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Research Article

Domain of the Double Sequential Band Matrix $B(\widetilde{r}, \widetilde{s})$ in the Sequence Space $\ell(p)$*

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The sequence space $\ell(p)$ was introduced by Maddox (1967). Quite recently, the domain of the generalized difference matrix $B(r, s)$ in the sequence space ℓ_p has been investigated by Kirişçi and Başar (2010). In the present paper, the sequence space $\ell(\widetilde{B}, p)$ of nonabsolute type has been studied which is the domain of the generalized difference matrix $B(\widetilde{r}, \widetilde{s})$ in the sequence space $\ell(p)$. Furthermore, the alpha-, beta-, and gamma-duals of the space $\ell(\widetilde{B}, p)$ have been determined, and the Schauder basis has been given. The classes of matrix transformations from the space $\ell(\widetilde{B}, p)$ to the spaces ℓ_∞, c and c_0 have been characterized. Additionally, the characterizations of some other matrix transformations from the space $\ell(\widetilde{B}, p)$ to the Euler, Riesz, difference, and so forth sequence spaces have been obtained by means of a given lemma. The last section of the paper has been devoted to conclusion.

1. Preliminaries, Background, and Notation

By w, we denote the space of all real valued sequences. Any vector subspace of w is called a *sequence space*. We write ℓ_∞, c, and c_0 for the spaces of all bounded, convergent, and null sequences, respectively. Also by bs, cs, ℓ_1, and ℓ_p, we denote the spaces of all bounded, convergent, absolutely convergent and p-absolutely convergent series, respectively, where $1 < p < \infty$.

A linear topological space X over the real field $\mathbb{R}$ is said to be a paranormed space if there is a subadditive function $g : X \to \mathbb{R}$ such that $g(\theta) = 0, g(x) = g(-x)$ and scalar multiplication is continuous; that is, $|\alpha_n - \alpha| \to 0$ and $g(x_n - x) \to 0$ imply $g(\alpha_n x_n - \alpha x) \to 0$ for all α's in $\mathbb{R}$ and all x's in X, where θ is the zero vector in the linear space X.

Assume here and after that (p_k) is a bounded sequence of strictly positive real numbers with $\sup p_k = H$ and $M = \max\{1, H\}$. Then, the linear spaces $\ell(p)$ were defined by Maddox [1] (see also Simons [2] and Nakano [3]) as follows:

$$\ell(p) = \left\{ x = (x_k) \in w : \sum_k |x_k|^{p_k} < \infty \right\}, \quad (0 < p_k \le H < \infty) \tag{1}$$

which is the complete space paranormed by

$$g(x) = \left(\sum_k |x_k|^{p_k} \right)^{1/M}. \tag{2}$$

For simplicity in notation, here and in what follows, the summation without limits runs from 0 to ∞. We assume throughout that $p_k^{-1} + (p_k')^{-1} = 1$ and denote the collection of all finite subsets of $\mathbb{N} = \{0, 1, 2, \ldots\}$ by $\mathscr{F}$ and use the convention that any term with negative subscript is equal to naught.

Let λ, μ be any two sequence spaces and let $A = (a_{nk})$ be an infinite matrix of real or complex numbers a_{nk}, where $n, k \in \mathbb{N}$. Then, we say that A defines a matrix mapping from λ into μ, and we denote it by writing $A : \lambda \to \mu$; if for every sequence $x = (x_k) \in \lambda$ the sequence $Ax = \{(Ax)_n\}$, the A-transform of x, is in μ, where

$$(Ax)_n = \sum_k a_{nk} x_k, \quad \text{for each } n \in \mathbb{N}. \tag{3}$$

By $(\lambda : \mu)$, we denote the class of all matrices A such that $A : \lambda \to \mu$. Thus, $A \in (\lambda : \mu)$ if and only if the series on the right side of (3) converges for each $n \in \mathbb{N}$ and every $x \in \lambda$, and we have $Ax = \{(Ax)_n\}_{n\in\mathbb{N}} \in \mu$ for all $x \in \lambda$. A sequence x is said to be A-summable to α if Ax converges to α which is called the A-limit of x.

The shift operator P is defined on ω by $(Px)_n = x_{n+1}$ for all $n \in \mathbb{N}$. A Banach limit L is defined on ℓ_∞, as a nonnegative linear functional, such that $L(Px) = L(x)$ and $L(e) = 1$, where $e = (1, 1, 1, \ldots)$. A sequence $x = (x_k) \in \ell_\infty$ is said to be almost convergent to the generalized limit l if all Banach limits of x are l and is denoted by $f - \lim x_k = l$. Lorentz [4] proved that

$$\begin{aligned} &f - \lim x_k = l \\ &\text{iff } \lim_{m\to\infty} \frac{1}{m+1} \sum_{k=0}^{m} x_{k+n} = l \text{ uniformly in } n. \end{aligned} \tag{4}$$

It is well known that a convergent sequence is almost convergent such that its ordinary and generalized limits are equal. By f, we denote the space of all almost convergent sequences; that is,

$$f := \left\{ x = (x_k) \in \omega : \exists l \in \mathbb{C} \ni \lim_{m\to\infty} \sum_{k=0}^{m} \frac{x_{n+k}}{m+1} = l \text{ uniformly in } n \right\}. \tag{5}$$

Define the double sequential band matrix $B(\tilde{r}, \tilde{s}) = \{b_{nk}(r_k, s_k)\}$ by

$$b_{nk}(r_k, s_k) = \begin{cases} r_k, & k = n, \\ s_k, & k = n-1, \\ 0, & \text{otherwise} \end{cases} \tag{6}$$

for all $k, n \in \mathbb{N}$, where $\tilde{r} = (r_k)$ and $\tilde{s} = (s_k)$ are the convergent sequences. We should note that the double sequential band matrices were firstly used by Srivastava and Kumar [5, 6], Panigrahi and Srivastava [7], and Akhmedov and El-Shabrawy [8].

The main purpose of this paper, which is a continuation of Kirişçi and Başar [9], is to introduce the sequence space $\ell(\widetilde{B}, p)$ of nonabsolute type consisting of all sequences whose $B(\tilde{r}, \tilde{s})$-transforms are in the space $\ell(p)$. Furthermore, the basis is constructed and the alpha-, beta-, and gamma-duals are computed for the space $\ell(\widetilde{B}, p)$. Moreover, the matrix transformations from the space $\ell(\widetilde{B}, p)$ to some sequence spaces are characterized. Finally, we note open problems and further suggestions.

It is clear that $\Delta^{(1)}$ can be obtained as a special case of $B(\tilde{r}, \tilde{s})$ for $\tilde{r} = e$ and $\tilde{s} = -e$ and it is also trivial that $B(\tilde{r}, \tilde{s})$ is reduced in the special case $\tilde{r} = re$ and $\tilde{s} = se$ to the generalized difference matrix $B(r, s)$. So, the results related to the matrix domain of the matrix $B(\tilde{r}, \tilde{s})$ are more general and more comprehensive than the corresponding consequences of the matrix domains of $\Delta^{(1)}$ and $B(r, s)$.

The rest of this paper is organized as follows. In Section 2, the linear sequence space $\ell(\widetilde{B}, p)$ is defined and proved that it is a complete paranormed space with a Schauder basis. Section 3 is devoted to the determination of alpha-, beta-, and gamma-duals of the space $\ell(\widetilde{B}, p)$. In Section 4, the classes $(\ell(\widetilde{B}, p) : \ell_\infty)$, $(\ell(\widetilde{B}, p) : f)$, $(\ell(\widetilde{B}, p) : c)$, and $(\ell(\widetilde{B}, p) : c_0)$ of infinite matrices are characterized. Additionally, the characterizations of some other classes of matrix transformations from the space $\ell(\widetilde{B}, p)$ to the Euler, Riesz, difference, and so forth sequence spaces are obtained by means of a given lemma. In the final section of the paper, open problems and further suggestions are noted.

2. The Sequence Space $\ell(\widetilde{B}, p)$ of Nonabsolute Type

In this section, we introduce the complete paranormed linear sequence space $\ell(\widetilde{B}, p)$.

The *matrix domain* λ_A of an infinite matrix A in a sequence space λ is defined by

$$\lambda_A = \{x = (x_k) \in \omega : Ax \in \lambda\}. \tag{7}$$

Choudhary and Mishra [10] defined the sequence space $\overline{\ell(p)}$ which consists of all sequences such that S-transforms of them are in the space $\ell(p)$, where $S = (s_{nk})$ is defined by

$$s_{nk} = \begin{cases} 1, & 0 \le k \le n, \\ 0, & k > n, \end{cases} \tag{8}$$

for all $k, n \in \mathbb{N}$. Başar and Altay [11] have recently examined the space $bs(p)$ which is formerly defined by Başar in [12] as the set of all series whose sequences of partial sums are in $\ell_\infty(p)$. More recently, Aydın and Başar [13] have studied the space $a^r(u, p)$ which is the domain of the matrix A^r in the sequence space $\ell(p)$, where the matrix $A^r = \{a_{nk}(r)\}$ is defined by

$$a_{nk}(r) = \begin{cases} \dfrac{1 + r^k}{n+1} u_k, & 0 \le k \le n, \\ 0, & k > n, \end{cases} \tag{9}$$

for all $k, n \in \mathbb{N}$, (u_k) such that $u_k \ne 0$ for all $k \in \mathbb{N}$ and $0 < r < 1$. Altay and Başar [14] have studied the sequence space $r^t(p)$ which is derived from the sequence space $\ell(p)$ of Maddox by the Riesz means R^t. With the notation of (7), the spaces $\overline{\ell(p)}, bs(p), a^r(u, p)$, and $r^t(p)$ can be redefined by

$$\begin{aligned} \overline{\ell(p)} &= [\ell(p)]_S, & bs(p) &= [\ell_\infty(p)]_S, \\ a^r(u, p) &= [\ell(p)]_{A^r}, & r^t(p) &= [\ell(p)]_{R^t}. \end{aligned} \tag{10}$$

Following Choudhary and Mishra [10], Başar and Altay [11], Altay and Başar [14–17], and Aydın and Başar [13, 18], we introduce the sequence space $\ell(\widetilde{B}, p)$ as the set of all sequences whose $B(\widetilde{r}, \widetilde{s})$-transforms are in the space $\ell(p)$; that is

$$\ell\left(\widetilde{B}, p\right) := \left\{(x_k) \in w : \sum_k \left|s_{k-1}x_{k-1} + r_k x_k\right|^{p_k} < \infty\right\}, \quad (0 < p_k \leq H < \infty). \tag{11}$$

It is trivial that in the case $p_k = p$ for all $k \in \mathbb{N}$, the sequence space $\ell(\widetilde{B}, p)$ is reduced to the sequence space $\widetilde{\ell}_p$ which is introduced by Kirişçi and Başar [9]. With the notation of (7), we can redefine the space $\ell(\widetilde{B}, p)$ as follows:

$$\ell\left(\widetilde{B}, p\right) := \left[\ell\left(p\right)\right]_{B(\widetilde{r},\widetilde{s})}. \tag{12}$$

Define the sequence $y = (y_k)$, which will be frequently used, as the $B(\widetilde{r}, \widetilde{s})$-transform of a sequence $x = (x_k)$; that is,

$$y_k = \{B\left(\widetilde{r}, \widetilde{s}\right) x\}_k = r_k x_k + s_{k-1}x_{k-1}, \quad \forall k \in \mathbb{N}. \tag{13}$$

Since the spaces $\ell(p)$ and $\ell(\widetilde{B}, p)$ are linearly isomorphic by Corollary 4, one can easily observe that $x = (x_k) \in \ell(\widetilde{B}, p)$ if and only if $y = (y_k) \in \ell(p)$, where the sequences $x = (x_k)$ and $y = (y_k)$ are connected with the relation (13).

Now, we may begin with the following theorem which is essential in the text.

Theorem 1. *$\ell(\widetilde{B}, p)$ is a complete linear metric space paranormed by the paranorm*

$$h\left(x\right) = \left(\sum_k \left|s_{k-1}x_{k-1} + r_k x_k\right|^{p_k}\right)^{1/M}. \tag{14}$$

Proof. It is easy to see that the space $\ell(\widetilde{B}, p)$ is linear with respect to the coordinate-wise addition and scalar multiplication. Therefore, we first show that it is a paranormed space with the paranorm h defined by (14).

It is clear that $h(\theta) = 0$ where $\theta = (0, 0, 0, \ldots)$ and $h(x) = h(-x)$ for all $x \in \ell(\widetilde{B}, p)$.

Let $x, y \in \ell(\widetilde{B}, p)$; then by Minkowski's inequality we have

$$\begin{aligned} h\left(x + y\right) &= \left[\sum_k \left|s_{k-1}\left(x_{k-1} + y_{k-1}\right) + r_k\left(x_k + y_k\right)\right|^{p_k}\right]^{1/M} \\ &= \left\{\sum_k \left[\left|s_{k-1}\left(x_{k-1} + y_{k-1}\right) + r_k\left(x_k + y_k\right)\right|^{p_k/M}\right]^M\right\}^{1/M} \\ &\leq \left(\sum_k \left|s_{k-1}x_{k-1} + r_k x_k\right|^{p_k}\right)^{1/M} \\ &\quad + \left(\sum_k \left|s_{k-1}y_{k-1} + r_k y_k\right|^{p_k}\right)^{1/M} \\ &= h\left(x\right) + h\left(y\right). \end{aligned} \tag{15}$$

Let (λ_n) be a sequence of scalars with $\lambda_n \to \lambda$, as $n \to \infty$, and let $(x^{(n)})_{n=0}^{\infty}$ be a sequence of elements $x^{(n)} \in \ell(\widetilde{B}, p)$ with $h(x^{(n)} - x) \to 0$, as $n \to \infty$. We observe that

$$\begin{aligned} h\left(\lambda_n x^{(n)} - \lambda x\right) \leq{}& h\left[\left(\lambda_n - \lambda\right)\left(x^{(n)} - x\right)\right] \\ &+ h\left[\lambda\left(x^{(n)} - x\right)\right] \\ &+ h\left[\left(\lambda_n - \lambda\right)x\right]. \end{aligned} \tag{16}$$

It follows from $\lambda_n \to \lambda$ $(n \to \infty)$ that $|\lambda_n - \lambda| < 1$ for all sufficiently large n; hence

$$\lim_{n\to\infty} h\left[\left(\lambda_n - \lambda\right)\left(x^{(n)} - x\right)\right] \leq \lim_{n\to\infty} h\left(x^{(n)} - x\right) = 0. \tag{17}$$

Furthermore, we have

$$\lim_{n\to\infty} h\left[\lambda\left(x^{(n)} - x\right)\right] \leq \max\left\{1, |\lambda|^M\right\} \lim_{n\to\infty} h\left(x^{(n)} - x\right) = 0. \tag{18}$$

Also, we have

$$\lim_{n\to\infty} h\left[\left(\lambda_n - \lambda\right)x\right] \leq \lim_{n\to\infty} \left|\lambda_n - \lambda\right| h\left(x\right) = 0. \tag{19}$$

Then, we obtain from (16), (17), (18), and (19) that $h(\lambda_n x^{(n)} - \lambda x) \to 0$, as $n \to \infty$. This shows that h is a paranorm on $\ell(\widetilde{B}, p)$.

Furthermore, if $h(x) = 0$, then $(\sum_k |s_{k-1}x_{k-1} + r_k x_k|^{p_k})^{1/M} = 0$. Therefore $|s_{k-1}x_{k-1} + r_k x_k|^{p_k} = 0$ for each $k \in \mathbb{N}$. If we put $k = 0$, since $s_{-1} = 0$ and $r_0 \neq 0$, we have $x_0 = 0$. For $k = 1$, since $x_0 = 0$ we have $x_1 = 0$. Continuing in this way, we obtain $x_k = 0$ for all $k \in \mathbb{N}$. That is, $x = \theta$. This shows that h is a total paranorm.

Now, we show that $\ell(\widetilde{B}, p)$ is complete. Let $\{x^n\}$ be any Cauchy sequence in $\ell(\widetilde{B}, p)$ where $x^n = \{x_0^{(n)}, x_1^{(n)}, x_2^{(n)}, \ldots\}$. Here and after, for short we write $\widetilde{B}$ instead of $B(\widetilde{r}, \widetilde{s})$. Then for a given $\varepsilon > 0$, there exists a positive integer $n_0(\varepsilon)$ such that $h(x^n - x^m) < \varepsilon$ for all $n, m > n_0(\varepsilon)$. Since for each fixed $k \in \mathbb{N}$

$$\begin{aligned} \left|\left(\widetilde{B}x^n\right)_k - \left(\widetilde{B}x^m\right)_k\right| &\leq \left[\sum_k \left|\left(\widetilde{B}x^n\right)_k - \left(\widetilde{B}x^m\right)_k\right|^{p_k}\right]^{1/M} \\ &= h\left(x^n - x^m\right) < \varepsilon \end{aligned} \tag{20}$$

for every $n, m > n_0(\varepsilon)$, $\{(\widetilde{B}x^0)_k, (\widetilde{B}x^1)_k, (\widetilde{B}x^2)_k, \ldots\}$ is a Cauchy sequence of real numbers for every fixed $k \in \mathbb{N}$. Since $\mathbb{R}$ is complete, it converges, say $(\widetilde{B}x^n)_k \to (\widetilde{B}x)_k$ as $n \to \infty$. Using these infinitely many limits $(\widetilde{B}x)_0, (\widetilde{B}x)_1, (\widetilde{B}x)_2, \ldots$ we define the sequence $\{(\widetilde{B}x)_0, (\widetilde{B}x)_1, (\widetilde{B}x)_2, \ldots\}$. For each $K \in \mathbb{N}$ and $n, m > n_0(\varepsilon)$

$$\left[\sum_{k=0}^{K} \left|\left(\widetilde{B}x^n\right)_k - \left(\widetilde{B}x^m\right)_k\right|^{p_k}\right]^{1/M} \leq h\left(x^n - x^m\right) < \varepsilon. \tag{21}$$

By letting $m, K \to \infty$, we have for $n > n_0(\varepsilon)$ that

$$h\left(x^n - x\right) = \left[\sum_k \left|\left(\widetilde{B}x^n\right)_k - \left(\widetilde{B}x\right)_k\right|^{p_k}\right]^{1/M} < \varepsilon. \tag{22}$$

This shows us $x^n - x \in \ell(\widetilde{B}, p)$. Since $\ell(\widetilde{B}, p)$ is a linear space, we conclude that $x \in \ell(\widetilde{B}, p)$; It follows that $x^n \to x$, as $n \to \infty$ in $\ell(\widetilde{B}, p)$, thus we have shown that $\ell(\widetilde{B}, p)$ is complete. □

Therefore, one can easily check that the absolute property does not hold on the space $\ell(\widetilde{B}, p)$; that is, $g_1(x) \neq g_1(|x|)$, where $|x| = (|x_k|)$. This says that $\ell(\widetilde{B}, p)$ is the sequence space of nonabsolute type.

Theorem 2. *Convergence in $\ell(\widetilde{B}, p)$ is stronger than coordinate-wise convergence.*

Proof. First we show that $h(x^n - x) \to 0$, as $n \to \infty$ implies $x_k^n \to x_k$; as $n \to \infty$ for every $k \in \mathbb{N}$. We fix k, then we have

$$\begin{aligned}\lim_{n\to\infty}\left|s_{k-1}x_{k-1}^{(n)} + r_k x_k^{(n)} - s_{k-1}x_{k-1} - r_k x_k\right|^{p_k} &\le \lim_{n\to\infty}\sum_k\left|s_{k-1}x_{k-1}^{(n)} + r_k x_k^{(n)} - s_{k-1}x_{k-1} - r_k x_k\right|^{p_k} \\ &= \lim_{n\to\infty}\left[h\left(x^n - x\right)\right]^M = 0.\end{aligned} \tag{23}$$

Hence, we have for $k = 0$ that

$$\lim_{n\to\infty}\left|s_{-1}x_{-1}^{(n)} + r_0 x_0^{(n)} - s_{-1}x_{-1} - r_0 x_0\right| = 0, \tag{24}$$

which gives the fact that $|x_0^{(n)} - x_0| \to 0$, as $n \to \infty$. Similarly, for each $k \in \mathbb{N}$, we have $|x_k^{(n)} - x_k| \to 0$, as $n \to \infty$. □

A sequence space λ with a linear topology is called a *K-space* provided each of the maps $p_i : \lambda \to \mathbb{C}$ defined by $p_i(x) = x_i$ is continuous for all $i \in \mathbb{N}$, where $\mathbb{C}$ denotes the complex field. A K-space λ is called an *FK-space* provided λ is complete linear metric space. An FK-space whose topology is normable is called a *BK-space*. Given a BK-space $\lambda \supset \phi$, we denote the nth *section of a sequence* $x = (x_k) \in \lambda$ by $x^{[n]} := \sum_{k=0}^{n} x_k e^{(k)}$, and we say that $x = (x_k)$ has the property AK if $\lim_{n\to\infty} \| x - x^{[n]}\|_\lambda = 0$. If AK property holds for every $x \in \lambda$, then we say that the space λ is called AK-space (cf. [19]). Now, we may give the following.

Theorem 3. $(\ell_p)_{\widetilde{B}}$ *is the linear space under the coordinatewise addition and scalar multiplication which is the BK-space with the norm*

$$\|x\| := \left(\sum_k\left|s_{k-1}x_{k-1} + r_k x_k\right|^p\right)^{1/p}, \quad \text{where } 1 \le p < \infty. \tag{25}$$

Proof. Because the first part of the theorem is a routine verification, we omit the detail. Since ℓ_p is the BK-space with respect to its usual norm (see [20, pages 217-218]) and $B(\widetilde{r}, \widetilde{s})$ is a normal matrix, Theorem 4.3.2 of Wilansky [21, page 61] gives the fact that $(\ell_p)_{\widetilde{B}}$ is the BK-space, where $1 \le p < \infty$. □

Let us suppose that $1 < p_k \le s_k$ for all $k \in \mathbb{N}$. Then, it is known that $\ell(p) \subset \ell(s)$ which leads us to the immediate consequence that $\ell(\widetilde{B}, p) \subset \ell(\widetilde{B}, s)$.

With the notation of (13), define the transformation T from $\ell(\widetilde{B}, p)$ to $\ell(p)$ by $x \mapsto y = Tx$. Since T is linear and bijection, we have the following.

Corollary 4. *The sequence space $\ell(\widetilde{B}, p)$ of nonabsolute type is linearly paranorm isomorphic to the space $\ell(p)$, where $0 < p_k \le H < \infty$ for all $k \in \mathbb{N}$.*

Theorem 5. *The space $\ell(\widetilde{B}, p)$ has AK.*

Proof. For each $x = (x_k) \in \ell(\widetilde{B}, p)$, we put

$$x^{\langle m\rangle} = \sum_{k=0}^{m} x_k e^{(k)}, \quad \forall m \in \{1, 2, \ldots\}. \tag{26}$$

Let $\varepsilon > 0$ and $x \in \ell(\widetilde{B}, p)$ be given. Then, there is $N = N(\varepsilon) \in \mathbb{N}$ such that

$$\sum_{k=N}^{\infty}\left|s_{k-1}x_{k-1} + r_k x_k\right|^{p_k} < \varepsilon^M. \tag{27}$$

Then we have for all $m \ge N$,

$$\begin{aligned}h\left(x - x^{\langle m\rangle}\right) = h\left(x - \sum_{k=1}^{m} x_k e^{(k)}\right) &= \left(\sum_{k=m+1}^{\infty}\left|s_{k-1}x_{k-1} + r_k x_k\right|^{p_k}\right)^{1/M} \\ &\le \left(\sum_{k=N}^{\infty}\left|s_{k-1}x_{k-1} + r_k x_k\right|^{p_k}\right)^{1/M} < \varepsilon.\end{aligned} \tag{28}$$

This shows that $x = \sum_k x_k e^{(k)}$.

Now we have to show that this representation is unique. We assume that $x = \sum_k \lambda_k e^{(k)}$. Then for each k,

$$\begin{aligned}&\left(\left|s_{k-1}\lambda_{k-1} + r_k\lambda_k - s_{k-1}x_{k-1} - r_k x_k\right|^{p_k}\right)^{1/M} \\ &\quad\le \left(\sum_k\left|s_{k-1}\lambda_{k-1} + r_k\lambda_k - s_{k-1}x_{k-1} - r_k x_k\right|^{p_k}\right)^{1/M} \\ &\quad= h\left(x - x\right) = 0.\end{aligned} \tag{29}$$

Hence, $s_{k-1}\lambda_{k-1} + r_k\lambda_k = s_{k-1}x_{k-1} + r_k x_k$ for each k.

For $k = 0$, $r_0\lambda_0 = r_0 x_0$. Since $r_0 \neq 0$, we have $\lambda_0 = x_0$.

For $k = 1$, $s_0\lambda_0 + r_1\lambda_1 = s_0 x_0 + r_1 x_1$. Since $r_1 \neq 0$, we also have $\lambda_1 = x_1$.

Continuing in this way, we obtain $\lambda_k = x_k$ for each k. Therefore, the representation is unique. □

We firstly define the concept of the Schauder basis for a paranormed sequence space and next give the basis of the sequence space $\ell(\widetilde{B}, p)$.

Let (X, g) be a paranormed space. A sequence (b_k) of the elements of X is called a *basis* for X if and only if, for each $x \in X$, there exists a unique sequence (α_k) of scalars such that

$$\lim_{n\to\infty} g\left(x - \sum_{k=0}^{n} \alpha_k b_k\right) = 0. \tag{30}$$

The series $\sum_k \alpha_k b_k$ which has the sum x is then called the expansion of x with respect to (b_n) and written as $x = \sum_k \alpha_k b_k$. Since it is known that the matrix domain λ_A of a sequence space λ has a basis if and only if λ has a basis whenever $A = (a_{nk})$ is a triangle (cf. [22, Remark 2.4]), we have the following.

Corollary 6. *Let $0 < p_k \le H < \infty$ and $\alpha_k = (\widetilde{B}x)_k$ for all $k \in \mathbb{N}$. Define the sequence $b^{(k)} = \{b_n^{(k)}\}_{n\in\mathbb{N}}$ of the elements of the space $\ell(\widetilde{B}, p)$ by*

$$b_n^{(k)} := \begin{cases} \dfrac{(-1)^{n-k}}{r_n} \displaystyle\prod_{j=k}^{n-1} \frac{s_j}{r_j}, & 0 \le k \le n, \\ 0, & k > n, \end{cases} \tag{31}$$

for every fixed $k \in \mathbb{N}$. Then, the sequence $\{b^{(k)}\}_{k\in\mathbb{N}}$ given by (31) *is a basis for the space $\ell(\widetilde{B}, p)$ and any $x \in \ell(\widetilde{B}, p)$ has a unique representation of the form $x := \sum_k \alpha_k b^{(k)}$.*

3. The Alpha-, Beta-, and Gamma-Duals of the Space $\ell(\widetilde{B},p)$

In this section, we state and prove the theorems determining the alpha-, beta-, and gamma-duals of the sequence space $\ell(\widetilde{B}, p)$ of nonabsolute type.

For the sequence spaces λ and μ, the set $S(\lambda, \mu)$ defined by

$$\begin{aligned} &S(\lambda, \mu) \\ &:= \{z = (z_k) \in \omega : xz = (x_k z_k) \in \mu \ \forall x = (x_k) \in \lambda\} \end{aligned} \tag{32}$$

is called the *multiplier space* of the spaces λ and μ. With the notation of (32), the alpha-, beta-, and gamma-duals of a sequence space λ, which are, respectively, denoted by λ^α, λ^β, and λ^γ, are defined by

$$\lambda^\alpha := S(\lambda, \ell_1), \qquad \lambda^\beta := S(\lambda, cs), \qquad \lambda^\gamma := S(\lambda, bs). \tag{33}$$

Since the case $0 < p_k \le 1$ may be established in similar way to the proof of the case $1 < p_k \le H < \infty$, we omit the detail of that case and give the proof only for the case $1 < p_k \le H < \infty$ in Theorems 10–12 below.

We begin with quoting three lemmas which are needed in proving Theorems 10–12.

Lemma 7 ([23, (i) and (ii) of Theorem 1]). *Let $A = (a_{nk})$ be an infinite matrix. Then, the following statements hold.*

(i) *Let $0 < p_k \le 1$ for all $k \in \mathbb{N}$. Then, $A \in (\ell(p) : \ell_\infty)$ if and only if*

$$\sup_{n,k\in\mathbb{N}} |a_{nk}|^{p_k} < \infty. \tag{34}$$

(ii) *Let $1 < p_k \le H < \infty$ for all $k \in \mathbb{N}$. Then, $A \in (\ell(p) : \ell_\infty)$ if and only if there exists an integer $M > 1$ such that*

$$\sup_{n\in\mathbb{N}} \sum_k \left|a_{nk} M^{-1}\right|^{p_k'} < \infty. \tag{35}$$

Lemma 8 ([23, Corollary for Theorem 1]). *Let $0 < p_k \le H < \infty$ for all $k \in \mathbb{N}$. Then, $A = (a_{nk}) \in (\ell(p) : c)$ if and only if* (34) *and* (35) *hold, and*

$$\lim_{n\to\infty} a_{nk} = \beta_k, \quad \forall k \in \mathbb{N}. \tag{36}$$

Lemma 9 ([24, Theorem 5.1.0]). *Let $A = (a_{nk})$ be an infinite matrix. Then, the following statements hold*

(i) *Let $0 < p_k \le 1$ for all $k \in \mathbb{N}$. Then, $A \in (\ell(p) : \ell_1)$ if and only if*

$$\sup_{N\in\mathcal{F}} \sup_{k\in\mathbb{N}} \left|\sum_{n\in N} a_{nk}\right|^{p_k} < \infty. \tag{37}$$

(ii) *Let $1 < p_k \le H < \infty$ for all $k \in \mathbb{N}$. Then, $A \in (\ell(p) : \ell_1)$ if and only if there exists an integer $M > 1$ such that*

$$\sup_{N\in\mathcal{F}} \sum_k \left|\sum_{n\in N} a_{nk} M^{-1}\right|^{p_k'} < \infty. \tag{38}$$

Theorem 10. *Define the sets $S_1(p)$ and $S_2(p)$ by*

$$\begin{aligned} S_1(p) = \bigcup_{M>1} \Bigg\{ a = (a_k) \in \omega : \\ \sup_{N\in\mathcal{F}} \sum_k \left|\sum_{n\in N} \frac{(-1)^{n-k}}{r_n} \prod_{j=k}^{n-1} \frac{s_j}{r_j} a_n M^{-1}\right|^{p_k'} < \infty \Bigg\}, \\ S_2(p) = \left\{ a = (a_k) \in \omega : \sup_{N\in\mathcal{F}} \sup_{k\in\mathbb{N}} \left|\sum_{n\in N} \frac{(-1)^{n-k}}{r_n} \prod_{j=k}^{n-1} \frac{s_j}{r_j} a_n\right|^{p_k} < \infty \right\}. \end{aligned} \tag{39}$$

Then,

$$\{\ell(\widetilde{B}, p)\}^\alpha = \begin{cases} S_1(p), & 1 < p_k \le H < \infty, \ \forall k \in \mathbb{N}, \\ S_2(p), & 0 < p_k \le 1, \ \forall k \in \mathbb{N}. \end{cases} \tag{40}$$

Proof. Let us take any $a = (a_n) \in \omega$. By using (13) we obtain that

$$x_n = \sum_{k=0}^{n} \frac{(-1)^{n-k}}{r_n} \prod_{j=k}^{n-1} \frac{s_j}{r_j} y_k \tag{41}$$

holds for all $n \in \mathbb{N}$ which leads us to

$$a_n x_n = \sum_{k=0}^{n} \frac{(-1)^{n-k}}{r_n} \prod_{j=k}^{n-1} \frac{s_j}{r_j} a_n y_k = (Cy)_n, \quad (n \in \mathbb{N}), \tag{42}$$

where $C = (c_{nk})$ is defined by

$$c_{nk} = \begin{cases} \dfrac{(-1)^{n-k}}{r_n} \displaystyle\prod_{j=k}^{n-1} \frac{s_j}{r_j} a_n, & 0 \le k \le n, \\ 0, & k > n \end{cases} \tag{43}$$

for all $k, n \in \mathbb{N}$. Thus, we observe by combining (42) with the condition (37) of Part (i) of Lemma 9 that $ax = (a_n x_n) \in \ell_1$ whenever $x = (x_k) \in \ell(\widetilde{B}, p)$ if and only if $Cy \in \ell_1$ whenever $y = (y_k) \in \ell(p)$. That means $\{\ell(\widetilde{B}, p)\}^{\alpha} = S_1(p)$. □

Theorem 11. *Define the sets $S_3(p)$, $S_4(p)$, and $S_5(p)$ by*

$$S_3(p) = \bigcup_{M>1} \left\{ a = (a_k) \in \omega : \sup_{n \in \mathbb{N}} \sum_k \left| \sum_{i=k}^{n} \frac{(-1)^{i-k}}{r_i} \prod_{j=k}^{i-1} \frac{s_j}{r_j} a_i M^{-1} \right|^{p_k'} < \infty \right\},$$

$$S_4(p) = \left\{ a = (a_k) \in \omega : \sum_{i=k}^{\infty} \frac{(-1)^{i-k}}{r_i} \prod_{j=k}^{i-1} \frac{s_j}{r_j} a_i < \infty \right\},$$

$$S_5(p) = \left\{ a = (a_k) \in \omega : \sup_{n,k \in \mathbb{N}} \left| \sum_{i=k}^{n} \frac{(-1)^{i-k}}{r_i} \prod_{j=k}^{i-1} \frac{s_j}{r_j} a_i \right|^{p_k} < \infty \right\}. \tag{44}$$

Then,

$$\{\ell(\widetilde{B}, p)\}^{\beta} = \begin{cases} S_3(p) \cap S_4(p), & 1 < p_k \le H < \infty \ \forall k \in \mathbb{N}, \\ S_4(p) \cap S_5(p), & 0 < p_k \le 1 \ \forall k \in \mathbb{N}. \end{cases} \tag{45}$$

Proof. Take any $a = (a_i) \in \omega$ and consider the equation obtained with (13) that

$$\begin{aligned} \sum_{i=0}^{n} a_i x_i &= \sum_{i=0}^{n} \left[\sum_{k=0}^{i} \frac{(-1)^{i-k}}{r_i} \prod_{j=k}^{i-1} \frac{s_j}{r_j} y_k \right] a_i \\ &= \sum_{k=0}^{n} \left[\sum_{i=k}^{n} \frac{(-1)^{i-k}}{r_i} \prod_{j=k}^{i-1} \frac{s_j}{r_j} a_i \right] y_k \\ &= (Dy)_n, \end{aligned} \tag{46}$$

where $D = (d_{nk})$ is defined by

$$d_{nk} = \begin{cases} \displaystyle\sum_{i=k}^{n} \frac{(-1)^{i-k}}{r_i} \prod_{j=k}^{i-1} \frac{s_j}{r_j} a_i, & 0 \le k \le n, \\ 0, & k > n \end{cases} \tag{47}$$

for all $k, n \in \mathbb{N}$. Thus, we deduce from Lemma 8 with (46) that $ax = (a_i x_i) \in cs$ whenever $x = (x_i) \in \ell(\widetilde{B}, p)$ if and only if $Dy \in c$ whenever $y = (y_k) \in \ell(p)$. Therefore, we derive from (35) and (36) that

$$\begin{gathered} \sup_{n \in \mathbb{N}} \sum_k \left| \sum_{i=k}^{n} \frac{(-1)^{i-k}}{r_i} \prod_{j=k}^{i-1} \frac{s_j}{r_j} a_i M^{-1} \right|^{p_k'} < \infty, \\ \sum_{i=k}^{\infty} \frac{(-1)^{i-k}}{r_i} \prod_{j=k}^{i-1} \frac{s_j}{r_j} a_i < \infty. \end{gathered} \tag{48}$$

This shows that $\{\ell(\widetilde{B}, p)\}^{\beta} = S_3(p) \cap S_4(p)$. □

Theorem 12.

$$\{\ell(\widetilde{B}, p)\}^{\gamma} = \begin{cases} S_3(p), & 1 < p_k \le H < \infty, \ \forall k \in \mathbb{N}, \\ S_5(p), & 0 < p_k \le 1, \ \forall k \in \mathbb{N}. \end{cases} \tag{49}$$

Proof. From Lemma 7 and (46), we obtain that $ax = (a_i x_i) \in bs$ whenever $x = (x_i) \in \ell(\widetilde{B}, p)$ if and only if $Dy \in \ell_{\infty}$ whenever $y = (y_k) \in \ell(p)$, where $D = (d_{nk})$ is defined by (47). Therefore, we obtain from (34) and (35) that $\{\ell(\widetilde{B}, p)\}^{\gamma} = S_3(p)$ for $1 < p_k$, $\{\ell(\widetilde{B}, p)\}^{\gamma} = S_5(p)$ for $p_k \le 1$. □

4. Matrix Transformations on the Sequence Space $\ell(\widetilde{B}, p)$

In this section, we characterize some matrix transformations on the space $\ell(\widetilde{B}, p)$. Theorem 13 gives the exact conditions of the general case $0 < p_k \le H < \infty$ by combining the cases $0 < p_k \le 1$ and $1 < p_k \le H < \infty$. We consider only the case $1 < p_k \le H < \infty$ and leave the case $0 < p_k \le 1$ to the reader because it can be proved in similar way.

Theorem 13. *Let $A = (a_{nk})$ be an infinite matrix. Then, the following statements hold.*

(i) *Let $1 < p_k \le H < \infty$ for all $k \in \mathbb{N}$. Then, $A \in (\ell(\widetilde{B}, p) : \ell_{\infty})$ if and only if there exists an integer $M > 1$ such that*

$$\sup_{n \in \mathbb{N}} \sum_k \left| \sum_{i=k}^{n} \frac{(-1)^{i-k}}{r_i} \prod_{j=k}^{i-1} \frac{s_j}{r_j} a_{ni} M^{-1} \right|^{p_k'} < \infty, \tag{50}$$

$$\sum_{i=k}^{\infty} \frac{(-1)^{i-k}}{r_i} \prod_{j=k}^{i-1} \frac{s_j}{r_j} a_{ni} < \infty. \tag{51}$$

(ii) *Let $0 < p_k \le 1$ for all $k \in \mathbb{N}$. Then, $A \in (\ell(\widetilde{B}, p) : \ell_{\infty})$ if and only if the condition (51) holds, and*

$$\sup_{n,k \in \mathbb{N}} \left| \sum_{i=k}^{n} \frac{(-1)^{i-k}}{r_i} \prod_{j=k}^{i-1} \frac{s_j}{r_j} a_{ni} \right|^{p_k} < \infty. \tag{52}$$

Proof. Suppose that the conditions (50) and (51) hold, and $x \in \ell(\widetilde{B}, p)$. In this situation, since $\{a_{nk}\}_{k \in \mathbb{N}} \in \{\ell(\widetilde{B}, p)\}^{\beta}$ for every fixed $n \in \mathbb{N}$, the A-transform of x exists. Consider the following equality obtained by using the relation (13) that

$$\sum_{k=0}^{m} a_{nk} x_k = \sum_{k=0}^{m} \sum_{i=k}^{m} \frac{(-1)^{i-k}}{r_i} \prod_{j=k}^{i-1} \frac{s_j}{r_j} a_{ni} y_k \tag{53}$$

for all $m, n \in \mathbb{N}$. Taking into account the hypothesis we derive from (53) as $m \to \infty$ that

$$\sum_k a_{nk} x_k = \sum_k \sum_{i=k}^{\infty} \frac{(-1)^{i-k}}{r_i} \prod_{j=k}^{i-1} \frac{s_j}{r_j} a_{ni} y_k, \quad \text{for each } n \in \mathbb{N}. \tag{54}$$

Now, by combining (54) with the following inequality (see [23]) which holds for any $M > 0$ and any $a, b \in \mathbb{C}$

$$|ab| \le M \left(\left| a M^{-1} \right|^{p'} + |b|^{p} \right), \tag{55}$$

where $p > 1$ and $p^{-1} + p'^{-1} = 1$, one can easily see that

$$\begin{aligned} \sup_{n\in\mathbb{N}} \left| \sum_k a_{nk} x_k \right| &\le \sup_{n\in\mathbb{N}} \sum_k \left| \sum_{i=k}^{\infty} \frac{(-1)^{i-k}}{r_i} \prod_{j=k}^{i-1} \frac{s_j}{r_j} a_{ni} \right| |y_k| \\ &\le \sup_{n\in\mathbb{N}} \sum_k M \left(\left| \sum_{i=k}^{\infty} \frac{1}{r_i} \prod_{j=k}^{i-1} \frac{s_j}{r_j} a_{ni} M^{-1} \right|^{p_k'} + |y_k|^{p_k} \right) \\ &\le M \left(\sup_{n\in\mathbb{N}} \sum_k \left| \sum_{i=k}^{\infty} \frac{1}{r_i} \prod_{j=k}^{i-1} \frac{s_j}{r_j} a_{ni} M^{-1} \right|^{p_k'} + \sum_k |y_k|^{p_k} \right) < \infty. \end{aligned} \tag{56}$$

Conversely, suppose that $A \in (\ell(\widetilde{B}, p) : \ell_\infty)$ and $1 < p_k \le H < \infty$ for all $k \in \mathbb{N}$. Then Ax exists for every $x \in \ell(\widetilde{B}, p)$ and this implies that $\{a_{nk}\}_{k\in\mathbb{N}} \in \{\ell(\widetilde{B}, p)\}^\beta$ for all $n \in \mathbb{N}$. Now, the necessity of (51) is immediate. Besides, we have from (54) that the matrix $B = (b_{nk})$ defined by $b_{nk} = \sum_{i=k}^{\infty}((-1)^{i-k}/r_i)\prod_{j=k}^{i-1}(s_j/r_j)a_{ni}$ for all $n, k \in \mathbb{N}$, is in the class $(\ell(p) : \ell_\infty)$. Then, B satisfies the condition (35) which is equivalent to (50).

This completes the proof. □

Lemma 14 ([25, Theorem 1]). *$A = (a_{nk}) \in (\ell(p) : f)$ if and only if* (34) *and* (35) *hold, and*

$$\exists \alpha_k \in \mathbb{C} \ni f - \lim a_{nk} = \alpha_k \quad \text{for every fixed } k \in \mathbb{N}. \tag{57}$$

Theorem 15. *Let the entries of the matrices $E = (e_{nk})$ and $F = (f_{nk})$ be connected with the relation*

$$e_{nk} := s_{k-1} f_{n,k-1} + r_k f_{nk} \quad \text{or} \quad f_{nk} := \sum_{i=k}^{\infty} \frac{(-1)^i}{r_i} \prod_{j=k}^{i-1} \frac{s_j}{r_j} e_{ni} \tag{58}$$

for all $k, n \in \mathbb{N}$. Then, $E \in (\ell(\widetilde{B}, p) : f)$ if and only if $F \in (\ell(p) : f)$ and

$$F^n \in (\ell(p) : c) \tag{59}$$

for every fixed $n \in \mathbb{N}$, where $F^n = (f_{mk}^{(n)})$ with

$$f_{mk}^{(n)} := \begin{cases} \sum_{i=k}^{m} \frac{(-1)^i}{r_i} \prod_{j=k}^{i-1} \frac{s_j}{r_j} e_{ni}, & 0 \le k \le m, \\ 0, & k > m, \end{cases} \tag{60}$$

for all $m, k \in \mathbb{N}$.

Proof. Let $E = (e_{nk}) \in (\ell(\widetilde{B}, p) : f)$ and take $x \in \ell(\widetilde{B}, p)$. Then, we obtain the equality

$$\begin{aligned} \sum_{k=0}^{m} e_{nk} x_k &= \sum_{k=0}^{m} e_{nk} \left[\sum_{i=0}^{k} \frac{(-1)^{k-i}}{r_i} \prod_{j=i}^{k-1} \frac{s_j}{r_j} y_i \right] \\ &= \sum_{k=0}^{m} \left[\sum_{i=k}^{m} \frac{(-1)^i}{r_i} \prod_{j=k}^{i-1} \frac{s_j}{r_j} \right] y_k = \sum_{k=0}^{m} f_{mk}^{(n)} y_k \end{aligned} \tag{61}$$

for all $m, n \in \mathbb{N}$. Since Ex exists, $F^n \in (\ell(p) : c)$. Letting $m \to \infty$ in the equality (61) we have $Ex = Fy$. Since $Ex \in f$, then $Fy \in f$. That is $F \in (\ell(p) : f)$.

Conversely, let $F \in (\ell(p) : f)$, and $F^n \in (\ell(p) : c)$, and take $x \in \ell(\widetilde{B}, p)$. Then, since $(f_{nk})_{k\in\mathbb{N}} \in \{\ell(p)\}^\beta$ and $F \in (\ell(p) : f)$ we have $(e_{nk})_{k\in\mathbb{N}} \in \{\ell(\widetilde{B}, p)\}^\beta$ for all $n \in \mathbb{N}$. So, Ex exists. Therefore we obtain from equality (61) as $m \to \infty$ that $Ex = Fy$, that is $E \in (\ell(\widetilde{B}, p) : f)$. □

Theorem 16. *Let $0 < p_k \le H < \infty$ for all $k \in \mathbb{N}$. Then, $A \in (\ell(\widetilde{B}, p) : c)$ if and only if* (50)–(52) *hold and*

$$\lim_{n\to\infty} \sum_{k=i}^{\infty} \frac{(-1)^{k-i}}{r_k} \prod_{j=i}^{k-1} \frac{s_j}{r_j} a_{nk} = \alpha_k, \quad \text{for every fixed } k \in \mathbb{N}. \tag{62}$$

Proof. Let $A \in (\ell(\widetilde{B}, p) : c)$ and $1 < p_k \le H < \infty$ for all $k \in \mathbb{N}$. Then, since the inclusion $c \subset \ell_\infty$ holds, the necessities of (50) and (51) are immediately obtained from part (i) of Theorem 13.

To prove the necessity of (62), consider the sequence $b^{(k)}$ defined by (31) which is in the space $\ell(\widetilde{B}, p)$ for every fixed $k \in \mathbb{N}$. Because the A-transform of every $x \in \ell(\widetilde{B}, p)$ exists and is in c by the hypothesis,

$$Ab^{(k)} = \left\{ \sum_{k=i}^{\infty} \frac{(-1)^{k-i}}{r_k} \prod_{j=i}^{k-1} \frac{s_j}{r_j} a_{nk} \right\}_{n\in\mathbb{N}} \in c \tag{63}$$

for every fixed $k \in \mathbb{N}$ which shows the necessity of (62).

Conversely suppose that conditions (50), (51), and (62) hold, and take any $x = (x_k)$ in the space $\ell(\widetilde{B}, p)$. Then, Ax exists. We observe for all $m, n \in \mathbb{N}$ that

$$\begin{aligned} &\sum_{k=0}^{m}\left|\sum_{k=i}^{m}\frac{(-1)^{k-i}}{r_k}\prod_{j=i}^{k-1}\frac{s_j}{r_j}a_{nk}M^{-1}\right|^{p_k'} \\ &\leq \sup_{n\in\mathbb{N}}\sum_{k}\left|\sum_{k=i}^{m}\frac{(-1)^{k-i}}{r_k}\prod_{j=i}^{k-1}\frac{s_j}{r_j}a_{nk}M^{-1}\right|^{p_k'} < \infty, \end{aligned} \tag{64}$$

which gives the fact that by letting $m, n \to \infty$ with (50) and (62) that

$$\begin{aligned} &\lim_{m,n\longrightarrow\infty}\sum_{k=0}^{m}\left|\sum_{k=i}^{m}\frac{(-1)^{k-i}}{r_k}\prod_{j=i}^{k-1}\frac{s_j}{r_j}a_{nk}M^{-1}\right|^{p_k'} \\ &\leq \sup_{n\in\mathbb{N}}\sum_{k}\left|\sum_{k=i}^{m}\frac{(-1)^{k-i}}{r_k}\prod_{j=i}^{k-1}\frac{s_j}{r_j}a_{nk}M^{-1}\right|^{p_k'} < \infty. \end{aligned} \tag{65}$$

This shows that $\sum_k |\alpha_k M^{-1}|^{p_k'} < \infty$ and so $(\alpha_k)_{k\in\mathbb{N}} \in \{\ell(\widetilde{B}, p)\}^{\beta}$ which implies that the series $\sum_k \alpha_k x_k$ converges for every $x \in \ell(\widetilde{B}, p)$.

Let us now consider the equality obtained from (54) with $a_{nk} - \alpha_k$ instead of a_{nk}

$$\begin{aligned} \sum_{k}(a_{nk}-\alpha_k)x_k &= \sum_{i}\sum_{k=i}\frac{(-1)^{k-i}}{r_k}\prod_{j=i}^{k-1}\frac{s_j}{r_j}(a_{nk}-\alpha_k)y_i \\ &= \sum_{k}c_{ni}y_i, \quad \forall n \in \mathbb{N}, \end{aligned} \tag{66}$$

where $C = (c_{ni})$ defined by $c_{ni} = \sum_{k=i}((-1)^{k-i}/r_k)\prod_{j=i}^{k-1}(s_j/r_j)(a_{nk} - \alpha_k)$ for all $n, i \in \mathbb{N}$. Therefore, we have at this stage from Lemma 8 that the matrix C belongs to the class $(\ell(p) : c_0)$ of infinite matrices. Thus, we see by (66) that

$$\lim_{n\to\infty}\sum_{k}(a_{nk}-\alpha_k)x_k = 0. \tag{67}$$

Equation (67) means that $Ax \in c$ whenever $x \in \ell(\widetilde{B}, p)$ and this is what we wished to prove. □

Therefore, we have the following

Corollary 17. *Let $0 < p_k \leq H < \infty$ for all $k \in \mathbb{N}$. Then, $A \in (\ell(\widetilde{B}, p) : c_0)$ if and only if (50)–(52) hold, and (62) also holds with $\alpha_k = 0$ for all $k \in \mathbb{N}$.*

Now, we give the following lemma given by Başar and Altay [26] which is useful for deriving the characterizations of the certain matrix classes via Theorems 13, 15, and 16 and Corollary 17.

Lemma 18 ([26, Lemma 5.3]). *Let λ, μ be any two sequence spaces, let A be an infinite matrix, and let B also be a triangle matrix. Then, $A \in (\lambda : \mu_B)$ if and only if $BA \in (\lambda : \mu)$.*

It is trivial that Lemma 18 has several consequences. Indeed, combining Lemma 18 with Theorems 13, 15, and 16 and Corollary 17, one can derive the following results.

Corollary 19. *Let $A = (a_{nk})$ be an infinite matrix and define the matrix $C = (c_{nk})$ by*

$$c_{nk} = \sum_{j=0}^{n}\binom{n}{j}(1-t)^{n-j}t^j a_{jk}, \quad \forall n, k \in \mathbb{N}. \tag{68}$$

Then, the necessary and sufficient conditions in order to A belongs to anyone of the classes $(\ell(\widetilde{B}, p) : e_\infty^t)$, $(\ell(\widetilde{B}, p) : e_c^t)$ and $(\ell(\widetilde{B}, p) : e_0^t)$ are obtained from the respective ones in Theorems 13, 16 and Corollary 17 by replacing the entries of the matrix A by those of the matrix C; where $0 < t < 1$, e_∞^t and e_c^t, e_0^t, respectively, denote the spaces of all sequences whose E^t-transforms are in the spaces ℓ_∞ and c, c_0 and are recently studied by Altay et al. [27] and Altay and Başar [28], where E^t denotes the Euler mean of order t.

Corollary 20. *Let $A = (a_{nk})$ be an infinite matrix and define the matrix $C = (c_{nk})$ by*

$$c_{nk} = s a_{n-1,k} + r a_{nk}, \quad \forall n, k \in \mathbb{N}. \tag{69}$$

Then, the necessary and sufficient conditions in order to A belongs to the class $(\ell(\widetilde{B}, p) : \widehat{f})$ is obtained from Theorem 15 by replacing the entries of the matrix A by those of the matrix C; where $r, s \in \mathbb{R} \setminus \{0\}$ and $\widehat{f}$ denotes the space of all sequences whose $B(r, s)$-transforms are in the space f and is recently studied by Başar and Kirişçi [29].

Corollary 21. *Let $A = (a_{nk})$ be an infinite matrix and define the matrix $C = (c_{nk})$ by*

$$c_{nk} = t a_{n-2,k} + s a_{n-1,k} + r a_{nk}, \quad \forall n, k \in \mathbb{N}. \tag{70}$$

Then, the necessary and sufficient conditions in order to A belongs to the class $(\ell(\widetilde{B}, p) : f(B))$ is obtained from Theorem 15 by replacing the entries of the matrix A by those of the matrix C; where $r, s, t \in \mathbb{R} \setminus \{0\}$ and $f(B)$ denotes the space of all sequences whose $B(r, s, t)$-transforms are in the space f and is recently studied by Sönmez [30].

Corollary 22. *Let $A = (a_{nk})$ be an infinite matrix and define the matrix $C = (c_{nk})$ by*

$$c_{nk} = \frac{1}{n+1}\sum_{j=0}^{n}a_{jk}, \quad \forall n, k \in \mathbb{N}. \tag{71}$$

Then, the necessary and sufficient conditions in order to A belongs to the class $(\ell(\widetilde{B}, p) : \widetilde{f})$ is obtained from Theorem 15 by replacing the entries of the matrix A by those of the matrix C, where $\widetilde{f}$ denotes the space of all sequences whose C_1-transforms are in the space f and is recently studied by Kayaduman and Şengönül [31].

Corollary 23. *Let $A = (a_{nk})$ be an infinite matrix and let $t = (t_k)$ be a sequence of positive numbers and define the matrix $C = (c_{nk})$ by*

$$c_{nk} = \frac{1}{T_n}\sum_{j=0}^{n} t_j a_{jk}, \quad \forall n, k \in \mathbb{N}, \tag{72}$$

where $T_n = \sum_{k=0}^{n} t_k$ for all $n \in \mathbb{N}$. Then, the necessary and sufficient conditions in order to A belongs to anyone of the classes $(\ell(\widetilde{B}, p) : r_\infty^t)$, $(\ell(\widetilde{B}, p) : r_c^t)$ and $(\ell(\widetilde{B}, p) : r_0^t)$ are obtained from the respective ones in Theorems 13, 16 and Corollary 17 by replacing the entries of the matrix A by those of the matrix C, where r_∞^t, r_c^t, and r_0^t are defined by Altay and Başar in [32] as the spaces of all sequences whose R^t-transforms are, respectively, in the spaces ℓ_∞, c, and c_0, and are derived from the paranormed spaces $r_\infty^t(p)$, $r_c^t(p)$ and $r_0^t(p)$ in the case $p_k = p$ for all $k \in \mathbb{N}$.

Since the spaces r_∞^t, r_c^t, and r_0^t reduce in the case $t = e$ to the Cesàro sequence spaces X_∞, $\widetilde{c}$, and $\widetilde{c}_0$ of nonabsolute type, respectively, Corollary 23 also includes the characterizations of the classes $(\ell(\widetilde{B}, p) : X_\infty)$, $(\ell(\widetilde{B}, p) : \widetilde{c})$, and $(\ell(\widetilde{B}, p) : \widetilde{c}_0)$, as a special case, where X_∞ and $\widetilde{c}$, $\widetilde{c}_0$ are the Cesàro spaces of the sequences consisting of C_1-transforms are in the spaces ℓ_∞ and c, c_0 and studied by Ng and Lee [33] and Şengönül and Başar [34], respectively, where C_1 denotes the Cesàro mean of order 1.

Corollary 24. *Let $A = (a_{nk})$ be an infinite matrix and define the matrix $C = (c_{nk})$ by $c_{nk} = a_{nk} - a_{n+1,k}$ for all $n, k \in \mathbb{N}$. Then, the necessary and sufficient conditions in order to A belongs to anyone of the classes $(\ell(\widetilde{B}, p) : \ell_\infty(\Delta))$, $(\ell(\widetilde{B}, p) : c(\Delta))$ and $(\ell(\widetilde{B}, p) : c_0(\Delta))$ are obtained from the respective ones in Theorems 13 and 16 and Corollary 17 by replacing the entries of the matrix A by those of the matrix C, where $\ell_\infty(\Delta)$, $c(\Delta)$, $c_0(\Delta)$ denote the difference spaces of all bounded, convergent, and null sequences and are introduced by Kızmaz [35].*

Corollary 25. *Let $A = (a_{nk})$ be an infinite matrix and define the matrix $C = (c_{nk})$ by $c_{nk} = \sum_{j=0}^{n} a_{jk}$ for all $n, k \in \mathbb{N}$. Then the necessary and sufficient conditions in order to A belongs to anyone of the classes $(\ell(\widetilde{B}, p) : bs)$, $(\ell(\widetilde{B}, p) : cs)$ and $(\ell(\widetilde{B}, p) : cs_0)$ are obtained from the respective ones in Theorems 13, 16 and Corollary 17 by replacing the entries of the matrix A by those of the matrix C, where cs_0 denotes the set of those series converging to zero.*

5. Conclusion

The difference spaces $\ell_\infty(\Delta)$, $c(\Delta)$, and $c_0(\Delta)$ were introduced by Kızmaz [35]. Since we essentially employ the infinite matrices which is more different than Kızmaz and the other authors following him, and use the technique of obtaining a new sequence space by the matrix domain of a triangle limitation method. Following this way, the domain of some triangle matrices in the sequence space $\ell(p)$ was recently studied and were obtained certain topological and geometric results by Altay and Başar [14, 16], Choudhary and Mishra [10], Başar et al. [36], and Aydın and Başar [13]. Although $bv(e, p) = [\ell(p)]_\Delta$ is investigated, since $B(1, -1) \equiv \Delta$, our results are more general than those of Başar et al. [36]. Also in case $p_k = p$ for all $k \in \mathbb{N}$ the results of the present study are reduced to the corresponding results of the recent paper of Kirişçi and Başar [9]. We should note that the difference spaces $\Delta c_0(p)$, $\Delta c(p)$ and $\Delta \ell_\infty(p)$ of Maddox's spaces $c_0(p)$, $c(p)$, and $\ell_\infty(p)$ were studied by Ahmad and Mursaleen [37]. Of course, a natural continuation of the present paper is to study the sequence spaces $[c_0(p)]_{B(\widetilde{r},\widetilde{s})}$, $[c(p)]_{B(\widetilde{r},\widetilde{s})}$ and $[\ell_\infty(p)]_{B(\widetilde{r},\widetilde{s})}$ to generalize the main results of Ahmad and Mursaleen [37] which fills up a gap in the existing literature.

It is clear that $\Delta^{(1)}$ can be obtained as a special case of $B(\widetilde{r}, \widetilde{s})$ for $\widetilde{r} = e$ and $\widetilde{s} = -e$ and it is also trivial that $B(\widetilde{r}, \widetilde{s})$ is reduced in the special case $\widetilde{r} = re$ and $\widetilde{s} = se$ to the generalized difference matrix $B(r, s)$. So, the results related to the domain of the matrix $B(\widetilde{r}, \widetilde{s})$ are much more general and more comprehensive than the corresponding consequences of the domain of the matrix $B(r, s)$. We should note from now that the main results of the present paper are given as an extended abstract without proof by Nergiz and Başar [38], and our next paper will be devoted to some geometric and topological properties of the space $\ell(\widetilde{B}, p)$.

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