ON THE DEGREE THEORY FOR DENSELY DEFINED MAPPINGS OF CLASS $(S_+)_L$

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Received 15 May 1999

We introduce a new construction of topological degree for densely defined mappings of monotone type. We also study the structure of the classes of mappings involved. Using the basic properties of the degree, we prove some abstract existence results that can be applied to elliptic problems.

1. Introduction

Topological degree theory is one of the main tools in the study of nonlinear problems. Since the pioneering works of Brouwer in 1912 [6] and Leray and Schauder in 1934 [11], numerous extensions have been published (cf. [2, 3, 4, 8, 13, 14, 15]).

In a recent paper [10], Kartsatos and Skrypnik introduced a topological degree for densely defined operators in reflexive Banach spaces. In the first part of [10] they considered densely defined mappings satisfying a generalized condition (S_+). This new condition, called $(S_+)_{0,L}$, is a natural extension of the standard (S_+)-property.

In [10] a construction of a new degree function is given. The authors start with a definition of condition $(S_+)_{0,L}$ for densely defined mappings. For this class, with a finite dimensional continuity condition, they define a Galerkin type approximation scheme. Using a limit process and the finite dimensional Brouwer degree, a new single valued degree is achieved. Only the basic properties of the new degree were proved. Most important is the homotopy invariance property. As an application, Kartsatos and Skrypnik consider Dirichlet problem for the elliptic operator

$$A(u) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left\{ e^{2u} \frac{\partial u}{\partial x_i} + a_i \left(x, u, \frac{\partial u}{\partial x_i} \right) \right\}, \tag{1.1}$$

where the coefficients a_i satisfy the standard growth and ellipticity conditions. More generally, the term e^u can be replaced by a function $\rho(u)$ satisfying a very mild growth condition.

Abstract and Applied Analysis 4:3 (1999) 141–152
1991 Mathematics Subject Classification: 47H11, 35J60
URL: http://aaa.hindawi.com/volume-4/S1085337599000111.html

Our paper is organized as follows. In Section 2, we recall the classical definitions of the different classes of everywhere defined mappings of monotone type. The structure and mutual relations of these classes are well known. For the readers' convenience we recall briefly the basic facts without proofs.

In Section 3, we define the class $(S_+)_{0,L}$ introduced by Kartsatos and Skrypnik in [10]. We also define new classes $(S_+)_L$, $(S_+)_{0,D(A)}$, and $(S_+)_{D(A)}$, which seem to be useful in the structural study of classes of operators. In Theorem 3.3 we show how these new classes are related to standard class (S_+). The invariance under perturbations is considered in Theorems 3.4 and 3.5.

Section 4 is devoted to the construction and properties of degree. We will first recall the construction given in [10]. Then we introduce an alternative construction. Indeed, instead of finite dimensional approximants, it is possible to use a Leray-Schauder type approximation scheme. This provides some advantage making the reasoning more transparent due to the fact that we are working in the same space at each stage of the process. We will close this section by a discussion about the basic properties of the degree.

In Section 5, we will consider bounded and locally bounded operators. It turns out that essentially no new results can be obtained in this case. Hence, any boundedness condition should be avoided.

Section 6 contains some abstract applications. We generalize the Borsuk's theorem and give sufficient conditions which make it possible to calculate the value of degree for an isolated zero, that is, the "index." We also give some abstract existence results.

2. Everywhere defined mappings of monotone type

Let X be a real separable reflexive Banach space, and let X^* be its dual space X^* with continuous pairing $\langle\cdot,\cdot\rangle$. By the results due to Trojanski [16] every reflexive Banach space has an equivalent norm such that X and X^* are both locally uniformly convex. Thus, we assume from now on that X and X^* are locally uniformly convex. This renorming is needed for the definition of the *duality mapping* $J : X \rightarrow X^*$. Indeed, it follows from the Hahn-Banach theorem and the convexity of the norm that the conditions

$$\|J(u)\| = \|u\|, \quad \langle J(u), u\rangle = \|u\|^2 \quad \forall u \in X \tag{2.1}$$

determine a unique map $J : X \rightarrow X^*$. We consider mappings acting from X into X^*. For simplicity we assume that all mappings are defined on the whole of X. The norm convergence is denoted by $\rightarrow$, the weak convergence by $\rightharpoonup$, and the continuous pairing between X and X^* by $\langle\cdot,\cdot\rangle$. We recall that a mapping $F : X \rightarrow X^*$ is

(1) *bounded* if it takes any bounded set into a bounded set,
(2) *locally bounded* if each point $u \in X$ has a neighborhood U such that $F(U)$ remains bounded,
(3) *demicontinuous* if $u_k \rightarrow u$ implies $F(u_k) \rightharpoonup F(u)$,
(4) *monotone* if $\langle F(u) - F(v), u - v\rangle \geq 0$ for all $u, v \in X$,
(5) *strongly monotone* if there exists a constant $\mu > 0$ such that for all $u, v \in X$

$$\langle N(u) - N(v), u - v\rangle \geq \mu\|u - v\|^2, \tag{2.2}$$

(6) *of class* (S_+) if for any sequence such that $u_k \rightharpoonup u$ in X^* and $\limsup\langle F(u_k), u_k - u\rangle \le 0$, it follows that $u_k \to u$,

(7) *pseudomonotone* if for any sequence such that $u_k \rightharpoonup u$ and $\limsup\langle F(u_k), u_k - u\rangle \le 0$, it follows that $F(u_k) \rightharpoonup F(u)$ and $\langle F(u_k), u_k - u\rangle \to 0$,

(8) *quasimonotone* if $u_k \rightharpoonup u$ implies $\limsup\langle F(u_k), u_k - u\rangle \ge 0$.

The definitions and notations given above are quite standard and frequently used in the literature. Different notations are used in [14], see also [9]. The following remarks and results are mostly well known.

(a) In applications, when a Galerkin method is used, even the demicontinuity is too strong requirement. It is sufficient that F is *hemicontinuous*, that is, $F(u+t_k v) \to F(u)$ as $t_k \to 0+$, or continuous on finite dimensional subspaces of X.

(b) The concept of pseudomonotonicity was originally introduced by Brezis [5]. It is not hard to see that the original definition coincides with the one given here.

(c) Any locally bounded pseudomonotone mapping is demicontinuous.

(d) Any demicontinuous map is locally bounded.

(e) Any monotone map is locally bounded.

(f) Any monotone hemicontinuous map is demicontinuous and pseudomonotone.

(g) The duality map $J : X \to X^*$ is continuous, bounded, and of class (S_+). Moreover, J is strictly monotone but generally not strongly monotone.

(h) Any strongly monotone map is of class (S_+).

(i) Any map of the form $I - C : X \to X$, where C is compact, is said to be *Leray-Schauder type* (denote $I - C \in$ (LS)). Note that in Hilbert space setting any Leray-Schauder type map belongs to the class (S_+).

We assume that all mappings appearing in this section from now on are demicontinuous. If $F : X \to X^*$ and $T : X \to X^*$ are strongly monotone, then $F+T$ and αF are strongly monotone for all $\alpha > 0$. In this sense, the class of strongly monotone maps is said to have a *conical structure*. Similarly, the class (S_+), the class of pseudomonotone maps (denote (PM)), the class of monotone maps (denote (MON)), and the class of quasimonotone maps (denote (QM)) have a conical structure. Note that the zero map is (PM) and in (QM) but not in (S_+) if the space is infinite dimensional. Denoting the class of compact maps by (COMP), we have the following inclusions:

(j) (COMP) $\subset$ (QM) and $(S_+) \subset$ (PM) $\subset$ (QM).

For demicontinuous mappings of class (S_+) there exists a classical topological degree theory. Its practical value depends on the number of admissible homotopies available. For instance, any affine homotopy $(1-t)F_0 + tF_1$, $0 \le t \le 1$ between (S_+)-maps is admissible. This fact is closely related to the conical structure of class (S_+). Another important property of the class (S_+) is that it is very stable under perturbations. Indeed, we have

(k) If $F \in (S_+)$ and $T \in$ (QM), then $F+T \in (S_+)$.

(l) If $F+T \in (S_+)$ for any $F \in (S_+)$, then $T \in$ (QM).

A standard application of the above concepts is a partial differential operator of the generalized divergence form

$$A(u)(x) = \sum_{|\alpha|\le m} (-1)^{\alpha} D^{\alpha} A_{\alpha}\big(x, u(x), \dots, D^m u(x)\big), \quad x \in \Omega, \tag{2.3}$$

where Ω is an open subset of $\mathbb{R}^N$. Assume that the Caratheodory functions A_α satisfy polynomial growth condition. Then there are natural conditions such that the divergence form (with some boundary conditions) generates a mapping of monotone type from a Sobolev-space into its dual space (see [7, 14]).

3. Densely defined mappings of monotone type

Let X be a real separable reflexive Banach space with dual space X^* and continuous pairing $\langle\cdot,\cdot\rangle$. We assume that X and X^* are locally uniformly convex. Let $A : D(A) \subset X \to X^*$ be a possibly nonlinear map. Following the notations of [10], we assume that there exists a subspace L of X such that

$$L \subset D(A), \qquad \overline{L} = X. \tag{3.1}$$

Denote by $\mathcal{F}(L)$ the set of all finite dimensional subspaces of L. Let $(F_j)_{j=1}^\infty$ be a sequence of subspaces such that for all $j \in \mathbb{Z}_+$

$$F_j \in \mathcal{F}(L), \quad F_j \subset F_{j+1}, \quad \dim F_j = j, \quad \overline{\cup_j F_j} = X. \tag{3.2}$$

For each sequence satisfying (3.2) we denote

$$L\big(F_j\big) = \cup_{j=1}^\infty F_j. \tag{3.3}$$

Definition 3.1. A mapping $A : D(A) \subset X \to X^*$ satisfies *condition* $(S_+)_{0,L}$ if (3.1) holds and for any sequence (F_j) of subspaces satisfying (3.2), the conditions

$$\begin{gathered} (u_k) \subset L, \quad u_k \rightharpoonup u_0, \quad \limsup_{k\to\infty}\big\langle A\big(u_k\big), u_k\big\rangle \le 0, \\ \lim_{k\to\infty}\big\langle A\big(u_k\big), v\big\rangle = 0 \quad \forall v \in L\big(F_j\big) \end{gathered} \tag{3.4}$$

imply that $u_k \to u_0, u_0 \in D(A)$, and $A(u_0) = 0$. In case $A - h$ satisfies condition $(S_+)_{0,L}$ for all $h \in X^*$, we say that A satisfies condition $(S_+)_L$.

If the condition "$(u_k) \subset L$" in Definition 3.1 is replaced by "$(u_k) \subset D(A)$," we say that the mapping satisfies *condition* $(S_+)_{0,D(A)}$. In case $A - h$ satisfies condition $(S_+)_{0,D(A)}$ for all $h \in X^*$, we say that A satisfies condition $(S_+)_{D(A)}$.

The condition $(S_+)_{0,L}$ is introduced in [10]. The subclass $(S_+)_{D(A)}$ of $(S_+)_L$ defined by us is a natural extension of the classical definition of class (S_+), (see Theorem 3.1). However, in relevant applications, it is easier to verify condition $(S_+)_{0,L}$ (cf. [10, Section 5]) than $(S_+)_{0,D(A)}$.

Example 3.2. We show that $(S_+)_{D(A)}$ is a proper subset of $(S_+)_L$. Indeed, assume that $X = H$, a real separable Hilbert space with orthonormal basis $\{e_1, e_2, \dots\}$. Now $X^* = H$ and take $L = \operatorname{span}\{e_1, e_2, \dots\}$. We consider the mapping $A = I + C : X \to X^*$, where I is the identity mapping and C is defined by setting

$$C(u) = \begin{cases} 0 & \text{if } u \in L, \\ -u & \text{if } u \notin L. \end{cases} \tag{3.5}$$

Now $D(A) = X$ and it is easy to see that $A \notin (S_+)_{0,D(A)}$ but $A \in (S_+)_L$.

The connection between classes (S_+), $(S_+)_{D(A)}$, and $(S_+)_L$ is the following.

Theorem 3.3. *Let $A : D(A) \subset X \to X^*$ be a bounded operator such that (3.1) holds. Then*

(A) (i) *$A \in (S_+)_{D(A)}$ if and only if*

(ii) *$D(A) = X$ and $A : X \to X^*$ is demicontinuous and of class (S_+).*

(B) (i)′ *$A \in (S_+)_L$ if and only if*

(ii)′ *$D(A) = X$ and $A : X \to X^*$ is demicontinuous and satisfies condition (S_+) for any sequence (u_k) on L, that is,*

$$(u_k) \subset L, \quad u_k \rightharpoonup u_0, \quad \limsup \langle A(u_k), u_k - u_0 \rangle \leq 0 \tag{3.6}$$

imply $u_k \to u_0$.

Proof. Assume (i) and let $u_0 \in X$ be given. There exists a sequence $(u_k) \subset D(A)$ such that $u_k \to u_0$ and since A is bounded, the sequence $(A(u_k))$ is bounded in X^*. We can write at least for a subsequence $A(u_k) \rightharpoonup w \in X^*$. Thus, $\limsup\langle A(u_k) - w, u_k\rangle = 0$ and $\lim\langle A(u_k) - w, v\rangle = 0$ for all $v \in L(F_j)$ imply that $u_0 \in D(A)$ and $A(u_0) = w$. Hence, $D(A) = X$ and A is demicontinuous. Assume now that

$$(u_k) \subset X, \quad u_k \rightharpoonup u_0, \quad \limsup \langle A(u_k), u_k - u_0 \rangle \leq 0. \tag{3.7}$$

For a subsequence $A(u_k) \rightharpoonup y \in X^*$ and hence

$$\limsup \langle A(u_k) - y, u_k \rangle \leq 0, \qquad \lim \langle A(u_k) - y, v \rangle = 0 \tag{3.8}$$

for all $v \in L(F_j)$. Consequently, $u_k \to u_0$ and (ii) is proved.

Assume now that (ii) is valid and $(u_k) \subset X$ such that

$$u_k \rightharpoonup u_0, \quad \limsup \langle A(u_k) - h, u_k \rangle \leq 0, \quad \lim \langle A(u_k) - h, v \rangle = 0 \tag{3.9}$$

for all $v \in L(F_j)$. Since $L(F_j)$ is dense in X and A is bounded, we conclude that $A(u_k) \rightharpoonup h$. Consequently, $\limsup\langle A(u_k), u_k - u_0\rangle \leq 0$ implying $u_k \to u_0$ and $A(u_0) = h$, which completes the proof.

The equivalence of (i)′ and (ii)′ can be proved analogously. □

Theorem 3.3 shows that in order to obtain new results, it is essential that A is not bounded. The case A is locally bounded on $D(A)$, that is, each $u \in D(A)$ has a neighborhood $B_\epsilon(u)$ such that $A(B_\epsilon(u) \cap D(A))$ is bounded, is treated separately in Section 5. We impose no boundedness assumption on A in the general case.

The structure of the usual class (S_+) is well known. Unfortunately, similar complete description for class $(S_+)_L$ seems impossible. However, some results into this direction can be achieved. In our next theorem we consider bounded perturbations.

Theorem 3.4. *Let $N : X \to X^*$ be a bounded mapping. Then*

(i) *N is demicontinuous and quasimonotone if and only if*

(ii) *$A + N \in (S_+)_{D(A)}$ for all $A \in (S_+)_{D(A)}$.*

Proof. Let $N : X \rightharpoonup X^*$ be a demicontinuous quasimonotone map and A satisfies condition $(S_+)_{D(A)}$. Assume that

$$
\begin{gathered}
(u_k) \subset D(A), \quad u_k \rightharpoonup u_0, \quad \limsup \langle A(u_k) + N(u_k) - h, u_k \rangle \leq 0, \\
\lim \langle A(u_k) + N(u_k) - h, v \rangle = 0 \quad \forall v \in L(F_j).
\end{gathered} \tag{3.10}
$$

At least for a subsequence we can assume that $N(u_k) \rightharpoonup w$. Then, due to the quasi-monotonicity of N, we easily obtain

$$
\limsup \langle A(u_k) + w - h, u_k \rangle \leq 0, \qquad \lim \langle A(u_k) + w - h, v \rangle = 0 \tag{3.11}
$$

for all $v \in L(F_j)$. Hence $u_k \to u_0 \in D(A)$ and $A(u_0) + w - h = 0$. Consequently, $A(u_0) + N(u_0) = h$ completes the first part of the proof.

On the other hand, if we assume (ii), then by Theorem 3.3 we also have $A + N$ is demicontinuous and of class (S_+) for any bounded demicontinuous $A \in (S_+)$. But this implies (cf. [1]) that N is demicontinuous and quasimonotone. □

From the first part of the above proof of Theorem 3.4, we immediately see that the following theorem holds.

THEOREM 3.5. *Let $N : X \rightharpoonup X^*$ be a bounded demicontinuous quasimonotone map and $A \in (S_+)_L$. Then $A + N \in (S_+)_L$.*

As a direct implication of the previous theorem, we notice that the following special cases are usually met in applications. If $A \in (S_+)_L$ and $N : X \rightharpoonup X^*$ is demicontinuous and bounded, then $A + N \in (S_+)_L$ in case N is strongly monotone, monotone, or compact.

4. On the construction of the degree

We say that A is an *admissible map* of class $(S_+)_L$ whenever the following conditions (A1) and (A2) hold. Recall that by (3.1) there exists a subspace L of X such that $L \subset D(A)$ and $\overline{L} = X$. We assume that $A : D(A) \subset X \rightharpoonup X^*$ satisfies the following conditions:

(A1) there exists a subspace L such that (3.1) holds and $A \in (S_+)_L$;

(A2) for every $F \in \mathcal{F}(L)$, $v \in L$ the mapping $a(\cdot, v) : F \rightharpoonup \mathbb{R}$ defined by $a(u, v) = \langle A(u), v \rangle$ is continuous.

Conditions (A1) and (A2) are adopted from [10]. Let (F_j) be a fixed sequence of subspaces of L satisfying (3.2) and let $(v_j) \subset L$ be a fixed sequence such that $F_j = \operatorname{span}(v_1, v_2, \ldots, v_j)$ for all $j \in \mathbb{Z}_+$. For every $j \in \mathbb{Z}_+$ a finite dimensional approximation A_j of the operator A is defined by

$$
A_j(u) = \sum_{i=1}^{j} \langle A(u), v_i \rangle v_i. \tag{4.1}
$$

Let G be an open bounded subset of X such that $A(u) \neq 0$ for all $u \in \partial G \cap D(A)$. It is proved in [10] that there exists $j_0 \in \mathbb{Z}_+$ such that the finite dimensional Brouwer degree $d_B(A_j, G\cap F_j, 0)$ is well defined and independent of j for all $j \geq j_0$. Thus, it is relevant to define a new degree function by setting

$$\deg(A, G, 0) = \lim_{j\to\infty} d_B\big(A_j, G\cap F_j, 0\big). \tag{4.2}$$

If $h \in X^*$ and $A(u) \neq h$ for all $u \in \partial G\cap D(A)$, then we naturally define

$$\deg(A, G, h) = \deg(A-h, G, 0). \tag{4.3}$$

Kartsatos and Skrypnik [10] also showed that the definition does not depend on the choice of the sequence (F_j).

It is interesting to notice that the degree defined above can be constructed in a different way via classical Leray-Schauder degree. Indeed, denoting $W_j = \overline{\operatorname{span}}(v_{j+1}, v_{j+2}, \ldots)$ we have $X = F_j \oplus W_j$ for all $j \in \mathbb{Z}_+$. Corresponding to this splitting, we define $P_j(f+w) = f$ for all $f + w \in F_j \oplus W_j$. The projection $P_j : X \to F_j$ is compact for each $j \in \mathbb{Z}_+$ due to the fact that $\dim F_j = j < \infty$. We define a family $\{T_j\}$ of mappings by setting

$$T_j = I - P_j + A_j \circ P_j : X \longrightarrow X, \quad j \in \mathbb{Z}_+, \tag{4.4}$$

where A_j is given by (4.1). If A is an admissible map of class $(S_+)_L$, then clearly $P_j - A_j \circ P_j$ is compact and consequently each T_j is a mapping of Leray-Schauder type. In fact $P_j - A_j \circ P_j$ is completely continuous and hence T_j is weakly continuous. Assume that $A(u) \neq 0$ for all $u \in \partial G\cap D(A)$, where $G \subset X$ is some open bounded set. Then there exists $j_0 \in \mathbb{Z}_+$ such that $d_{\mathrm{LS}}(T_j, G, 0)$ is well defined and constant in j for all $j \geq j_0$. Hence we can define

$$\deg(A, G, 0) = \lim_{j\to\infty} d_{\mathrm{LS}}\big(T_j, G, 0\big). \tag{4.5}$$

It is tedious but not difficult to prove that the degree defined by (4.5) and the degree constructed by Kartsatos and Skrypnik in [10] coincide. This fact follows from the uniqueness of the Brouwer degree. In view of Theorem 3.3, any bounded demicontinuous map $N : X \to X^*$ of class (S_+) is also admissible and hence the restriction of the degree function (deg) to bounded demicontinuous maps of class (S_+) coincides with the classical topological degree introduced in [7, 15], which is unique [1].

The degree function defined above has the properties of a classical topological degree (see [7, 12]). In [10], only homotopy invariance property is considered. In what follows, A is an admissible map of class $(S_+)_L$, G is an open bounded subset of X and $h \in X^*$ such that $A(u) \neq h$ for all $u \in \partial G\cap D(A)$. The first of the basic properties (a)–(d) is

(a) If $\deg(A, G, h) \neq 0$, then the equation $A(u) = h$ admits at least one solution on G.

Clearly (a) follows from the definition of "deg." In fact, (a) follows also from the additivity property:

(b) (Additivity) Let G_1 and G_2 be open disjoint subsets of G. If $h \notin A((\overline{G}\setminus(G_1 \cup G_2))\cap D(A))$, then

$$\deg(A, G, h) = \deg\big(A, G_1, h\big) + \deg\big(A, G_2, h\big). \tag{4.6}$$

The additivity property is a consequence of the corresponding property of the Brouwer degree and the definition (4.2).

(c) (Invariance under homotopies) We first recall the definition of admissible homotopies given in [10]. Let A_t, $0 \le t \le 1$, be a family of operators from $D(A_t) \subset X$ to X^*. We assume that there exist a subspace L of X and a sequence (F_j) satisfying condition (3.2) such that

$$L \subset D(A_t) \subset X, \qquad \overline{L} = X. \tag{4.7}$$

The family A_t, $0 \le t \le 1$ satisfies *condition* $(S_+)_{0,L}^{(t)}$ if the conditions

$$\begin{gathered} (u_k) \subset L(F_j), \quad u_k \rightharpoonup u_0, \quad t_k \longrightarrow t_0, \quad \limsup \langle A_{t_k}(u_k), u_k \rangle \le 0, \\ \lim \langle A_{t_k}(u_k), v \rangle = 0 \quad \forall v \in L(F_j) \end{gathered} \tag{4.8}$$

imply that $u_k \rightarrow u_0 \in D(A_{t_0})$ and $A_{t_0}(u_0) = 0$. In case $A_t - h_t$, $0 \le t \le 1$ satisfies condition $(S_+)_{0,L}$ for any continuous curve $h_t, 0 \le t \le 1$ in X^*, we say that A_t, $0 \le t \le 1$ is a *homotopy of class* $(S_+)_L^{(t)}$. Moreover, we say that a homotopy is *admissible* if

(H) for every $F \subset L(F_j)$, $v \in L(F_j)$ the mapping $a(\cdot,\cdot,v) : F \times [0,1] \rightarrow \mathbb{R}$ defined by $a(u,t,v) = \langle A_t(u), v\rangle$ is continuous.

Note that for fixed $t \in [0,1]$, A_t is not necessarily an admissible map of class $(S_+)_L$ whenever A_t, $0 \le t \le 1$, is an admissible homotopy of class $(S_+)_L$. This is due to the fact that in the definition of condition $(S_+)_{0,L}^{(t)}$, the sequence (F_j) of subspaces of L is fixed. In case A_t, $0 \le t \le 1$, is an admissible homotopy and $A_0, A_1 \in (S_+)_L$, the mappings A_0 and A_1 are called "homotopic" [10]. Essentially the following result is proved in [10].

Theorem 4.1. *Let A_t, $0 \le t \le 1$, be an admissible homotopy of class $(S_+)_L^{(t)}$ and $G \subset X$ an open bounded set such that $A_t(u) \ne 0$ for all $u \in \partial G \cap D(A_t)$, $0 \le t \le 1$. If A_0 and A_1 are admissible maps of class $(S_+)_L$, then*

$$\deg(A_0, G, 0) = \deg(A_1, G, 0). \tag{4.9}$$

(d) (Normalizing) For the duality mapping $J : X \rightarrow X^*$ we have

$$\deg(J, G, h) = \begin{cases} +1 & \text{if } h \in J(G), \\ 0 & \text{if } h \notin J(\overline{G}), \end{cases} \tag{4.10}$$

where $G \subset X$ is any open bounded set.

The proof of (d) is based on the use of affine homotopy between the approximant J_j and the identity map in the finite dimensional space F_j.

5. On locally bounded operators

As we pointed out in Section 3, in order to obtain new results, it is not relevant to consider bounded mappings of class $(S_+)_{D(A)}$. However, assume that A is locally bounded on $D(A)$, that is, each $u \in D(A)$ has a neighborhood $B_\epsilon(u)$ such that $A(B_\epsilon(u) \cap D(A))$ is bounded. The following observation is not hard to see.

LEMMA 5.1. *Let $A : D(A) \subset X \to X^*$ be a mapping of class $(S_+)_{D(A)}$ which is locally bounded on $D(A)$. Then A satisfies condition (A2), thus being admissible.*

Hence, the degree theory for locally bounded maps of class $(S_+)_{D(A)}$ is a special case of the degree theory given in Section 4. However, it is interesting to notice that in this case there is another more simple way to construct the degree. Moreover, the degree also turns out to be unique. Indeed, assume that $A : D(A) \subset X \to X^*$ is a locally bounded map of class $(S_+)_{D(A)}$. Let $G \subset X$ be a bounded open set such that $A(u) \neq 0$ for all $u \in \partial G \cap D(A)$. It is easy to prove the following lemma.

LEMMA 5.2. *The set $K := A^{-1}(0) \cap G$ is compact and there exists an open set $G_0 \subset G$ such that*

(a) $K \subset G_0 \subset D(A)$,
(b) *the restriction of A to $\overline{G}_0$ is a bounded demicontinuous map of class (S_+).*

Hence the (S_+)-degree $d_{S+}(A, G_0, 0)$ is well defined and by the additivity its value is independent of the choice of the set G_0 satisfying conditions (a) and (b) of Lemma 5.2. Thus, it is relevant to define a degree function by setting

$$\deg(A, G, 0) = d_{S_+}\big(A, G_0, 0\big). \tag{5.1}$$

This new degree function has the properties of a classical topological degree and is unique by the uniqueness of d_{S_+}. Especially, it coincides with the degree function constructed in Section 4 whenever both are defined.

6. Applications

Let $A : D(A) \subset X \to X^*$ be an admissible map of class $(S_+)_L$. There exists a rich variety of existence results for mappings of class (S_+) (cf. [7, 14]). However, the situation here is more difficult due to the lack of admissible homotopies. For instance, affine homotopies of the form $(1-t)A_0 + tA_1$ between mappings A_0 and A_1 of class $(S_+)_L$ cannot be used in general. We start with a result which provides a sufficient condition for the degree of an isolated zero, that is, the index, to be nonzero.

THEOREM 6.1. *Let $A : D(A) \subset X \to X^*$ be an admissible mapping of class $(S_+)_{0,L}$ such that $A(0) = 0$. Assume that there exists $r > 0$ such that $A(u) \neq 0$ for all $u \in D(A)$, $\|u\| = r$, and*

$$\langle A(u), u\rangle \geq 0 \quad \forall u \in D(A). \tag{6.1}$$

Then $\deg(A, B_r(0), 0) = +1$.

Proof. Let us consider homotopy

$$A_t = (1-t)A + tJ, \quad 0 \leq t \leq 1. \tag{6.2}$$

Since $\langle A_t(u), u\rangle \geq t\|u\|^2$ for all $u \in D(A)$, the condition $A_t(u) \neq 0$ holds for all $u \in D(A)$, $\|u\| = r$, $0 \leq t \leq 1$. Hence, it is sufficient to show that $A_t = (1-t)A + tJ$, $0 \leq t \leq 1$, defines an admissible homotopy in the sense of Section 4. Indeed, the continuity condition (H) clearly holds and assume that for some sequence (F_j) satisfying (3.2)

$$\begin{gathered}(u_k) \subset L(F_j), \quad u_k \rightharpoonup u_0, \quad t_k \longrightarrow t_0, \quad \limsup \langle A_{t_k}(u_k), u_k \rangle \le 0, \\ \lim \langle A_{t_k}(u_k), v \rangle = 0 \quad \forall v \in L(F_j).\end{gathered} \tag{6.3}$$

Now $\langle A_{t_k}(u_k), u_k \rangle \ge t_k \|u_k\|^2$ and if $t_0 > 0$, we have $u_k \to 0$ completing the proof. If $t_0 = 0$, then $\limsup \langle A(u_k), u_k \rangle \le 0$ and $\lim \langle A(u_k), v \rangle = 0$ for all $v \in L(F_j)$. Hence, $u_k \to u_0 \in D(A)$ and $A(u_0) = A_0(u_0) = 0$. □

As an application of the previous result we have the following three theorems.

THEOREM 6.2. *Let $A : D(A) \subset X \to X^*$ be an admissible mapping of class $(S_+)_L$ such that $A(0) = 0$. Assume that there exists $r > 0$ such that $A(u) \ne 0$ for all $u \in D(A)$, $\|u\| = r$, and*

$$\langle A(u), u \rangle \ge 0 \quad \forall u \in D(A). \tag{6.4}$$

Let $N : X \to X^*$ be a bounded demicontinuous quasimonotone map such that

$$\frac{\|A(u)\|}{\|N(u)\|} \ge 1 \quad \forall u \in D(A), \|u\| = r. \tag{6.5}$$

Then the equation $A(u) + N(u) = 0$ admits at least one solution $u \in D(A) \cap \overline{B}_r(0)$.

Proof. If $A(u) + tN(u) = 0$ for some $u \in D(A)$, $\|u\| = r$, and $t \in]0, 1[$, then $\|A(u)\| < \|N(u)\|$, which is a contradiction. Hence, the proof is complete as soon as we have proved that $A_t(u) = A(u) + tN(u)$, $0 \le t \le 1$, defines an admissible homotopy. Assume that

$$\begin{gathered}(u_k) \subset L(F_j), \quad u_k \rightharpoonup u_0, \quad t_k \to t_0, \quad \limsup \langle A_{t_k}(u_k), u_k \rangle \le 0, \\ \lim \langle A_{t_k}(u_k), v \rangle = 0 \quad \forall v \in L(F_j).\end{gathered} \tag{6.6}$$

Taking a subsequence, if necessary, we can assume that $N(u_k) \rightharpoonup w \in X^*$. Thus $\limsup \langle A(u_k) + t_0 w, u_k \rangle \le -t_0 \liminf \langle N(u_k), u_k - u_0 \rangle \le 0$ and $\lim \langle A(u_k) + t_0 w, v \rangle = 0$ for all $v \in L(F_j)$ implying that $u_k \to u_0 \in D(A)$ and $A(u_0) + t_0 w = 0$. Consequently, $w = N(u_0)$ and thus $A_{t_0}(u_0) = 0$. By Theorem 6.1 we obtain

$$\deg(A + N, B_r(0), 0) = \deg(A, B_r(0), 0) = +1, \tag{6.7}$$

which proves the assertion. □

THEOREM 6.3. *Let $A : D(A) \subset X \to X^*$ be an admissible mapping of class $(S_+)_L$ such that $A(0) = 0$. Assume that there exists $r > 0$ such that $A(u) \ne 0$ for all $u \in D(A)$, $\|u\| \ge r$, and*

$$\langle A(u), u \rangle \ge 0 \quad \forall u \in D(A). \tag{6.8}$$

Let $N : X \to X^*$ be a bounded demicontinuous quasimonotone map such that

$$\langle N(u), u \rangle > 0 \quad \forall u \in D(A), \|u\| \ge r. \tag{6.9}$$

Then the equation $A(u)+N(u)=0$ *admits at least one solution* $u\in D(A)\cap B_r(0)$.

Proof. By Theorem 6.2, we can use the admissible homotopy $A+tN$, $0\le t\le 1$. It is clear that $A(u)+tN(u)\neq 0$ for all $u\in D(A)$, $\|u\|\ge r$, and $0\le t\le 1$. Hence, the assertion follows from Theorem 6.1. □

THEOREM 6.4. *Let* $A: D(A)\subset X \longrightarrow X^*$ *be an admissible mapping of class* $(S_+)_L$ *such that* $A(0)=0$. *Assume that there exists* $r>0$ *such that* $A(u)\neq 0$ *for all* $u\in D(A)$, $\|u\|\ge r$, *and*

$$\langle A(u),u\rangle \ge 0 \quad \forall u\in D(A). \tag{6.10}$$

Let $N: X\longrightarrow X^*$ *be a bounded demicontinuous quasimonotone map such that*

$$\frac{\langle N(u),u\rangle}{\|u\|} \longrightarrow \infty \quad \text{as } u\in D(A),\ \|u\| \longrightarrow \infty. \tag{6.11}$$

Then the equation $A(u)+N(u)=h$ *admits at least one solution* $u\in D(A)$ *for each* $h\in X^*$.

Proof. It is not hard to see that considering $N-h$ instead of N, there exists $r'\ge r$ such that the conditions of Theorem 6.3 hold. □

We close this section by giving a generalization of the Borsuk's theorem.

THEOREM 6.5. *Assume that* $A: D(A)\subset X\longrightarrow X^*$ *is an admissible map of class* $(S_+)_{0,L}$ *such that* $D(A)$ *is symmetric with respect to the origin and there exists* $R>0$ *such that* $A(-u)=-A(u)$ *for all* $u\in D(A)$, $\|u\|=R$. *Then there exists a solution for the equation* $A(u)=0$ *on* $\overline{B}_R(0)$. *Moreover,* $\deg(A,B_R(0),0)$ *is an odd number whenever defined.*

Proof. Since the approximant A_j is odd on $\partial B_R(0)\cap F_j$ for all $j\in\mathbb{Z}_+$, the assertion follows from the Borsuk's theorem in finite dimensional space (see [12]) and the definition of $\deg(A,B_R(0),0)$. □

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