ON MODULI OF k-CONVEXITY

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We establish the continuity of some moduli of k-convexity. Let X be a Banach space. We denote by X^* the dual space of X and by B_X the unit ball of X. Several moduli of convexity for the norm of X have been defined; the last two definitions in the following are valid for spaces having dimension $\geq k$:

$$\delta_X(\epsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : x, y \in B_X, \|x-y\| \geq \epsilon\right\} \quad \text{(see [2])},$$
$$\delta_X^{(k)}(\epsilon) = \inf\left\{1 - \frac{\|x_1+\cdots+x_{k+1}\|}{k+1} : x_1, \dots, x_{k+1} \in B_X, A\big(x_1, \dots, x_{k+1}\big) \geq \epsilon\right\} \quad \text{(see [10])},$$
$$\Delta_X^{(k)}(\epsilon) = \inf_{\|x\|=1} \inf_{\substack{Y\subset X\\ \dim(Y)=k}} \sup_{\substack{\|y\|=1\\ y\in Y}} \big\{\|x+\epsilon y\| - 1\big\} \quad \text{(see [9])}, \tag{1}$$

where

$$A\big(x_1, \dots, x_{k+1}\big) = \frac{1}{k!} \sup\left\{\begin{vmatrix} 1 & \cdots & 1 \\ f_1\big(x_1\big) & \cdots & f_1\big(x_{k+1}\big) \\ \vdots & \cdots & \vdots \\ f_k\big(x_1\big) & \cdots & f_k\big(x_{k+1}\big) \end{vmatrix} : f_1, \dots, f_k \in B_{X^*}\right\}. \tag{2}$$

Evidently, by subtracting the first column from the other columns, the determinant can be replaced by

$$\begin{vmatrix} f_1\big(x_2 - x_1\big) & \cdots & f_1\big(x_{k+1} - x_1\big) \\ \vdots & \cdots & \vdots \\ f_k\big(x_2 - x_1\big) & \cdots & f_k\big(x_{k+1} - x_1\big) \end{vmatrix}. \tag{3}$$

Also $A(x_1, \dots, x_{k+1})$ can be thought of as the "volume" of the convex hull of $x_1, \dots,$ x_{k+1} since that is the case in Euclidean spaces.

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X is called uniformly convex if $\delta_X(\epsilon) > 0$ for $\epsilon > 0$ and k-uniformly convex if $\delta_X^{(k)}(\epsilon) > 0$ for $\epsilon > 0$. Note that $\delta_X(\epsilon) = \delta_X^{(1)}(\epsilon)$; so 1-uniform convexity coincides with uniform convexity. Lin [8] proved that $\Delta_X^{(k)}(\epsilon) > 0$ for $\epsilon > 0$ is equivalent to k-uniform convexity. Gurariĭ [5] proved that $\delta_X(\epsilon)$ is continuous on $[0,2)$ and there exist spaces of which $\delta_X(\epsilon) = 0$ for $0 \le \epsilon < 2$ and $\delta_X(2) = 1$. The continuity problem of $\delta_X^{(k)}$ was mentioned in Kirk [6]. Let $\mu_X^{(k)} = \sup\{A(x_1,\dots,x_{k+1}) : x_1,\dots,x_{k+1} \in B_X\}$. Note that $\mu_X^{(1)} = 2$. In this paper, we prove that $\delta_X^{(k)}(\epsilon)$ is continuous on $\big[0,\mu_X^{(k)}\big)$. It is quite evident that $\Delta_X^{(k)}(\epsilon)$ satisfy the Lipschitz condition with constant 1.

Definition 1. Let $k \ge 1$ and $0 \le a < b \le \infty$. A function $f(\epsilon)$ on (a,b) is called *k-convex* if

$$f\left(\left(\lambda\epsilon_2^{1/k} + (1-\lambda)\epsilon_1^{1/k}\right)^k\right) \le \lambda f\big(\epsilon_2\big) + (1-\lambda) f\big(\epsilon_1\big) \tag{4}$$

for every $\epsilon_1, \epsilon_2 \in (a,b)$, $0 \le \lambda \le 1$.

Obviously 1-convexity is simply the ordinary convexity.

Lemma 2. *Let $0 \le a < b \le \infty$ and let f be a nondecreasing k-convex function on (a,b) with $M = \sup_{a<x<y<b}(f(y)-f(x)) < \infty$. Let $\epsilon_1 < \epsilon_2$, $\epsilon_1, \epsilon_2 \in (a,b)$. Then*

$$\frac{f(c)-f\big(\epsilon_1\big)}{c-\epsilon_1} \le \frac{M}{k\big(\epsilon_2^{1/k}-\epsilon_1^{1/k}\big)\epsilon_1^{1-1/k}} \tag{5}$$

for every $\epsilon_1 < c < \epsilon_2$.

Proof. Let $z(x)$, $\epsilon_1 \le x \le \epsilon_2$ be the function whose graph is defined by

$$\begin{aligned} x &= \left(\lambda\epsilon_2^{1/k} + (1-\lambda)\epsilon_1^{1/k}\right)^k \\ y &= \lambda f\big(\epsilon_2\big) + (1-\lambda) f\big(\epsilon_1\big) \end{aligned} \quad 0 \le \lambda \le 1. \tag{6}$$

By direct computations, we have

$$z'(x) = \frac{f\big(\epsilon_2\big)-f\big(\epsilon_1\big)}{k\big(\epsilon_2^{1/k}-\epsilon_1^{1/k}\big)\big(\lambda\epsilon_2^{1/k}+(1-\lambda)\epsilon_1^{1/k}\big)^{k-1}} \le \frac{M}{k\big(\epsilon_2^{1/k}-\epsilon_1^{1/k}\big)\epsilon_1^{1-1/k}}. \tag{7}$$

If $\epsilon_1 < c < \epsilon_2$, then by the k-convexity of f and the mean-value theorem,

$$\frac{f(c)-f\big(\epsilon_1\big)}{c-\epsilon_1} \le \frac{z(c)-z\big(\epsilon_1\big)}{c-\epsilon_1} = z'(\psi) \le \frac{M}{k\big(\epsilon_2^{1/k}-\epsilon_1^{1/k}\big)\epsilon_1^{1-1/k}}. \tag{8}$$

□

The inequality in the following lemma is a consequence of a more general result proved in Bernal-Sullivan [1].

LEMMA 3. *Let X be a Banach space and $x_1, \ldots, x_{k+1} \in X$. Then*

$$A\big(x_1, \ldots, x_{k+1}\big) \le \frac{1}{k!} k^{k/2} \big\|x_2 - x_1\big\| \cdots \big\|x_{k+1} - x_1\big\|. \tag{9}$$

Proof. Hadamard inequality says that if $r_1, r_2, \ldots, r_k$ are the rows (or columns) of a $k \times k$ matrix, then

$$\det\big(r_1, r_2, \ldots, r_k\big) \le \big\|r_1\big\|_2 \big\|r_2\big\|_2 \cdots \big\|r_k\big\|_2. \tag{10}$$

Here $\|\cdot\|_2$ denotes the Euclidean norm in $\mathbb{R}^k$. Since the Euclidean norm of the jth column of the determinant in (3) is $\le k^{1/2}\|x_{j-1} - x_1\|$, the inequality follows. □

The inequality in the next theorem for the case $k = 1$ improves the one obtained in [5]. The general idea is similar to that in Goebel [3]. However, the reader should be aware that the assertion of Lemma 1 in that paper (that $\delta(\epsilon)$ is convex) is incorrect; a counterexample can be found in [7] or [4].

THEOREM 4. *Let X be a Banach space. Then*

$$\frac{\delta_X^{(k)}(c) - \delta_X^{(k)}\big(\epsilon_1\big)}{c - \epsilon_1} \le \frac{1}{k\big(\epsilon_2^{1/k} - \epsilon_1^{1/k}\big)\epsilon_1^{1-1/k}} \tag{11}$$

for every $0 < \epsilon_1 < c < \epsilon_2 < \mu_X^{(k)}$.

Proof. For simplicity, in the following we will consider $k = 2$ and will indicate how to generalize to general k. Note that if $A(x_1, x_2, x_3) > 0$, then $x_2 - x_1$ and $x_3 - x_1$ are linearly independent.

For unit vectors u, u_{21}, u_{31}, and u_{32} in X, with $\{u_{21}, u_{31}\}$ linearly independent, consider the set

$$\begin{aligned} N\big(u, u_{21}, u_{31}, u_{32}; \epsilon\big) = \Big\{ &\big(x_1, x_2, x_3\big) \in X^3 : x_1 + x_2 + x_3 = \lambda u, x_2 - x_1 = \lambda_{21} u_{21}, \\ &x_3 - x_1 = \lambda_{31} u_{31}, x_3 - x_2 = \lambda_{32} u_{32} \\ &\text{for some } \lambda, \lambda_{21}, \lambda_{31}, \lambda_{32} \ge 0 \text{ and } A\big(x_1, x_2, x_3\big) \ge \epsilon \Big\}, \end{aligned} \tag{12}$$

and define

$$\delta\big(u, u_{21}, u_{31}, u_{32}; \epsilon\big) = \inf\left\{1 - \frac{\big\|x_1 + x_2 + x_3\big\|}{3} : \big(x_1, x_2, x_3\big) \in N\big(u, u_{21}, u_{31}, u_{32}; \epsilon\big)\right\}. \tag{13}$$

Obviously, $\delta(u, u_{21}, u_{31}, u_{32}; \epsilon)$ is nondecreasing and has values in $[0, 1]$.

If $(x_1, x_2, x_3) \in N(u, u_{21}, u_{31}, u_{32}; \epsilon_1)$, $(y_1, y_2, y_3) \in N(u, u_{21}, u_{31}, u_{32}; \epsilon_2)$, and

$$\begin{aligned} &x_1 + x_2 + x_3 = \lambda u, \quad x_2 - x_1 = \lambda_{21} u_{21}, \quad x_3 - x_1 = \lambda_{31} u_{31}, \quad x_3 - x_2 = \lambda_{32} u_{32}, \\ &y_1 + y_2 + y_3 = \alpha u, \quad y_2 - y_1 = \alpha_{21} u_{21}, \quad y_3 - y_1 = \alpha_{31} u_{31}, \quad y_3 - y_2 = \alpha_{32} u_{32} \end{aligned} \tag{14}$$

for some $\lambda, \lambda_{ij}, \alpha, \alpha_{ij} \geq 0$, then by linear independence of $\{u_{21}, u_{31}\}$, there exists $c \geq 0$ such that

$$\alpha_{21} = c\lambda_{21}, \quad \alpha_{31} = c\lambda_{31}, \quad \alpha_{32} = c\lambda_{32}. \tag{15}$$

Indeed, $\lambda_{32}u_{32} = x_3 - x_2 = (x_3 - x_1) - (x_2 - x_1) = \lambda_{31}u_{31} - \lambda_{21}u_{21}$ and $\alpha_{32}u_{32} = \alpha_{31}u_{31} - \alpha_{21}u_{21}$ imply

$$\left(\alpha_{32}\lambda_{31} - \lambda_{32}\alpha_{31}\right)u_{31} - \left(\alpha_{32}\lambda_{21} - \lambda_{32}\alpha_{21}\right)u_{21} = 0 \tag{16}$$

from which it follows that $\alpha_{31}/\lambda_{31} = \alpha_{32}/\lambda_{32} = \alpha_{21}/\lambda_{21}$.

Let

$$C\left(u_{21}, u_{31}\right) = \sup\left\{\begin{vmatrix} f_1\left(u_{21}\right) & f_1\left(u_{31}\right) \\ f_2\left(u_{21}\right) & f_2\left(u_{31}\right) \end{vmatrix} : f_1, f_2 \in B_{X^*}\right\}. \tag{17}$$

Then $A(x_1, x_2, x_3) = \lambda_{21}\lambda_{31}C(u_{21}, u_{31}) \geq \epsilon_1$ and $A(y_1, y_2, y_3) = c^2\lambda_{21}\lambda_{31}C(u_{21}, u_{31}) \geq \epsilon_2$.

For $0 \leq \zeta \leq 1$, let $z_i = \zeta x_i + (1-\zeta)y_i$, $i = 1, 2, 3$. Then

$$\begin{aligned} z_2 - z_1 &= \left(\zeta\lambda_{21} + (1-\zeta)c\lambda_{21}\right)u_{21} = \left(\zeta + (1-\zeta)c\right)\lambda_{21}u_{21}, \\ z_3 - z_1 &= \left(\zeta + (1-\zeta)c\right)\lambda_{31}u_{31}, \\ z_3 - z_2 &= \left(\zeta + (1-\zeta)c\right)\lambda_{32}u_{32}, \\ z_1 + z_2 + z_3 &= \left(\zeta\lambda + (1-\zeta)\alpha\right)u, \\ A\left(z_1, z_2, z_3\right) &= \left(\zeta + (1-\zeta)c\right)^2\lambda_{21}\lambda_{31}C\left(u_{21}, u_{31}\right) \geq \left(\zeta\epsilon_1^{1/2} + (1-\zeta)\epsilon_2^{1/2}\right)^2, \end{aligned} \tag{18}$$

$$\begin{aligned} 1 - \frac{\|z_1 + z_2 + z_3\|}{3} &= 1 - \frac{\|\zeta\left(x_1 + x_2 + x_3\right) + (1-\zeta)\left(y_1 + y_2 + y_3\right)\|}{3} \\ &= 1 - \frac{\|\zeta\lambda u + (1-\zeta)\alpha u\|}{3} \\ &= 1 - \frac{\zeta\lambda + (1-\zeta)\alpha}{3} \\ &= \zeta\left(1 - \frac{\lambda}{3}\right) + (1-\zeta)\left(1 - \frac{\alpha}{3}\right) \\ &= \zeta\left(1 - \frac{\|x_1 + x_2 + x_3\|}{3}\right) + (1-\zeta)\left(1 - \frac{\|y_1 + y_2 + y_3\|}{3}\right). \end{aligned} \tag{19}$$

Hence

$$\begin{aligned} &\delta\left(u, u_{21}, u_{31}, u_{32}; \left(\zeta\epsilon_1^{1/2} + (1-\zeta)\epsilon_2^{1/2}\right)^2\right) \\ &\qquad \leq \zeta\delta\left(u, u_{21}, u_{31}, u_{32}; \epsilon_1\right) + (1-\zeta)\delta\left(u, u_{21}, u_{31}, u_{32}; \epsilon_2\right). \end{aligned} \tag{20}$$

Since

$$\begin{aligned} \delta_X^{(2)}(\epsilon) = \inf\{\delta\left(u, u_{21}, u_{31}, u_{32}; \epsilon\right) : \|u\| = \|u_{21}\| = \|u_{31}\| = \|u_{32}\| = 1, \\ \{u_{21}, u_{31}\} \text{ linearly independent}\}, \end{aligned} \tag{21}$$

and the inequality in Lemma 2 is preserved under passing to infimum, inequality (11) for $k = 2$ follows.

For general k, we have $\binom{k+1}{2} + 1$ unit vectors $u, u_{21}, \dots$ and the proof is similar to the one above. □

COROLLARY 5. *Let X be a Banach space. Then $\delta_X^{(k)}(\epsilon)$ is continuous on $\left[0, \mu_X^{(k)}\right)$.*

Proof. Take $\|x_1\| = 1$ and $x_2, \dots, x_{k+1}$ in a small ball centered at x_1. Then, by Lemma 3, $A(x_1, \dots, x_{k+1})$ is small. Since $1 - \|x_1 + \dots + x_{k+1}\|/(k+1)$ is close to 0, we see that $\delta_X^{(k)}(\epsilon)$ is continuous at 0.

Continuity of $\delta_X^{(k)}(\epsilon)$ on $\left(0, \mu_X^{(k)}\right)$ follows immediately from the inequality (11). □

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