all $x \in X$.

(A_3) There exist constants a_1, a_2, b_1, and b_2, satisfying $\|Ag(t,\chi)\| \le a_1\|\chi\|_{\mathfrak{B}_h} + a_2$, $|g(t,\chi)| \le b_1\|\chi\|_{\mathfrak{B}_h} + b_2$, and $t \in J, \chi \in \mathfrak{B}_h$.

(A_4) $F : J \times \mathfrak{B}_h \to P_{bd,cv,cl}(X); (t,\chi) \longmapsto F(t,\chi)$ is measurable with respect to t for every $\chi \in \mathfrak{B}_h$, upper semicontinuous with respect to χ for every $t \in J$, and for every fixed $\chi \in \mathfrak{B}_h$

$$S_{F,\chi} = \left\{ f \in L^1(J,X) : f(t) \in F(t,\chi), \text{a.e. } t \in J \right\} \tag{5}$$

is nonempty, or equivalently, $\inf\{|f| : f(t) \in F(t,\chi)\} \in L^1\{J,X\}$.

(A_5) There is an integrable function $\varphi : J \to [0,\infty)$ and a continuous and nondecreasing function $\psi : [0,\infty) \to (0,\infty)$, such that

$$\|F(t,\chi)\| = \sup\{|f| : f \in F(t,\chi)\} \le \varphi(t)\psi\left(\|\chi\|_{\mathfrak{B}_h}\right), \tag{6}$$

$$t \in J, \chi \in \mathfrak{B}_h.$$

Lemma 7 (see [28]). *Let I be a compact real interval, and let, E be a Banach space. Let F be a multivalued map satisfying (A_4), and let Γ be a linear continuous mapping from $L^1(I,E) \to C(I,E)$. Then the operator*

$$\begin{aligned} \Gamma \circ S_F : C(I,E) &\longrightarrow P_{bd,cv,cl}(C(I,E)), x \\ &\longmapsto (\Gamma \circ S_F)(x) = \Gamma\left(S_{F,x}\right) \end{aligned} \tag{7}$$

is a closed graph operator in $C(I,E) \times C(I,E)$.

We denote the Banach space $\mathfrak{B}_b = \{x(t) : (-\infty,b] \to X : x_k \in C(J_k,X),\ k = 1,2,\dots,m$, such that x is left continuous and right limits$\}$, with seminorm defined by $\|x\|_{\mathfrak{B}_b} := \|x_0\|_{\mathfrak{B}_h} + \sup_{0\le s\le b}|x(s)|$, where x_k is the restriction of x to $J_k = (t_k, t_{k+1}]$.

Lemma 8. *If $x \in \mathfrak{B}_b$ and $\eta = \int_{-\infty}^0 h(s)ds < +\infty$, then for $x_t \in \mathfrak{B}_h$, $\eta|x(t)| \le \|x_t\|_{\mathfrak{B}_h} \le \|x_0\|_{\mathfrak{B}_h} + \eta\sup_{0\le s\le t}|x(s)|$ holds.*

Proof. On the one hand, we have

$$\begin{aligned} \|x_t\|_{\mathfrak{B}_h} &= \int_{-\infty}^0 h(s)\sup_{s\le\theta\le 0}|x_t(\theta)|\,ds \\ &= \int_{-\infty}^0 h(s)\sup_{t+s\le\theta\le t}|x(\theta)|\,ds \\ &= \int_{-\infty}^{-t} h(s)\sup_{t+s\le\theta\le t}|x(\theta)|\,ds \\ &\quad + \int_{-t}^0 h(s)\sup_{t+s\le\theta\le t}|x(\theta)|\,ds \\ &\le \int_{-\infty}^{-t} h(s)\left[\sup_{t+s\le\theta\le 0}|x(\theta)| + \sup_{0\le\theta\le t}|x(\theta)|\right]ds \\ &\quad + \int_{-t}^0 h(s)\sup_{0\le\theta\le t}|x(\theta)|\,ds \\ &\le \int_{-\infty}^{-t} h(s)\sup_{t+s\le\theta\le 0}|x(\theta)|\,ds \\ &\quad + \int_{-\infty}^0 h(s)\sup_{0\le\theta\le t}|x(\theta)|\,ds \\ &\le \int_{-\infty}^{-t} h(s)\sup_{s\le\theta\le 0}|x(\theta)|\,ds + \eta\sup_{0\le s\le t}|x(s)| \\ &\le \int_{-\infty}^{0} h(s)\sup_{s\le\theta\le 0}|x(\theta)|\,ds + \eta\sup_{0\le s\le t}|x(s)| \\ &= \|x_0\|_{\mathfrak{B}_h} + \eta\sup_{0\le s\le t}|x(s)|. \end{aligned} \tag{8}$$

On the other hand, $\|x_t\|_{\mathfrak{B}_h} = \int_{-\infty}^0 h(s)\sup_{s\le\theta\le 0}|x_t(\theta)|ds \ge |x_t(0)|\int_{-\infty}^0 h(s)ds = \eta|x(t)|$.

The proof is thus completed. □

For any $x_1 \in X$, we design the control function $u(t)$ in system (1) as

$$\begin{aligned} u(t) = W^{-1}\Bigg\{ & x_1 - T(b)[\phi(0) - g(0,\phi)] - g(b,x_b) \\ & - \int_0^b AT(b-\xi)g\left(\xi,x_\xi\right)d\xi \\ & - \int_0^b T(b-\xi)f(\xi)\,d\xi \\ & - \sum_{k=1}^m T(b-t_k)\mathcal{I}_k(x(t_k^-))\Bigg\}(t). \end{aligned} \tag{9}$$

Then we consider the multi-valued function $\mathscr{F}:\mathfrak{B}_b\to 2^{\mathfrak{B}_b}$, for any $x\in\mathfrak{B}_b$,

$$\mathscr{F}x(t)=\begin{cases}\phi(t), & t\in(-\infty,0]\\ T(t)[\phi(0)-g(0,\phi)]+g(t,x_t)\\ \quad+\int_0^t AT(t-s)g(s,x_s)\,ds\\ \quad+\int_0^t T(t-s)f(s)\,ds\\ \quad+\int_0^t T(t-s)(Bu)(s)\,ds\\ \quad+\sum_{0<t_k<t}T(t-t_k)\mathscr{I}_k(x(t_k^-)), & t\in J,\end{cases}\tag{10}$$

where $u(t)$ is described by (9), $f(t)\in S_{F,x}$, $\mathscr{I}_k(x(t_k^-))\in I_k(x(t_k^-))$.

If we define function $\widetilde{\phi}:(-\infty,b]\to\mathfrak{B}_b$ as

$$\widetilde{\phi}(t)=\begin{cases}\phi(t), & t\in(-\infty,0],\\ T(t)\phi(0), & t\in[0,b],\end{cases}\tag{11}$$

where $\phi(t)\in\mathfrak{B}_h$, and denote $y(t)=x(t)-\widetilde{\phi}(t)$, then $x(t)$ is a mild solution of system (1) if and only if $y_0=0$ and

$$\begin{aligned}y(t)=&-T(t)g(0,\phi)+g(t,x_t)\\ &+\int_0^t AT(t-s)g(s,x_s)\,ds\\ &+\int_0^t T(t-s)f(s)\,ds\\ &+\int_0^t T(t-s)(Bu)(s)\,ds\\ &+\sum_{0<t_k<t}T(t-t_k)\mathscr{I}_k(x(t_k^-)),\quad t\in J,\end{aligned}\tag{12}$$

where $x_t=y_t+\widetilde{\phi}_t$, $x(t_k^-)=y(t_k^-)+\widetilde{\phi}(t_k^-)$.

Now we denote $\mathfrak{B}_b^0=\{y(t)\in\mathfrak{B}_b: y_0=0\in\mathfrak{B}_h\}$, and define norm $\|y(t)\|_{\mathfrak{B}_b^0}=\sup_{0\le t\le b}|y(t)|$, thus $\mathfrak{B}_b^0$ is a Banach space. Then we can define another multi-valued function $\mathscr{F}_1:\mathfrak{B}_b^0\to 2^{\mathfrak{B}_b^0}$; $\mathscr{F}_1y(t)=\mathscr{F}x(t)-\widetilde{\phi}(t)$, so

$$\mathscr{F}_1y(t)=\begin{cases}0, & t\in(-\infty,0]\\ -T(t)g(0,\phi)+g(t,x_t)\\ \quad+\int_0^t AT(t-s)g(s,x_s)\,ds\\ \quad+\int_0^t T(t-s)f(s)\,ds\\ \quad+\int_0^t T(t-s)(Bu)(s)\,ds\\ \quad+\sum_{0<t_k<t}T(t-t_k)\mathscr{I}_k(x(t_k^-)), & t\in J,\end{cases}\tag{13}$$

where $u(t)$ is the same as that in (10). Thus $\mathscr{F}$ has a fixed point in $\mathfrak{B}_b$ if and only if $\mathscr{F}_1$ has a fixed point in $\mathfrak{B}_b^0$, $y(t)=x(t)-\widetilde{\phi}(t)$.

Let $\mathfrak{B}_r=\{y(t)\in\mathfrak{B}_b^0:\|y(t)\|_{\mathfrak{B}_b^0}=\sup_{0\le s\le b}|y(s)|\le r\}$, and r is a positive constant. It is true that $\mathfrak{B}_r$ is a closed subspace of $\mathfrak{B}_b^0$, so $\mathfrak{B}_r$ is also a Banach space. Next we show that $\mathscr{F}_1$ has a fixed point in $\mathfrak{B}_r$.

Lemma 9. *$\mathscr{F}_1$ is bounded, convex, and closed on $\mathfrak{B}_r$.*

Proof. (I) $\mathscr{F}_1$ is bounded on $\mathfrak{B}_r$. Let $y\in\mathfrak{B}_r$, thanks to Lemma 8,

$$\begin{aligned}\|x_t\|_{\mathfrak{B}_h}&\le\|y_t\|_{\mathfrak{B}_h}+\|\widetilde{\phi}_t\|_{\mathfrak{B}_h}\\ &\le\eta\sup_{0\le s\le t}|y(s)|+\|y_0\|_{\mathfrak{B}_h}\\ &\quad+\eta\sup_{0\le s\le t}|\widetilde{\phi}(s)|+\|\widetilde{\phi}_0\|_{\mathfrak{B}_h}\\ &\le\eta(r+M_1|\phi(0)|)+\|\phi\|_{\mathfrak{B}_h}\\ &\triangleq r_1,\end{aligned}\tag{14}$$

$$\begin{aligned}|u(t)|=&\Bigg|W^{-1}\Bigg\{x_1-T(b)[\phi(0)-g(0,\phi)]\\ &-g(b,x_b)-\int_0^b AT(b-\xi)g(\xi,x_\xi)\,d\xi\\ &-\int_0^b T(b-\xi)f(\xi)\,d\xi\\ &-\sum_{k=1}^m T(b-t_k)\mathscr{I}_k(x(t_k^-))\Bigg\}(t)\Bigg|\\ \le& M_3\Bigg[|x_1|+M_1(|\phi(0)|+|g(0,\phi)|)\\ &+|g(b,x_b)|+M_1b(a_1r_1+a_2)\\ &+M_1\int_0^b|f(\xi)|\,d\xi+M_1\sum_{k=1}^m|\mathscr{I}_k(x(t_k^-))|\Bigg]\\ \le& M_3\Bigg[|x_1|+M_1(|\phi(0)|+|g(0,\phi)|)\\ &+b_1r_1+b_2+M_1b(a_1r_1+a_2)\\ &+M_1\int_0^b\varphi(\xi)\psi(\|x_\xi\|_{\mathfrak{B}_h})\,ds+M_1\sum_{k=1}^m\alpha_k\Bigg].\end{aligned}\tag{15}$$

Then we have

$$\begin{aligned}|\mathscr{F}_1y(t)|\le&|T(t)g(0,\phi)|+|g(t,x_t)|\\ &+\int_0^t|AT(t-s)g(s,x_s)|\,ds\\ &+\int_0^t|T(t-s)f(s)|\,ds\end{aligned}$$

$$\begin{aligned}
&+ \int_0^t \left| T(t-s)(Bu)(s) \right| ds \\
&+ \sum_{0<t_k<t} \left| T(t-t_k) \mathcal{I}_k \left(x\left(t_k^-\right)\right)\right| \\
&\leq M_1 \left| g(0,\phi) \right| + M_1 b \left(a_1 r_1 + a_2 \right) \\
&+ M_1 \int_0^b \varphi(s) \psi\left(\|x_s\|_{\mathfrak{B}_h} \right) ds \\
&+ b M_1 M_2 M_3 \\
&\times \Bigg[|x_1| + M_1 \left(|\phi(0)| + |g(0,\phi)| \right) \\
&\quad + (1 + bM_2) \left(a_1 \|x_b\|_{\mathfrak{B}_h} + a_2 \right) \\
&\quad + M_1 \int_0^b \varphi(s) \psi\left(\|x_s\|_{\mathfrak{B}_h} \right) ds \\
&\quad + M_1 \sum_{k=1}^m \alpha_k \Bigg] + M_1 \sum_{k=1}^m \alpha_k \\
&\leq M_1 \left| g(0,\phi) \right| + (1 + bM_2)\left(a_1 r_1 + a_2 \right) \\
&+ M_1 \psi(r_1) \int_0^b \varphi(s) ds + b M_1 M_2 M_4 \\
&\times \Bigg[|x_1| + M_1 \left(|\phi(0)| + |g(0,\phi)| \right) \\
&\quad + b_1 r_1 + b_2 + M_1 b \left(a_1 r_1 + a_2 \right) \\
&\quad + M_1 \psi(r_1) \int_0^b \varphi(s) ds + M_1 \sum_{k=1}^m \alpha_k \Bigg] \\
&+ M_1 \sum_{k=1}^m \alpha_k \\
&\triangleq M_4.
\end{aligned} \tag{16}$$

Consequently, $\|\mathcal{F}_1 y(t)\|_{\mathfrak{B}_b} \leq M_4$.

(II) $\mathcal{F}_1$ is convex on $\mathfrak{B}_r$. Let $y \in \{\mathfrak{B}_r\}$ and $z_1, z_2 \in \mathcal{F}_1 y$, there must be $f_1, f_2 \in S_{F,x}$ and $\mathcal{I}_k^1, \mathcal{I}_k^2 \in I_k(x(t_k^-))$, such that

$$\begin{aligned}
z_\gamma = & - T(t) g(0,\phi) + g(t, x_t) \\
&+ \int_0^t AT(t-s) g(s, x_s) ds \\
&+ \int_0^t T(t-s) f_\gamma(s) ds \\
&+ \int_0^t T(t-s) BW^{-1} \\
&\times \Bigg\{ x_1 - T(b) \left[\phi(0) - g(0,\phi) \right] \\
&\quad - g(b, x_b) - \int_0^b AT(b-\xi) g\left(\xi, x_\xi\right) d\xi \\
&\quad - \int_0^b T(b-\xi) f_\gamma(\xi) d\xi \\
&\quad - \sum_{k=1}^m T(b-t_k) \mathcal{I}_k^\gamma \left(x\left(t_k^-\right)\right) \Bigg\} s ds \\
&+ \sum_{0<t_k<t} T(t-t_k) \; \mathcal{I}_k^\gamma \left(x\left(t_k^-\right)\right),
\end{aligned} \tag{17}$$

for $\gamma = 1, 2$. Then for any $\tau \in [0,1]$, we have

$$\begin{aligned}
&\left[\tau z_1 + (1-\tau) z_2 \right](t) \\
&\quad = -T(t) g(0,\phi) + g(t, x_t) \\
&\quad + \int_0^t AT(t-s) g(s, x_s) ds \\
&\quad + \int_0^t T(t-s) \left[\tau f_1 + (1-\tau) f_2 \right](s) ds \\
&\quad + \int_0^t T(t-s) BW^{-1} \\
&\quad \times \Big\{ x_1 - T(b) \left[\phi(0) - g(0,\phi) \right] \\
&\qquad - g(b, x_b) - \int_0^b AT(b-\xi) g\left(\xi, x_\xi\right) d\xi \\
&\qquad - \int_0^b T(b-\xi) \left[\tau f_1 + (1-\tau) f_2 \right](\xi) d\xi \\
&\qquad - \sum_{k=1}^m T(b-t_k) \\
&\qquad\quad \times \left[\tau \mathcal{I}_k^1 + (1-\tau) \mathcal{I}_k^2 \right] \left(x\left(t_k^-\right)\right) \Big\}(s) ds \\
&\quad + \sum_{0<t_k<t} T(t-t_k) \left[\tau \mathcal{I}_k^1 + (1-\tau) \mathcal{I}_k^2 \right] \left(x\left(t_k^-\right)\right)(s) ds.
\end{aligned} \tag{18}$$

Combining $S_{F,x}$ and $I_k(x(t_k^-))$ is convex, and $\mathcal{F}_1$ is convex.

(III) $\mathcal{F}_1$ is closed on $\mathfrak{B}_r$. Let $y \in \{\mathfrak{B}_r\}$. Here we should proof that if there are sequences $\{z_n\}$ satisfying $z_n \to z^*$ and $z_n \in \mathcal{F}_1 y$, then $z^* \in \mathcal{F}_1 y$ holds.

For every $z_n \in \mathcal{F}_1 y$, there is a $f_n \in S_{F,x}$ and a $\mathcal{I}_k^n \in I_k(x(t_k^-))$, such that

$$\begin{aligned}
z_n = & - T(t) g(0,\phi) + g(t, x_t) \\
&+ \int_0^t AT(t-s) g(s, x_s) ds \\
&+ \int_0^t T(t-s) f_n(s) ds
\end{aligned}$$

$$+\int_0^t T(t-s)BW^{-1} \times \Big\{x_1 - T(b)[\phi(0)-g(0,\phi)] - g(b,x_b) - \int_0^b AT(b-\xi)g(\xi,x_\xi)d\xi - \int_0^b T(b-\xi)f_n(\xi)d\xi - \sum_{k=1}^m T(b-t_k)\mathcal{I}_k^n(x(t_k^-))\Big\}(s)ds + \sum_{0<t_k<t} T(t-t_k)\mathcal{I}_k^n(x(t_k^-)). \tag{19}$$

And for z^*, we should prove that there must be some $f^* \in S_{F,x}$ and some $\mathcal{I}_k^* \in I_k(x(t_k^-))$, such that

$$\begin{aligned} z^* = & -T(t)g(0,\phi) + g(t,x_t) + \int_0^t AT(t-s)g(s,x_s)ds + \int_0^t T(t-s)f^*(s)ds + \int_0^t T(t-s)BW^{-1} \\ & \times \Big\{x_1 - T(b)[\phi(0)-g(0,\phi)] - g(b,x_b) - \int_0^b AT(b-\xi)g(\xi,x_\xi)d\xi - \int_0^b T(b-\xi)f^*(\xi)d\xi \\ & - \sum_{k=1}^m T(b-t_k)\mathcal{I}_k^*(x(t_k^-))\Big\}(s)ds + \sum_{0<t_k<t} T(t-t_k)\mathcal{I}_k^*(x(t_k^-)). \end{aligned} \tag{20}$$

Considering calculation of (20) subtracting (19), we get

$$\begin{aligned} z^* - z_n = & \int_0^t T(t-s) \times \Big[BW^{-1}\Big(\int_0^b T(b-\xi)(f_n(\xi)-f^*(\xi))d\xi\Big) + \sum_{k=1}^m T(t-t_k)(\mathcal{I}_k^n(x(t_k^-)) - \mathcal{I}_k^*(x(t_k^-))) + (f_n - f^*)\Big](s)ds \\ & + \sum_{0<t_k<t} T(t-t_k)(\mathcal{I}_k^n(x(t_k^-)) - \mathcal{I}_k^*(x(t_k^-))) \longrightarrow 0, \quad n \longrightarrow +\infty. \end{aligned} \tag{21}$$

Meanwhile I_k is a closed multi-valued map, there truly exists some $\mathcal{I}_k^* \in I_k(x(t_k^-))$ for (20).

Accordingly, (21) can be transformed to

$$\int_0^t T(t-s)\Big[BW^{-1}\Big(\int_0^b T(b-\xi)(f_n - f^*)(\xi)d\xi\Big) + (f_n - f^*)\Big](s)ds \longrightarrow 0, \quad n \longrightarrow +\infty. \tag{22}$$

We construct a linear and continuous operator like

$$\Gamma : L^1(J,X) \longrightarrow C(J,X); \Gamma \circ S_F(x) = \Gamma(f)(t) = \int_0^t T(t-s)\Big[BW^{-1}\Big(\int_0^b T(b-\xi)f(\xi)d\xi\Big) \times (s) + f(s)\Big]ds, \tag{23}$$

moreover,

$$\|\Gamma(f)(t)\|_{L^1} \leq (bM_1M_2M_3+1)M_1\|f\|_{L^1}. \tag{24}$$

From Lemma 7, $\Gamma \circ S_F$ is a closed graph operator. Then (22) implies that $\int_0^t T(t-s)[BW^{-1}(\int_0^b T(b-\xi)f^*(\xi)d\xi)(s) + f^*(s)]ds \in \Gamma(S_{F,x})$, with $f^* \in S_{F,x}$.

From the foregoing, (20) holds, which means $z^* \in \mathcal{F}_1 y$. Thus $\mathcal{F}_1$ is closed.

The proof is thus completed. □

Theorem 10. *Assume that hypotheses* (A_0)–(A_5) *hold, and*

$$\begin{aligned} M_1 &\left|g(0,\phi)\right| + b_1 r_1 + b_2 + M_1 b\left(a_1 r_1 + a_2\right) \\ &+ M_1 \psi\left(r_1\right) \int_0^b \varphi(s)\,ds + bM_1M_2M_3 \\ &\times \Bigg[\left|x_1\right| + M_1\left(\left|\phi(0)\right| + \left|g(0,\phi)\right|\right) \\ &\quad + b_1 r_1 + b_2 + M_1 b\left(a_1 r_1 + a_2\right) \\ &\quad + M_1 \psi\left(r_1\right) \int_0^b \varphi(s)\,ds + M_1 \sum_{k=1}^{m} \alpha_k \Bigg] \\ &+ M_1 \sum_{k=1}^{m} \alpha_k < r, \end{aligned} \tag{25}$$

then system (1) *is controllable on* J *under control function* (9).

Proof. First, $\mathscr{F}_1$ is a condensing map on $\mathfrak{B}_b^0$. Considering Remark 5, we just have to prove that for any bounded subset E of $\mathfrak{B}_b^0$, $\operatorname{diam}(\mathscr{F}_1(E)) < \operatorname{diam}(E)$. It is obvious, because $\|\mathscr{F}_1 x(t)\| \le M_4 < r = \operatorname{diam}(\mathfrak{B}_r)$ for any $x(t) \in \mathfrak{B}_r$ by combining inequalities (19) and (25).

Second, here we show that the set $\Omega = \{y \in \mathfrak{B}_b^0 : \lambda y \in \mathscr{F}_1 y$ for some $\lambda > 1\}$ is bounded. Let $y \in \Omega$, then there are $f \in S_{F,x}$ and $\mathscr{I}_k \in I_k(x(t_k^-))$, such that

$$\begin{aligned} y(t) = \frac{1}{\lambda} \Bigg[& -T(t)\, g(0,\phi) + g\left(t, x_t\right) \\ & + \int_0^t AT(t-s)\, g\left(s, x_s\right) ds \\ & + \int_0^t T(t-s)\, f(s)\, ds \\ & + \int_0^t T(t-s)\,(Bu)(s)\, ds \\ & + \sum_{0<t_k<t} T\left(t-t_k\right) \mathscr{I}_k\left(x\left(t_k^-\right)\right) \Bigg]. \end{aligned} \tag{26}$$

So $|y(t)| \le (1/\lambda) M_4 < M_4$. Due to Lemma 6, $\mathscr{F}_1$ has a fixed point in $\mathfrak{B}_r$. Consequently, $\mathscr{F}$ has a fixed point, which means system (1) is controllable.

The proof is thus completed. □

Remark 11. In case $\{I_k\}$ in system (1) are single-valued maps, then the system (1) degenerates into the system (1.1) in [11]. Accordingly, our degenerated assumptions for the controllability Theorem 3.1 [11] are less conservative, which means the following: firstly, the hypothesis $(H_4)(i)$ [11] is unnecessary; secondly, the very complex hypothesis (H_6) [11] can be replaced by inequality (25); finally, the assumption $|AT| \le M_2$ [11] is replaced by $\|Ag(t,\chi)\| \le a_1 \|\chi\|_{\mathfrak{B}_h} + a_2$, so A is not necessary to be bounded; otherwise, our results can only be applied to finite Banach spaces [29].

4. Conclusion

In this paper, we have investigated the controllability of impulsive neutral functional differential inclusions in Banach spaces. Based on a fixed point theorem with regard to condensing map, sufficient conditions for the controllability of the impulsive neutral functional differential inclusions in Banach spaces have been derived. Moreover, a remark has been given to explain less conservative criteria for special cases. We have found an effective method to solve the controllability problem of impulsive neutral functional differential inclusions with multi-valued jump sizes in Banach spaces. Work has been improved in the previous literature.

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