A RIESZ REPRESENTATION THEOREM FOR CONE-VALUED FUNCTIONS

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Received 29 November 1999

We consider Borel measures on a locally compact Hausdorff space whose values are linear functionals on a locally convex cone. We define integrals for cone-valued functions and verify that continuous linear functionals on certain spaces of continuous cone-valued functions endowed with an inductive limit topology may be represented by such integrals.

1. Introduction

The theory of locally convex cones, as developed in [3], deals with ordered cones that are not necessarily embeddable in vector spaces. A topological structure is introduced using order theoretical concepts. We will review some of the main concepts and globally refer to [3] for details and proofs.

An *ordered cone* is a set $\mathcal{P}$ endowed with an addition and a scalar multiplication for nonnegative real numbers. The addition is associative and commutative, and there is a neutral element $0 \in \mathcal{P}$. For the scalar multiplication the usual associative and distributive properties hold, that is, $\alpha(\beta a) = (\alpha\beta)a$, $(\alpha+\beta)a = \alpha a + \beta a$, $\alpha(a+b) = \alpha a + \alpha b$, $1a = a$ and $0a = 0$ for all $a, b \in \mathcal{P}$ and $\alpha, \beta \geq 0$. The *cancellation law*, stating that $a + c = b + c$ implies $a = b$, however, is not required in general. It holds if and only if the cone $\mathcal{P}$ may be embedded into a real vector space. Also, $\mathcal{P}$ carries a (partial) order, that is, a reflexive transitive relation $\leq$ such that $a \leq b$ implies $a + c \leq b + c$ and $\alpha a \leq \alpha b$ for all $a, b, c \in \mathcal{P}$ and $\alpha \geq 0$. As equality in $\mathcal{P}$ is such a relation, all results about ordered cones apply to cones without order structures as well.

A *linear functional* on a cone $\mathcal{P}$ is a mapping $\mu : \mathcal{P} \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ such that $\mu(a+b) = \mu(a) + \mu(b)$ and $\mu(\alpha a) = \alpha\mu(a)$ for all $a, b \in \mathcal{P}$ and $\alpha \geq 0$. In $\overline{\mathbb{R}}$ we consider the usual algebraic operations, in particular $\alpha + \infty = +\infty$ for all $\alpha \in \overline{\mathbb{R}}$, $\alpha \cdot (+\infty) = +\infty$ for all $\alpha > 0$ and $0 \cdot (+\infty) = 0$. Note that linear functionals can assume only finite values at invertible elements of $\mathcal{P}$.

Abstract and Applied Analysis 4:4 (1999) 209–229
1991 Mathematics Subject Classification: 46A13, 46E40
URL: http://aaa.hindawi.com/volume-4/S1085337599000160.html

A *full locally convex cone* $(\mathcal{P},\mathcal{V})$ is an ordered cone $\mathcal{P}$ that contains an *abstract neighborhood system* $\mathcal{V}$, that is, a subset of positive elements which is directed downward, closed for addition and multiplication by strictly positive scalars. The elements v of $\mathcal{V}$ define *upper*, respectively *lower neighborhoods* for the elements of $\mathcal{P}$ by

$$v(a)=\{b\in\mathcal{P}\mid b\le a+v\},\quad \text{respectively } (a)v=\{b\in\mathcal{P}\mid a\le b+v\}, \tag{1.1}$$

creating the *upper*, respectively *lower topologies* on $\mathcal{P}$. Their common refinement is called the *symmetric topology.* All elements of $\mathcal{P}$ are supposed to be *bounded below*, that is, for every $a\in\mathcal{P}$ and $v\in\mathcal{V}$ we have $0\le a+\lambda v$ for some $\lambda\ge 0$. Finally, a *locally convex cone* $(\mathcal{P},\mathcal{V})$ is a subcone of a full locally convex cone not necessarily containing the abstract neighborhood system $\mathcal{V}$. Every locally convex ordered topological vector space is a locally convex cone in this sense, as it may be canonically embedded into a full locally convex cone (see [3, Example I.2.7]). Endowed with the neighborhood system $\mathcal{W}=\{\varepsilon\in\mathbb{R}\mid\varepsilon>0\}$, $\overline{\mathbb{R}}$ is a full locally convex cone.

The *polar* v° of a neighborhood $v\in\mathcal{V}$ consists of all linear functionals μ on a locally convex cone $(\mathcal{P},\mathcal{V})$ satisfying $\mu(a)\le\mu(b)+1$, whenever $a\le b+v$ for $a,b\in\mathcal{P}$. The union of all polars of neighborhoods forms the *dual cone* $\mathcal{P}^*$ of $\mathcal{P}$. The functionals belonging to $\mathcal{P}^*$ are said to be *(uniformly) continuous*. Continuity requires that μ is monotone, and for a full cone $\mathcal{P}$ it means just that $\mu(v)\le 1$ holds for some $v\in\mathcal{V}$ in addition. We endow $\mathcal{P}^*$ with the topology $w(\mathcal{P}^*,\mathcal{P})$ of pointwise convergence of the elements of $\mathcal{P}$, considered as functions on $\mathcal{P}^*$ with values in $\overline{\mathbb{R}}$ with its usual topology. The polar v° of a neighborhood $v\in\mathcal{V}$ is seen to be $w(\mathcal{P}^*,\mathcal{P})$-compact and convex (see [3, Theorem II.2.4]). Hahn-Banach type extension and separation theorems for locally convex cones were established in [3, 6]. Theorem II.2.9 from [3] (a more general version is Theorem 4.1 from [6]) states that, for a subcone $(\mathcal{Q},\mathcal{V})$ of $(\mathcal{P},\mathcal{V})$ every linear functional in $v^\circ\subset\mathcal{Q}^*$ extends to an element of $v^\circ\subset\mathcal{P}^*$.

While all elements of a locally convex cone are bounded below, they need not to be bounded above. An element $a\in\mathcal{P}$ is called *bounded (above)* (see [3, Section I.2.3]) if for every $v\in\mathcal{V}$ there is $\lambda\ge 0$ such that $a\le\lambda v$. All invertible elements of $\mathcal{P}$ are bounded. Continuous linear functionals in particular take only finite values on bounded elements.

In Section 2, we introduce functional-valued Borel measures on a locally compact Hausdorff space and measurable cone-valued functions. We take advantage of the fact that every locally convex cone, hence every locally convex vector space, may be embedded in a full cone which contains sufficiently many positive elements in order to apply the concepts of measure theory for real-valued functions. In Section 3, we investigate integrals for cone-valued functions. In Section 4, we derive a generalized Riesz representation theorem for spaces of continuous cone-valued functions endowed with an inductive limit topology, our main result.

2. Measures and measurable functions

Throughout the following, let X be a locally compact Hausdorff space, $\mathcal{B}$ the σ-algebra of all Borel subsets of X, and $\mathcal{K}$ the family of all compact subsets of X. Let $(\mathcal{P},\mathcal{V})$ be a locally convex cone, $\mathcal{P}^*$ its dual. Endowed with the pointwise algebraic operations and order the $\mathcal{P}$-valued functions on X form again a cone, denoted by $\mathcal{F}(X,\mathcal{P})$. The *support* of $f \in \mathcal{F}(X,\mathcal{P})$ is the closure $\operatorname{supp}(f)$ in X of the set $\{x \in X \mid f(x) \neq 0\}$. For a positive real-valued function φ on X and $f \in \mathcal{F}(X,\mathcal{P})$, we denote by $\varphi \otimes f \in \mathcal{F}(X,\mathcal{P})$ the mapping

$$x \longmapsto \varphi(x)f(x) : X \longrightarrow \mathcal{P}. \tag{2.1}$$

For an element $a \in \mathcal{P}$ we also use its symbol to denote the constant function $x \mapsto a$, hence $\varphi \otimes a$ for $x \mapsto \varphi(x)a$. Note that even if φ is continuous real-valued, the mapping $\varphi \otimes a \in \mathcal{F}(X,\mathcal{P})$ need not to be continuous with respect to any of our topologies on $\mathcal{P}$ if the element a is not bounded. We denote the subcone of $\mathcal{F}(X,\mathcal{P})$ of those functions, that are continuous with respect to the symmetric topology of $\mathcal{P}$, by $\mathcal{C}(X,\mathcal{P})$.

As usual, χ_E stands for the characteristic function on X of a subset $E \subset X$, and $\mathcal{S}(X,\mathcal{P})$ is the subcone of $\mathcal{F}(X,\mathcal{P})$ of all $\mathcal{P}$-valued step functions on X; that is, functions $h = \sum_{i=1}^{n} \chi_{E_i} \otimes a_i$ with $E_i \in \mathcal{B}$ and $a_i \in \mathcal{P}$. If the sets E_i are pairwise disjoint and their union is X, we call the above the *standard representation* for the step function h.

2.1. Inductive limit neighborhoods for cone-valued functions. We may adjoin infinite elements v_∞ to the neighborhood system $\mathcal{V}$ in the following way: for $v \in \mathcal{V}$ and $a, b \in \mathcal{P}$, we set $a \leq b + v_\infty$ if $a \leq b + \lambda v$ for some $\lambda \geq 0$. Also we use a maximal element ∞ such that $a \leq b + \infty$ for all $a, b \in \mathcal{P}$. We may add and multiply these infinite elements in a canonical way, that is, $\alpha \cdot v_\infty = v_\infty$ for all $\alpha > 0$, $v_\infty + w_\infty = (v+w)_\infty$, etc. We obtain the extended neighborhood system $\mathcal{V}_\infty = \mathcal{V} \cup \{v + w_\infty \mid v, w \in \mathcal{V}\} \cup \{\infty\}$. An *inductive limit neighborhood* for $\mathcal{F}(X,\mathcal{P})$ is a convex set $\mathfrak{v}$ of $\mathcal{V}_\infty$-valued functions on X, such that for every $K \in \mathcal{K}$ there is $v_K \in \mathcal{V}$ and $s \in \mathfrak{v}$ such that $\chi_K \otimes v_K \leq s$. For functions $f, g \in \mathcal{F}(X,\mathcal{P})$ we say that

$$f \leq g + \mathfrak{v} \quad \text{if } f \leq g + s \text{ for some } s \in \mathfrak{v}. \tag{2.2}$$

We denote by $\mathfrak{W}$ the family of all inductive limit neighborhoods for $\mathcal{F}(X,\mathcal{P})$.

2.2. Functional-valued measures. Let $\mathcal{B}_\mathcal{K}$ denote the ring of all relatively compact Borel subsets of X. A $\mathcal{P}^*$*-valued Borel measure* ϑ on X is a set function

$$E \longmapsto \vartheta_E : \mathcal{B}_\mathcal{K} \longrightarrow \mathcal{P}^* \tag{2.3}$$

such that $\vartheta(\emptyset) = 0$ and for each $a \in \mathcal{P}$ the $\overline{\mathbb{R}}$-valued set function ϑ^a, that is,

$$E \longmapsto \vartheta^a(E) = \vartheta_E(a) : \mathcal{B}_\mathcal{K} \longrightarrow \overline{\mathbb{R}} \tag{2.4}$$

is σ-additive on $\mathcal{B}_\mathcal{K}$, namely $\vartheta^a(\cup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \vartheta^a(E_i)$ holds for all $a \in \mathcal{P}$ and disjoint sets $E_i \in \mathcal{B}_\mathcal{K}$ such that $\cup_{i=1}^{\infty} E_i \in \mathcal{B}_\mathcal{K}$. If the neighborhood $v \in \mathcal{V}$ is an element of $\mathcal{P}$, then for every $E \in \mathcal{B}_\mathcal{K}$ it is clear that $\vartheta^v(E) = \vartheta_E(v)$ is the infimum of all constants $0 \leq \rho \leq +\infty$ such that

$$\sum_{i=1}^{n} \vartheta^{a_i}(E_i) \leq \sum_{i=1}^{n} \vartheta^{b_i}(E_i) + \rho \tag{2.5}$$

holds for any choice of disjoint Borel sets $E_i \subset E$ and $a_i, b_i \in \mathcal{P}$ such that $a_i \leq b_i + v$. If, on the other hand, the neighborhood $v \notin \mathcal{P}$, then we may use the above to define $\vartheta^v(E)$ as the infimum of all such constants ρ. With this definition, it is straightforward to check that for fixed v and varying $E \in \mathcal{B}_{\mathcal{K}}$, ϑ^v is also σ-additive on $\mathcal{B}_{\mathcal{K}}$. Moreover, for fixed E and v we may use the above to define an $\overline{\mathbb{R}}$-valued monotone linear functional ϑ_E on the full cone $\widetilde{\mathcal{P}} = \{a + \lambda v \mid a \in \mathcal{P},\ \lambda \geq 0\}$. (The order and the algebraic operations for $\widetilde{\mathcal{P}}$ are canonical.) We set

$$\vartheta_E(a + \lambda v) = \vartheta^{a+\lambda v}(E) = \vartheta^a(E) + \lambda \vartheta^v(E) \tag{2.6}$$

for $a \in \mathcal{P}$ and $\lambda \geq 0$. In a similar way, ϑ_E may be extended to an $\overline{\mathbb{R}}$-valued monotone linear functional on a full cone that contains all neighborhoods of $\mathcal{P}$ (see [3, Chapter I.5]). However, for this linear functional to be continuous, hence an element of the dual of this full cone, we need to require that ϑ is *bounded on* E, namely that $\vartheta_E(v) < +\infty$ for at least one neighborhood $v \in \mathcal{V}$.

In this vein, we say that a $\mathcal{P}^*$-valued Borel measure ϑ is $\mathcal{K}$-*bounded* if it is bounded on all compact subsets, hence also on all relatively compact subsets of X. This requirement for a $\mathcal{P}^*$-valued measure corresponds to Dieudonné's notion of *p-domination* in [1] and to Prolla's of *finite p-semivariation* in [5, Chapter 5.5] for measures with values in the dual of a locally convex vector space.

In what follows, we will always require functional-valued measures to be $\mathcal{K}$-bounded. We may therefore assume, without loss of generality, that the locally convex cone $(\mathcal{P}, \mathcal{V})$ is full, as otherwise the measure ϑ may be extended to a larger full cone. We take advantage of the fact that full cones contain sufficiently many positive elements. For a fixed positive element $0 \leq a \in \mathcal{P}$, for example, ϑ^a may be canonically extended to an $\overline{\mathbb{R}}$-valued Borel measure on X, setting for every Borel set $E \in \mathcal{B}$

$$\vartheta^a(E) = \sup\left\{\vartheta^a(F) \mid F \subset E,\ F \in \mathcal{B}_{\mathcal{K}}\right\}. \tag{2.7}$$

Note, however, that for a fixed nonrelatively compact Borel set E and varying $0 \leq a \in \mathcal{P}$ the above formula defines a monotone linear functional on the positive cone in $\mathcal{P}$, but this functional might not be extendable into an element of $\mathcal{P}^*$.

2.3. R-continuous cone-valued functions. For the remainder of Section 2 and for Section 3, let $(\mathcal{P}, \mathcal{V})$ be a full locally convex cone, and let $\mathcal{P}_+$ denote the subcone of all positive elements of $\mathcal{P}$. Due to the presence of unbounded elements, continuity with respect to the symmetric topology of $\mathcal{P}$ is a rather restrictive requirement for cone-valued functions. We will therefore use the following slightly more generous concept: a positive function $f \in \mathcal{F}(X, \mathcal{P}_+)$ is said to be *relatively continuous* (*r-continuous*, for short) on X if for every $x \in X$, $v \in \mathcal{V}$, and $\gamma > 1$ there is a neighborhood U of x such that

$$f(x) \leq \gamma f(y) + v, \quad f(y) \leq \gamma f(x) + v, \quad \forall y \in U. \tag{2.8}$$

A not necessarily positive function $f \in \mathcal{F}(X, \mathcal{P})$ is r-continuous if for every $v \in \mathcal{V}$ there is a positive real-valued continuous function φ on X such that $f + \varphi \otimes v$ is positive and satisfies the above definition. We denote the cone of all r-continuous functions in $\mathcal{F}(X, \mathcal{P})$ by $\mathcal{C}^r(X, \mathcal{P})$. For a function $f \in \mathcal{C}(X, \mathcal{P})$ and for $v \in \mathcal{V}$ we may choose the continuous real-valued function $\varphi(x) = 1 + \inf\{\rho \geq 0 \mid f(x) + \rho v \geq 0\}$ and realize that $f + \varphi \otimes v$ satisfies our definition. Thus $\mathcal{C}(X, \mathcal{P}) \subset \mathcal{C}^r(X, \mathcal{P})$, and both cones indeed coincide if all elements of $\mathcal{P}$ are bounded. For a positive real-valued continuous function φ and an unbounded element $a \in \mathcal{P}$, the function $\varphi \otimes a \in \mathcal{F}(X, \mathcal{P})$ is generally not continuous, but r-continuous at all points $x \in X$, where $\varphi(x) > 0$. For the latter, assume first that $a \in \mathcal{P}_+$. For $\varphi(x) > 0$, $v \in \mathcal{V}$, and $\gamma > 1$ there is a neighborhood U of x such that $\varphi(y) \leq \gamma\varphi(z)$ for all $y, z \in U$. Then $\varphi \otimes a(y) = \varphi(y)a \leq \gamma\varphi(z)a = \gamma\varphi \otimes a(z)$. For any $a \in \mathcal{P}$ there is $\lambda \geq 0$ such that $b = a + \lambda v \geq 0$. The function $\varphi \otimes b = \varphi \otimes a + (\lambda\varphi) \otimes v$ is r-continuous at x by the above, thus $\varphi \otimes a$ is also r-continuous by our definition.

For an illustration of this, let $\mathcal{P}$ be the cone of all real-valued continuous functions on $\mathbb{R}$ which are uniformly bounded below, endowed with the pointwise algebraic operations and order. With the neighborhood system $\mathcal{V}$ consisting of all strictly positive constant functions in $\mathcal{P}$, then $(\mathcal{P}, \mathcal{V})$ becomes a full locally convex cone. The function $f \in \mathcal{P}$ such that $f(t) = t^2$ for $t \in \mathbb{R}$ is obviously not bounded in $\mathcal{P}$. If we choose $X = [0, 1)$ and the real-valued function $\varphi(x) = x$ on X, then the $\mathcal{P}$-valued function $\varphi \otimes f$ is r-continuous on $(0, 1)$ but not at $x = 0$. Continuity with respect to any of the given locally convex topologies on $\mathcal{P}$ fails at all points of X.

To further illustrate some implications of r-continuity, let us consider the following: for a fixed neighborhood $v \in V$, and $a, b \in \mathcal{P}$ we say that b is v-bounded relative to a and write $b \in B_v(a)$ if there are $\lambda, \rho \geq 0$ such that $b \leq \lambda a + \rho v$. Thus, $0 \in \mathcal{B}_v(a)$ for every $a \in \mathcal{P}$, and $\bigcap_{v \in \mathcal{V}} B_v(0)$ consists of all bounded elements of $\mathcal{P}$. It is straightforward to verify that $B_v(a)$ is a subcone and a face in $\mathcal{P}$, closed with respect to the lower topology of $\mathcal{P}$. Moreover, $B_v(a)$ consists exactly of those elements $b \in \mathcal{P}$ such that $\mu(b) < +\infty$ whenever $\mu(a) < +\infty$ for a linear functional $\mu \in v^\circ$. Likewise, the set $(a)B_v = \{b \in \mathcal{P} \mid a \in B_v(b)\}$ is closed with respect to the upper topology, and $\alpha a + b \in (a)B_v$ whenever $a \in (a)B_v$, $b \in \mathcal{P}$ and $\alpha > 0$. Moreover, $(a)B_v$ consists exactly of those elements $b \in \mathcal{P}$ such that $\mu(b) = +\infty$ whenever $\mu(a) = +\infty$ for some $\mu \in v^\circ$. As an immediate consequence of our definition we realize that for an r-continuous function the inverse images of all sets $B_v(a)$ and $(a)B_v$ are both open and closed in X.

2.4. Measurable cone-valued functions. Measurability for vector-valued functions has been introduced in various places (cf. Dunford et al. [2, Definition III.2.10]). A suitable adaptation for cone-valued functions needs to consider the presence of unbounded elements in $\mathcal{P}$ and the absence of negatives. We will therefore define measurability only for positive-valued functions. As $\mathcal{P}$ is a full cone, hence contains sufficiently many positive elements, this concept will prove adequate for our upcoming integration theory.

For a sequence of functions $(f_n)_{n \in \mathbb{N}}$ and f in $\mathcal{F}(X, \mathcal{P})$ and a subset E of X, we will write $f_n \nearrow f$ *on* E if $(f_n)_{n \in \mathbb{N}}$ converges to f pointwise on E with respect to the lower topology of $\mathcal{P}$, that is, if for every $x \in E$ and $v \in \mathcal{V}$ there is $n_0 \in \mathbb{N}$ such that $f(x) \leq f_n(x) + v$ for all $n \geq n_0$.

A function $f \in \mathcal{F}(X, \mathcal{P}_+)$ is said to be *(Borel) measurable* if for every $K \in \mathcal{K}$, $v \in \mathcal{V}$, and $\gamma > 1$ there is a sequence $(h_n)_{n \in \mathbb{N}}$ of step functions in $\mathcal{S}(X, \mathcal{P}_+)$ such that

$$h_n \nearrow f \quad \text{on } K, \qquad h_n \leq \gamma f + v. \tag{2.9}$$

For an element $a \in \mathcal{P}_+$ and a neighborhood $v \in \mathcal{V}$ we denote

$$\Downarrow a = \big\{ b \in \mathcal{P} \mid b \leq \gamma a + \varepsilon v \ \forall \gamma > 1 \text{ and } \varepsilon > 0 \big\}. \tag{2.10}$$

This set is decreasing and closed with respect to the lower topology of $\mathcal{P}$ and coincides with the set of all $b \in \mathcal{P}$ such that $\mu(b) \leq \mu(a)$ for all $\mu \in v^\circ$. The latter follows from a Hahn-Banach type separation theorem (see [6, Corollary 4.6]). Likewise, the set

$$\Uparrow a = \big\{ b \in \mathcal{P} \mid (1/\gamma) a \leq b + \varepsilon v \ \forall \gamma > 1 \text{ and } \varepsilon > 0 \big\} \tag{2.11}$$

is increasing, closed with respect to the upper topology and coincides with the set of all $b \in \mathcal{P}$ such that $\mu(b) \geq \mu(a)$ for all $\mu \in v^\circ$. Moreover, it is straightforward to check that for an r-continuous function $f \in \mathcal{C}^r(X,\mathcal{P})$, the inverse images of sets $\Downarrow a$ and $\Uparrow a$ are closed in X.

Lemma 2.1. *If $f \in \mathcal{F}(X,\mathcal{P}_+)$ is measurable, then $f^{-1}(\Downarrow a) \in \mathcal{B}$ and $f^{-1}(\Uparrow a) \in \mathcal{B}$ for all $a \in \mathcal{P}_+$ and $v \in \mathcal{V}$.*

Proof. Let $a \in \mathcal{P}_+$, $v \in \mathcal{V}$, and $K \in \mathcal{K}$. For $m \in \mathbb{N}$, set $\varepsilon_m = 1/m$ and $\gamma_m = 1 + \varepsilon_m$ and choose a sequence $(h_n^m)_{n\in\mathbb{N}}$ of step functions such that $h_n^m \nearrow f$ on K and $h_n^m \leq \gamma_m f + \varepsilon_m v$. Set

$$F_n^m = \big\{ x \in K \mid h_n^m(x) \leq \gamma_m^2 a + 2\varepsilon_m v \big\} \in \mathcal{B} \tag{2.12}$$

and $F = \bigcap_{m,n\in\mathbb{N}} F_n^m \in \mathcal{B}$. As $h_n^m(x) \leq \gamma_m f(x) + \varepsilon_m v$ for all $x \in K$, we realize that $f(x) \in \Downarrow a$ implies that $x \in F_n^m$ for all $m,n \in \mathbb{N}$, hence $x \in F$. On the other hand, for $x \in F$ and every $m \in \mathbb{N}$ there is $n \in \mathbb{N}$ such that $f(x) \leq h_n^m(x) + \varepsilon_m v \leq \gamma_m^2 a + 3\varepsilon_m v$, hence $f(x) \in \Downarrow a$. Thus $f^{-1}(\Downarrow a) \cap K = F \in \mathcal{B}$. This holds for all $K \in \mathcal{K}$, hence $f^{-1}(\Downarrow a) \in \mathcal{B}$ (see [7, Lemma 13.9]). Similarly, for the second statement set

$$E_n^m = \left\{ x \in K \mid \left(\frac{1}{\gamma_m}\right) a \leq h_n^m(x) + 2\varepsilon_m v \right\} \in \mathcal{B} \tag{2.13}$$

and $E = \bigcap_{m\in\mathbb{N}} (\bigcup_{n\in\mathbb{N}} E_n^m)$. As for every $x \in K$ and $m \in \mathbb{N}$ there is $n \in \mathbb{N}$ such that $f(x) \leq h_n^m(x) + \varepsilon_m v$, clearly $f(x) \in \Uparrow a$ implies that $x \in E$. On the other hand, for $x \in E$ and every $m \in \mathbb{N}$ we have $(1/\gamma_m) a \leq h_n^m(x) + 2\varepsilon_m v \leq \gamma_m f(x) + 3\varepsilon_m v$ for some $n \in \mathbb{N}$, hence $f(x) \in \Uparrow a$. This shows that $f^{-1}(\Uparrow a) \cap K \in \mathcal{B}$. □

Lemma 2.2. *A function $f \in \mathcal{F}(X,\overline{\mathbb{R}}_+)$ is measurable if and only if it is Borel measurable in the usual sense.*

Proof. Let $f \in \mathcal{F}(X,\overline{\mathbb{R}}_+)$. If f is Borel measurable in the usual sense, then it is the pointwise limit of an increasing sequence $(\psi_n)_{n\in\mathbb{N}}$ of real-valued step functions. For any $K \in \mathcal{K}$, the $\overline{\mathbb{R}}_+$-valued step functions $h_n(x) = \psi_n(x)$ if $f(x) < +\infty$ and $h_n(x) = +\infty$ else, then fulfill our criterion. If, on the other hand, f is measurable in our sense, then $f^{-1}(\Downarrow a) \in \mathcal{B}$ and $f^{-1}(\Uparrow a) \in \mathcal{B}$ for all $a \in \overline{\mathbb{R}}_+$ by Lemma 2.1. But $\Downarrow a = (-\infty, a]$ and $\Uparrow a = [a, +\infty]$. Thus f is Borel measurable in the usual sense. □

LEMMA 2.3. *The measurable functions form a subcone of $\mathcal{F}(X,\mathcal{P}_+)$, closed with respect to the symmetric topology of compact convergence.*

Proof. The first statement is obvious. For the second, let $f \in \mathcal{F}(X,\mathcal{P}_+)$ be an accumulation point of the subcone of measurable functions, let $K \in \mathcal{K}$, $v \in \mathcal{V}$, and $1 < \gamma \le 2$. There is a measurable function $g \in \mathcal{F}(X,\mathcal{P}_+)$ such that $g(x) \le f(x)+v$ and $f(x) \le g(x)+v$ for all $x \in K$. We find a sequence $(h_n)_{n\in\mathbb{N}}$ in $\mathcal{S}(X,\mathcal{P})$ such that $h_n \nearrow g$ on K and $h_n \le \gamma g + v$. We may assume that the step functions h_n are supported by K and set $l_n = h_n + v \in \mathcal{S}(X,\mathcal{P})$. Then $l_n \nearrow f$ on K and $l_n \le \gamma(\chi_K \otimes g) + 2v \le \gamma(\chi_K \otimes f) + 4v$, demonstrating that f is indeed measurable. □

THEOREM 2.4. *For a function $f \in \mathcal{F}(X,\mathcal{P}_+)$, the following statements are equivalent:*

(a) *f is measurable.*

(b) *For every $K \in \mathcal{K}$, $v \in \mathcal{V}$, and $\gamma > 1$ there is a sequence $(f_n)_{n\in\mathbb{N}}$ of measurable functions in $\mathcal{F}(X,\mathcal{P}_+)$ such that $f_n \nearrow f$ on K and $f_n \le \gamma f + v$.*

(c) *For every $K \in \mathcal{K}$, $v \in \mathcal{V}$, and $\gamma > 1$ there are $E_i \in \mathcal{B}_\mathcal{K}$ such that $\cup_{i\in\mathbb{N}} E_i = K$ and $f(x) \le \gamma f(y) + v$ whenever $x, y \in E_i$ for some $i \in \mathbb{N}$.*

Proof. (b) follows from (a) trivially with $f_n = f$. Now suppose that (b) holds and let $K \in \mathcal{K}$. For $v \in \mathcal{V}$ and $1 < \gamma \le 2$, there is a sequence $(f_n)_{n\in\mathbb{N}}$ of measurable functions such that $f_n \nearrow f$ on K and $f_n \le \gamma f + v$. By our definition of measurability, for every f_n there is a sequence $(h_n^m)_{m\in\mathbb{N}}$ of step functions such that $h_n^m \nearrow f_n$ on K and $h_n^m \le \gamma f_n + v$. We may assume that $\operatorname{supp}(h_n^m) \subset K$ and represent each of these step functions through finitely many characteristic functions of disjoint Borel sets whose union is K. Altogether, there are only countably many terms $\chi_{F_i} \otimes a_i$ involved in building these step functions. Consider the corresponding family $\{(F_i, a_i) \mid i \in \mathbb{N}\}$. Thus

$$a_i = h_n^m(x) \le \gamma f_n(x) + v \le \gamma\big(\gamma f(x) + v\big) + v \le \gamma^2 f(x) + 3v \tag{2.14}$$

holds for all $x \in F_i$. We set $E_i^n = f_n^{-1}(\Downarrow(\gamma a_i + 5v)) \cap F_i \in \mathcal{B}_\mathcal{K}$ and $E_i = \cap_{n\in\mathbb{N}} E_i^n \in \mathcal{B}_\mathcal{K}$. Then for every $x \in K$ there is $k \in \mathbb{N}$ such that $f(x) \le f_k(x) + v$, and for f_k there is (F_i, a_i) such that $x \in F_i$ and $f_k(x) \le a_i + v$. Thus $f(x) \le a_i + 2v$ and $f_n(x) \le \gamma f(x) + v \le \gamma a_i + 5v$ for all $n \in \mathbb{N}$. This shows $x \in E_i$, hence $\cup_{i\in\mathbb{N}} E_i = K$. Second, for any choice of $x, y \in E_i$ we have $f(x) \le g_n(x) + v$ for some $n \in \mathbb{N}$, hence

$$\begin{aligned} f(x) &\le \gamma\big(\gamma a_i + 5v\big) + v \le \gamma^2 a_i + 11v \\ &\le \gamma^2\big(\gamma^2 f(y) + 3v\big) + 11v \le \gamma^4 f(y) + 23v. \end{aligned} \tag{2.15}$$

Thus (b) implies (c), indeed. Now suppose that (c) holds and let $K \in \mathcal{K}$, $v \in \mathcal{V}$, and $1 < \gamma < 2$. By (c) there is a disjoint partition of K into Borel sets $(E_i)_{i\in\mathbb{N}}$ such that $f(x) \le \gamma f(y) + v$ holds for all $x, y \in E_i$. We choose the step functions $h_n = \sum_{i=1}^n \chi_{E_i} \otimes a_i$, where $a_i = \gamma f(x_i) + v$ for some $x_i \in E_i$. Then $h_n \nearrow f$ on K and $h_n(x) \le \gamma^2 f(x) + 3v$ for all $x \in K$, hence $h_n \le \gamma^2 f + 3v$. Thus f is indeed measurable. □

We proceed to identify some measurable functions in $\mathcal{F}(X,\mathcal{P})$.

THEOREM 2.5. *Every* $f\in\mathcal{C}^r(X,\mathcal{P}_+)$ *is measurable.*

Proof. Let $K\subset\mathcal{K}$. Given $v\in\mathcal{V}$ and $\gamma>1$, a simple compactness argument shows that there are finitely many disjoint Borel sets $E_1,\dots,E_n$ whose union is K such that $f(y)\le\gamma f(x)+v$ whenever $x,y\in E_i$ for any $i=1,\dots,n$. Criterion (c) of Theorem 2.4 is therefore satisfied. □

THEOREM 2.6. *If* $f\in\mathcal{F}(X,\mathcal{P}_+)$ *is measurable and* φ *is a positive real-valued Borel measurable function on* X*, then the function* $\varphi\otimes f\in\mathcal{F}(X,\mathcal{P}_+)$ *is also measurable.*

Proof. Our claim is obvious if φ is a real-valued step function, as for any sequence $(h_n)_{n\in\mathbb{N}}$ of set functions approaching f, the sequence $(\varphi\otimes h_n)_{n\in\mathbb{N}}$ will approach $\varphi\otimes f$ in the same manner. Generally, there is an increasing sequence $(\psi_n)_{n\in\mathbb{N}}$ of real-valued step functions that converges pointwise to φ. For $\gamma>1$ set $f_n=\gamma(\psi_n\otimes f)$. All the functions f_n are measurable by the above, and $\psi_n\le\gamma(\varphi\otimes f)$ holds for all $n\in\mathbb{N}$. Moreover, for every $x\in X$ there is $n_0\in\mathbb{N}$ such that $\gamma\psi_n(x)\ge\varphi(x)$, hence $f_n(x)\ge\varphi\otimes f$, for all $n\ge n_0$. This shows $f_n\nearrow\varphi\otimes f$, and by Theorem 2.4(b) the function $\varphi\otimes f$ is seen to be measurable. □

3. Integrals for cone-valued functions

We may now define integrals for measurable functions in $\mathcal{F}(X,\mathcal{P}_+)$ with respect to $\mathcal{K}$-bounded Borel measure ϑ. The values of these integrals will be in $\overline{\mathbb{R}}$. Corresponding to ϑ we identify an inductive limit neighborhood $\mathfrak{v}_\vartheta\in\mathfrak{W}$ as follows: for each $K\in\mathcal{K}$ we choose a neighborhood $v_K\in\mathcal{V}$ such that $\vartheta_K(v_K)\le 1$, and let $\mathfrak{v}_\vartheta$ be the convex hull of the functions $\chi_K\otimes v_K\in\mathcal{S}(X,\mathcal{V})$. We denote the cone of all $\mathcal{P}$-valued step functions with compact support by $\mathcal{S}_\mathcal{K}(X,\mathcal{P})$. First for a function $h=\sum_{i=1}^n\chi_{E_i}\otimes a_i\in\mathcal{S}_\mathcal{K}(X,\mathcal{P}_+)$ with $E_i\in\mathcal{B}_\mathcal{K}$ and $a_i\in\mathcal{P}_+$, we define the $\overline{\mathbb{R}}$-valued integral with respect to ϑ by

$$\int_X h\,d\vartheta=\sum_{i=1}^n\vartheta_{E_i}(a_i)=\sum_{i=1}^n\vartheta^{a_i}(E_i). \tag{3.1}$$

It is straightforward to check that the sum on the right-hand side is independent of the representation for h. Moreover, the integral is monotone and linear on $\mathcal{S}_\mathcal{K}(X,\mathcal{P}_+)$, and $\int_X h_1\,d\vartheta\le\int_X h_2\,d\vartheta+1$ holds whenever $h_1\le h_2+\mathfrak{v}_\vartheta$ for $h_1,h_2\in\mathcal{S}_\mathcal{K}(X,\mathcal{P}_+)$. Indeed, the latter means $h_1\le h_2+s$ for some $s=\sum_{i=1}^n\lambda_i(\chi_{K_i}\otimes v_{K_i})\in\mathfrak{v}_\vartheta$, that is, $0\le\lambda_i\le 1$ and $\sum_{i=1}^n\lambda_i=1$. Thus $\int_X s\,d\vartheta=\sum_{i=1}^n\lambda_i\vartheta_{K_i}(v_{K_i})\le 1$ and

$$\int_X h_1\,d\vartheta\le\int_X(h_2+s)\,d\vartheta=\int_X h_2\,d\vartheta+\int_X s\,d\vartheta\le\int_X h_2\,d\vartheta+1. \tag{3.2}$$

Using this, for an arbitrary measurable function $f \in \mathcal{F}(X, \mathcal{P}_+)$ we define its *integral* over X as

$$\int_X f\,d\vartheta = \inf_{\mathfrak{v}\in\mathfrak{W}} \sup\left\{\int_X h\,d\vartheta \mid h \in \mathcal{S}_{\mathcal{K}}\big(X, \mathcal{P}_+\big),\ h \le f + \mathfrak{v}\right\}. \tag{3.3}$$

We will verify that this formula defines a monotone linear functional on the subcone of measurable functions in $\mathcal{F}(X, \mathcal{P}_+)$, continuous relative to the neighborhood $\mathfrak{v}_\vartheta \in \mathfrak{W}$. For a Borel subset Y of X and a measurable function $f \in \mathcal{F}(X, \mathcal{P}_+)$ we set $\int_Y f\,d\vartheta = \int_X \chi_Y \otimes f\,d\vartheta$ and infer as an immediate consequence of our definition.

Lemma 3.1. *$\int_X f\,d\vartheta = \sup_{K\in\mathcal{K}} \int_K f\,d\vartheta$ for every measurable function $f \in \mathcal{F}(X, \mathcal{P}_+)$.*

Lemma 3.2. *$\int_X f\,d\vartheta \le \int_X g\,d\vartheta + 1$ whenever $f \le g + \mathfrak{v}_\vartheta$ for measurable functions $f, g \in \mathcal{F}(X, \mathcal{P}_+)$.*

Proof. Let $f \le g + \mathfrak{v}_\vartheta$, that is, $f \le g + s$ for some $s \in \mathfrak{v}_\vartheta$. For $\varepsilon > 0$ and $\gamma > 1$, let $\mathfrak{u} \in \mathfrak{W}$ such that $\mathfrak{u} \le \varepsilon\mathfrak{v}_\vartheta$ and

$$\sup\left\{\int_X h\,d\vartheta \mid h \in \mathcal{S}_{\mathcal{K}}\big(X, \mathcal{P}_+\big),\ h \le g + \mathfrak{u}\right\} \le \int_X g\,d\vartheta + \varepsilon. \tag{3.4}$$

Now, let $l \in \mathcal{S}_{\mathcal{K}}(X, \mathcal{P}_+)$ be any step function with compact support $K \in \mathcal{K}$ such that $l \le f + \mathfrak{u}$, that is, $l \le f + t$ for some $t \in \varepsilon\mathfrak{v}_\vartheta$. Both s and t are $\mathcal{V}$-valued step functions and $\int_X s\,d\vartheta \le 1$ and $\int_X t\,d\vartheta \le \varepsilon$. There is a function $\chi_K \otimes v \in \mathfrak{u}$ for some $v \in \mathcal{V}$, and by Theorem 2.4(c) there are disjoint Borel sets $E_i \in \mathcal{B}_{\mathcal{K}}$ such that $\cup_{i\in\mathbb{N}} E_i = K$ and $g(x) \le \gamma g(y) + v$ whenever $x, y \in E_i$ for some $i \in \mathbb{N}$. We set $a_i = g(x_i)$ for some $x_i \in E_i$. Then

$$l(x) \le g(x) + s(x) + t(x) \le \gamma a_i + s(x) + t(x) + v \quad \forall x \in E_i. \tag{3.5}$$

We define step functions in $\mathcal{S}_{\mathcal{K}}(X, \mathcal{P})$

$$h_n = \left(\frac{1}{\gamma}\right)\sum_{i=1}^{n} \chi_{E_i} \otimes a_i, \qquad l_n = \sum_{i=1}^{n} \chi_{E_i} \otimes l. \tag{3.6}$$

Using the σ-additivity of the involved measures we infer $\int_X l\,d\vartheta = \lim_{n\to\infty} \int_X l_n\,d\vartheta$. Moreover, $h_n(x) \le g(x) + v$ holds for all $x \in K$, thus $h_n \le g + \mathfrak{u}$ and $\int_X h_n\,d\vartheta \le \int_X g\,d\vartheta + \varepsilon$ by our choice of the neighborhood $\mathfrak{u}$. Then $l_n \le \gamma^2 h_n + s + t + \chi_K \otimes v$ yields

$$\int_X l_n\,d\vartheta \le \gamma^2 \int_X h_n\,d\vartheta + 1 + 2\varepsilon \le \gamma^2\left(\int_X g\,d\vartheta + \varepsilon\right) + 1 + 2\varepsilon \quad \forall n \in \mathbb{N}. \tag{3.7}$$

By the definition of the integral, this shows that

$$\int_X f\,d\vartheta \le \gamma^2\left(\int_X g\,d\vartheta + \varepsilon\right) + 1 + 2\varepsilon. \tag{3.8}$$

As $\gamma > 1$ and $\varepsilon > 0$ are arbitrary, our claim follows. □

We may now prove a version of Fatou's lemma.

THEOREM 3.3. *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of measurable functions in $\mathcal{F}(X,\mathcal{P}_+)$ such that $f_n \nearrow f$ for a measurable function $f \in \mathcal{F}(X,\mathcal{P}_+)$. Then*

$$\int_X f\,d\vartheta \le \underline{\lim}_n \int_X f_n\,d\vartheta. \tag{3.9}$$

Proof. For $\varepsilon > 0$ let $\mathfrak{u} \in \mathfrak{W}$ be such that $\mathfrak{u} \le \varepsilon \mathfrak{v}_\vartheta$. Let

$$h = \sum_{i=1}^{n} \chi_{E_i} \otimes a_i \in \mathcal{S}_{\mathcal{K}}\big(X,\mathcal{P}_+\big) \tag{3.10}$$

be a step function supported by the compact set $K \in \mathcal{K}$ such that $h \le f + \mathfrak{u}$, that is, $h \le f + s$ for some $s \in \varepsilon\mathfrak{v}_\vartheta$. There is $\chi_K \otimes v \in \mathfrak{u}$ for some $v \in \mathcal{V}$, that is, $\vartheta_K(v) \le \varepsilon$. The functions $g_n = f_n + s + \chi_K \otimes v$ are also measurable and

$$g_n \nearrow f + s + \chi_K \otimes v, \quad g_n \le f_n + 2\mathfrak{u}. \tag{3.11}$$

Now for $i = 1,\dots,n$ and $k \in \mathbb{N}$, we set $F_i^k = g_k^{-1}(\Uparrow a_i) \cap E_i \in \mathcal{B}_{\mathcal{K}}$. Then for every $x \in E_i$ there is $m \in \mathbb{N}$ such that $x \in F_i^k$ for all $k \ge m$. Thus if we set $E_i^m = \cap_{k\ge m} F_k^i \in \mathcal{B}_{\mathcal{K}}$, then $E_i = \cup_{m\in\mathbb{N}} E_i^m$. We set

$$h_m = \sum_{i=1}^{n} \chi_{E_i^m} \otimes a_i \tag{3.12}$$

and have $(1/\gamma)h_m \le g_k + \mathfrak{u} \le f_k + 3\mathfrak{u}$ for all $k \ge m$ and $\gamma > 1$. Thus, using Lemma 3.2,

$$\int_X h_m\,d\vartheta = \sum_{i=1}^{n} \vartheta^{a_i}\big(E_i^m\big) \le \gamma\left(\int_X f_k\,d\vartheta + 3\varepsilon\right) \quad \forall k \ge m,\ \gamma > 1. \tag{3.13}$$

Therefore,

$$\sum_{i=1}^{n} \vartheta^{a_i}\big(E_i^m\big) \le \underline{\lim}_n \int_X f_n\,d\vartheta + 3\varepsilon \tag{3.14}$$

holds as well. From the σ-additivity of the measures ϑ^{a_i} we deduce that $\vartheta^{a_i}(E_i) = \lim_{m\to\infty} \vartheta^{a_i}(E_i^m)$. Thus

$$\int_X h\,d\vartheta = \sum_{i=1}^{n} \vartheta^{a_i}\big(E_i\big) = \lim_{m\to\infty}\left(\sum_{i=1}^{n} \vartheta^{a_i}\big(E_i^m\big)\right) \le \underline{\lim}_n \int_X f_n\,d\vartheta + 3\varepsilon. \tag{3.15}$$

By the definition of the integral for the function $f \in \mathcal{F}(X,\mathcal{P})$, and as $\varepsilon > 0$ was arbitrary, this shows that

$$\int_X f\,d\vartheta \le \underline{\lim}_n \int_X f_n\,d\vartheta \tag{3.16}$$

as claimed. □

THEOREM 3.4. *Let $f, g \in \mathcal{F}(X, \mathcal{P}_+)$ be measurable functions, $0 \le \alpha \in \mathbb{R}$. Then*

(a) $\int_X \alpha f \, d\vartheta = \alpha \int_X f \, d\vartheta$,

(b) $\int_X (f+g) \, d\vartheta = \int_X f \, d\vartheta + \int_X g \, d\vartheta$.

Proof. Part (a) is trivial. For (b), let $f, g \in \mathcal{F}(X, \mathcal{P}_+)$ be measurable functions and let K be a compact subset of X. Given $\varepsilon > 0$ and $\gamma > 1$ we choose $\mathfrak{u} \in \mathfrak{W}$ such that $\mathfrak{u} \le \varepsilon \mathfrak{v}_\vartheta$. There are sequences $(h_n)_{n\in\mathbb{N}}$ and $(l_n)_{n\in\mathbb{N}}$ in $\mathcal{S}_{\mathcal{K}}(X, \mathcal{P}_+)$ such that $h_n \nearrow \chi_K \otimes f$ and $h_n \le \gamma(\chi_K \otimes f) + \mathfrak{u}$, as well as $l_n \nearrow \chi_K \otimes g$ and $l_n \le \gamma(\chi_K \otimes g) + \mathfrak{u}$. Thus $(h_n + l_n) \nearrow \chi_K \otimes (f+g)$ and $(h_n + l_n) \le \gamma \, \chi_K \otimes (f+g) + 2\tilde{v}$. Using Theorem 3.3 and the additivity of the integral for step functions with compact support, we obtain

$$\begin{aligned}
\int_K (f+g) \, d\vartheta &\le \underline{\lim}_n \int_K \big(h_n + l_n\big) \, d\vartheta \le \overline{\lim}_n \int_K h_n \, d\vartheta + \overline{\lim}_n \int_K l_n \, d\vartheta \\
&\le \gamma \int_K f \, d\vartheta + \gamma \int_K g \, d\vartheta + 2\varepsilon, \\
\int_K f \, d\vartheta + \int_K g \, d\vartheta &\le \underline{\lim}_n \int_K h_n \, d\vartheta + \underline{\lim}_n \int_K l_n \, d\vartheta \le \underline{\lim}_n \int_K \big(h_n + l_n\big) \, d\vartheta \\
&\le \gamma \int_K (f+g) \, d\vartheta + 2\varepsilon,
\end{aligned} \tag{3.17}$$

for any choice of $\varepsilon > 0$ and $\gamma > 1$. Using Lemma 3.1, this proves our claim. □

Now we are in a position to define integrability for nonpositive functions as well. We say that a function $f \in \mathcal{F}(X, \mathcal{P})$ is *integrable with respect to* ϑ if there is a measurable function $g \in \mathcal{F}(X, \mathcal{P}_+)$ such that $\int_X g \, d\vartheta < +\infty$, and $f + g$ is in $\mathcal{F}(X, \mathcal{P}_+)$ and measurable. We set

$$\int_X f \, d\vartheta = \int_X (f+g) \, d\vartheta - \int_X g \, d\vartheta. \tag{3.18}$$

The integral is well defined and Theorem 3.3 and Lemma 3.2 yield the following theorem.

THEOREM 3.5. *The integrable functions form a subcone of $\mathcal{F}(X, \mathcal{P})$. If $f, g \in \mathcal{F}(X, \mathcal{P})$ are integrable and $0 \le \alpha \in \mathbb{R}$, then*

(a) $\int_X (\alpha f) \, d\vartheta = \alpha \int_X f \, d\vartheta$,

(b) $\int_X (f+g) \, d\vartheta = \int_X f \, d\vartheta + \int_X g \, d\vartheta$,

(c) $f \le g + \mathfrak{v}_\vartheta$ *implies* $\int_X f \, d\vartheta \le \int_X g \, d\vartheta + 1$.

THEOREM 3.6. *If $f \in \mathcal{C}^r(X, \mathcal{P})$ attains nonpositive values only on a relatively compact subset of X, then f is integrable.*

Proof. There is $E \in \mathcal{B}_{\mathcal{K}}$ such that $f(x) \ge 0$ for all $x \in X \setminus E$ and $v_E \in \mathcal{V}$ such that $\vartheta_E(v_E) = \int_X \chi_E \otimes v_E \, d\vartheta < +\infty$. Now a simple compactness argument shows that r-continuity implies that f is uniformly bounded below on E, that is, to say there is $\lambda \ge 0$ such that $f + \lambda(\chi_E \otimes v_E) \ge 0$. This function is measurable by Theorem 2.5, hence our claim follows. □

THEOREM 3.7. *If $f \in \mathcal{F}(X,\mathcal{P})$ is integrable and φ is a bounded positive real-valued Borel measurable function on X, then the function $\varphi \otimes f \in \mathcal{F}(X,\mathcal{P})$ is also integrable.*

Proof. There is a measurable function $g \in \mathcal{F}(X,\mathcal{P}_+)$ such that $f+g \in \mathcal{F}(X,\mathcal{P}_+)$ is also measurable and $\int_X g\,d\vartheta < +\infty$. As φ is bounded, there is $\rho \geq 0$ such that $0 \leq \varphi(x) \leq \rho$ for all $x \in X$. The functions $\varphi \otimes g$ and $\varphi \otimes f + \varphi \otimes g$ are positive and measurable by Theorem 2.6. Our claim follows, as $\varphi \otimes g \leq \rho g$ implies that $\int_X \varphi \otimes g\,d\vartheta < +\infty$. □

THEOREM 3.8. *Let $a \in \mathcal{P}_+$ and let φ be a positive real-valued Borel measurable function with compact support. Then*

$$\int_X (\varphi \otimes a)\,d\vartheta = \int_X \varphi\,d\vartheta^a. \tag{3.19}$$

Proof. The function $\varphi \otimes a$ is measurable by Theorem 2.6. There is an increasing sequence $(\psi)_{n\in\mathbb{N}}$ of real-valued step functions with compact support that converges pointwise to φ. Then for every $\gamma > 1$ we have

$$\gamma\big(\psi_n \otimes a\big) \nearrow \varphi \otimes a, \quad \psi_n \otimes a \leq \varphi \otimes a. \tag{3.20}$$

Following Theorem 3.3 this shows that

$$\begin{aligned}\int_X (\varphi \otimes a)\,d\vartheta &\leq \gamma \underline{\lim}_n \int_X \big(\psi_n \otimes a\big)\,d\vartheta \leq \gamma \overline{\lim}_n \int_X \big(\psi_n \otimes a\big)\,d\vartheta \\ &\leq \gamma \int_X (\varphi \otimes a)\,d\vartheta \quad \forall \gamma > 1.\end{aligned} \tag{3.21}$$

Hence

$$\int_X (\varphi \otimes a)\,d\vartheta = \lim_{n\to\infty} \int_X \big(\psi_n \otimes a\big)\,d\vartheta = \lim_{n\to\infty} \int_X \psi_n\,d\vartheta^a = \int_X \varphi\,d\vartheta^a. \tag{3.22}$$

□

3.1. Regularity of functional-valued Borel measures. Following the usual terminology, we say that a functional-valued Borel measure $\vartheta : \mathcal{B}_{\mathcal{K}} \to \mathcal{P}^*$ is *inner regular* on $\mathcal{B}_{\mathcal{K}}$ if

$$\vartheta^a(E) = \sup\big\{\vartheta^a(K) \mid K \subset E,\ K \in \mathcal{K}\big\} \tag{3.23}$$

holds for all $a \in \mathcal{P}_+$ and $E \in \mathcal{B}_{\mathcal{K}}$. Correspondingly, ϑ is *outer regular* on $\mathcal{B}_{\mathcal{K}}$ if

$$\vartheta^a(E) = \inf\big\{\vartheta^a(O) \mid E \subset O,\ O \in \mathcal{B}_{\mathcal{K}} \text{ open}\big\}. \tag{3.24}$$

An outer regular measure which is also inner regular for all open sets in $\mathcal{B}_{\mathcal{K}}$ is called *quasiregular.* An inner regular measure may be extended to all Borel sets by

$$\vartheta^a(E) = \sup\big\{\vartheta^a(F) \mid F \subset E,\ F \in \mathcal{B}_{\mathcal{K}}\big\} \quad \text{for every } E \in \mathcal{B},\ a \in \mathcal{P}_+. \tag{3.25}$$

For a quasiregular measure the corresponding procedure requires two steps: for $a \in \mathcal{P}_+$ we set

$$\vartheta^a(O) = \sup\big\{\vartheta^a(K) \mid K \subset E,\ K \in \mathcal{K}\big\} \tag{3.26}$$

for all open sets $O \subset X$ and

$$\vartheta^a(E) = \inf\left\{\vartheta^a(O) \mid E \subset O,\ O \text{ open}\right\} \tag{3.27}$$

for any Borel set $E \in \mathcal{B}$. For a nonrelatively compact Borel set E, ϑ_E is just a monotone linear functional on $\mathcal{P}_+$, not necessarily contained in the dual of $\mathcal{P}$. Note that all possible extension for ϑ coincide if X is σ-compact.

Example 3.9. (a) We obtain classical integration theory if we choose $\mathcal{P} = \overline{\mathbb{R}}$ with the canonical order and the neighborhoods $\mathcal{V} = \{\varepsilon \in \mathbb{R} \mid \varepsilon > 0\}$. The dual $\overline{\mathbb{R}}^*$ of $\overline{\mathbb{R}}$ consists of all positive reals (via the usual multiplication) and the singular functional $\bar{0}$ such that $\bar{0}(\alpha) = 0$ for all $\alpha \in \mathbb{R}$ and $\bar{0}(+\infty) = +\infty$. The monotone linear functionals on the positive cone $\overline{\mathbb{R}}_+$ in $\overline{\mathbb{R}}$ further include the element $+\infty$. A $\mathcal{K}$-bounded $\mathcal{P}^*$-valued Borel measure ϑ on a locally compact space may therefore take only real values on compact subsets of X, whereas an extension to all Borel sets may take the value $+\infty$. Any such extension is the sum of an $\overline{\mathbb{R}}_+$-valued Borel measure ϑ_1 in the usual sense and a measure ϑ_2 that takes only the values 0 and $\bar{0}$. The only unbounded element of the locally convex cone $\overline{\mathbb{R}}$ is $+\infty$, which remains unchanged under multiplication by scalars $\gamma > 0$. Thus the concepts of continuity and r-continuity coincide for $\overline{\mathbb{R}}$-valued functions. A continuous function may take the value $+\infty$ only on a subset of X which is both open and closed.

(b) Let $(E, \|\ \|)$ be a normed space with unit ball $\mathbb{B}$ and dual E'. Since E is generally not a full cone, nor carries a canonical order structure, we shall use the full cone containing E

$$\mathcal{P} = \{a + \lambda\mathbb{B} \mid a \in E,\ \lambda \geq 0\} \quad \text{with } \mathcal{V} = \{\varepsilon\mathbb{B} \mid \varepsilon > 0\} \tag{3.28}$$

and the set inclusion as order, that is, $a + \lambda\mathbb{B} \leq b + \rho\mathbb{B}$ if $\|a - b\| \leq \rho - \lambda$. In particular, $a + \lambda\mathbb{B} \in \mathcal{P}_+$ if $\|a\| \leq \lambda$. The dual of $\mathcal{P}$ is

$$\mathcal{P}^* = \left\{\mu \oplus \rho \mid \mu \in E',\ \|\mu\| \leq \rho\right\} \tag{3.29}$$

such that $(\mu \oplus \rho)(a + \lambda\mathbb{B}) = \mu(a) + \rho\lambda$. A $\mathcal{P}^*$-valued measure $\vartheta : \mathcal{B}_{\mathcal{K}} \to E'$ is required to be σ-additive on $\mathcal{B}_{\mathcal{K}}$. In this case, $\mathcal{K}$-boundedness means that for every $K \in \mathcal{K}$, hence for every $E \in \mathcal{B}_{\mathcal{K}}$ there is $\rho \geq 0$ such that $\sum_{i=1}^n \vartheta^{a_i}(E_i) \leq \rho$ for every choice of disjoint Borel sets $E_i \subset E$ and $a_i \in \mathbb{B}$. Correspondingly, we define $\vartheta^{\mathbb{B}}(E)$ to be the infimum of all these constants $\rho \geq 0$ (see Subsection 2.2). Thus for a step function $h = \sum_{i=1}^n \chi_{E_i} \otimes (a_i + \lambda_i\mathbb{B}) \in \mathcal{S}_{\mathcal{K}}(X, \mathcal{P}_+)$, that is, $\|a_i\| \leq \lambda_i$ for all $i = 1, \dots, n$, we have $\int_X h\, d\vartheta = \sum_{i=1}^n (\vartheta^{a_i}(E_i) + \lambda_i \vartheta^{\mathbb{B}}(E_i))$. Let $f : X \to E$ be a bounded function with compact support, that is, $\|f(x)\| \leq \rho$ for all $x \in X$. Then $f + \rho\mathbb{B} \in \mathcal{F}(X, \mathcal{P}_+)$, and an inspection of our definition of measurability in Subsection 2.4 shows that this function is measurable if and only if f may be uniformly approximated by a sequence of E-valued step functions. Hence the integral of f over X is the limit of the integrals for such a sequence.

4. A Riesz representation theorem

4.1. Lower semicontinuous inductive limit topologies. Let $(\mathcal{P},\mathcal{V})$ be a (not necessarily full) locally convex cone, and let $X,\mathcal{B}$, and $\mathcal{K}$ be as before. An inductive limit neighborhood $\mathfrak{v}$ for $\mathcal{F}(X,\mathcal{P})$ is called *lower semicontinuous* if all its elements $s\in\mathfrak{v}$ are $\mathcal{V}_\infty$-valued functions of the following type: $s(x)=\infty$ on an open subset O_s of X, and $s(x)=\sum_{i=1}^n\varphi_i(x)v_i$ for all $x\in X\setminus O_s$, where the φ_i are $\overline{\mathbb{R}}_+$-valued lower semicontinuous functions and $v_i\in\mathcal{V}$. In this context, $+\infty\cdot v$ represents the infinite element $v_\infty\in\mathcal{V}_\infty$ as defined in Subsection 2.1. An *inductive limit topology* on $\mathcal{F}(X,\mathcal{P})$ is generated by a system $\mathfrak{V}$ of inductive limit neighborhoods, closed for addition and multiplication by strictly positive scalars and directed downward with respect to the order relation

$$\mathfrak{v}\le\mathfrak{u}\quad\text{if for every } s\in\mathfrak{v} \text{ there is } t\in\mathfrak{u} \text{ such that } s\le t. \tag{4.1}$$

This topology is called *lower semicontinuous* if it contains a base of lower semicontinuous neighborhoods. Let $\mathcal{F}_{\mathfrak{V}}(X,\mathcal{P})$ be the subcone of $\mathcal{F}(X,\mathcal{P})$ of all functions f that are bounded below with respect to all neighborhoods in $\mathfrak{V}$, that is, for every $\mathfrak{v}\in\mathfrak{V}$ there is $\lambda\ge 0$ such that $0\le f+\lambda\mathfrak{v}$. In this way $(\mathcal{F}_{\mathfrak{V}}(X,\mathcal{P}),\mathfrak{V})$ becomes a locally convex cone.

We obtain a wide variety of topologies, the finest of which is the standard inductive limit topology for functions on a locally compact space, that is, $\mathfrak{V}=\mathfrak{W}$, consisting of all inductive limit neighborhoods. $\mathfrak{W}$ is lower semicontinuous, as for every $\mathfrak{v}\in\mathfrak{W}$ there is a lower semicontinuous neighborhood $\mathfrak{u}\in\mathfrak{W}$ such that $\mathfrak{u}\le\mathfrak{v}$. Indeed, for every relatively compact open set $O\subset X$ choose $v_O\in\mathcal{V}$ such that $\chi_O\otimes v_O\le s$ for some $s\in\mathfrak{v}$, and let $\mathfrak{u}$ consist of all convex combinations of the functions $\chi_O\otimes v_O$.

If, for another example, the elements of $\mathfrak{V}$ are just the singleton sets containing the constant mappings $x\mapsto v$ for $v\in\mathcal{V}$, then $\mathfrak{V}$ generates the topology of uniform convergence. If these singleton sets consist of mappings $x\mapsto v$ for $x\in K$ and $x\mapsto\infty$ else, for some $K\in\mathcal{K}$, then we obtain the topology of compact convergence. If we use finite sets instead of compact ones in the last example, the topology of pointwise convergence emerges. All these topologies are obviously lower semicontinuous.

We consider certain subcones of r-continuous functions in $\mathcal{F}_{\mathfrak{V}}(X,\mathcal{P})$. A simple compactness argument shows that every r-continuous $\mathcal{P}$-valued function with compact support is bounded below with respect to all neighborhoods in $\mathfrak{W}$, hence the cone $\mathcal{C}^r_{\mathcal{K}}(X,\mathcal{P})$ of these functions is contained in $\mathcal{F}_{\mathfrak{V}}(X,\mathcal{P})$. Finally, by $\mathcal{C}^r_{\mathfrak{V}}(X,\mathcal{P})$ we denote the closure of $\mathcal{C}^r_{\mathcal{K}}(X,\mathcal{P})$ in $\mathcal{C}^r(X,\mathcal{P})$ with respect to the symmetric topology generated by $\mathfrak{V}$, that is, the cone of all $f\in\mathcal{C}^r(X,\mathcal{P})$ such that for every $\mathfrak{v}\in\mathfrak{V}$ there is $g\in\mathcal{C}^r_{\mathcal{K}}(X,\mathcal{P})$ such that $f\le g+\mathfrak{v}$ and $g\le f+\mathfrak{v}$. Obviously $\mathcal{C}^r_{\mathfrak{V}}(X,\mathcal{P})\subset\mathcal{F}_{\mathfrak{V}}(X,\mathcal{P})$. Reviewing the above examples we observe that $\mathcal{C}^r_{\mathfrak{W}}(X,\mathcal{P})=\mathcal{C}^r_{\mathcal{K}}(X,\mathcal{P})$. For the topology of uniform convergence, on the other hand, $\mathcal{C}^r_{\mathfrak{V}}(X,\mathcal{P})$ consists of the functions $f\in\mathcal{C}^r(X,\mathcal{P})$ that *vanish at infinity,* that is, functions f such that for every $v\in\mathcal{V}$ there is a compact subset K of X such that

$$f(x)\le v,\quad 0\le f(x)+v,\quad \forall x\in X\setminus K. \tag{4.2}$$

For the topologies of compact and of pointwise convergence we have $\mathcal{C}^r_{\mathfrak{V}}(X,\mathcal{P})=\mathcal{C}^r(X,\mathcal{P})$.

We say that a $\mathcal{P}^*$-valued Borel measure ϑ on X is *$\mathfrak{V}$-continuous* if it is *continuous relative to a neighborhood* $\mathfrak{v} \in \mathfrak{V}$, that is, if

$$\int_X f\,d\vartheta \leq \int_X g\,d\vartheta + 1 \quad \text{whenever } f \leq g + \mathfrak{v} \tag{4.3}$$

for integrable functions $f, g \in \mathcal{F}(X, \mathcal{P})$. Every such measure is $\mathcal{K}$-bounded, as for $K \in \mathcal{K}$ there is $\chi_K \otimes v \leq \mathfrak{v}$ for some $v \in \mathcal{V}$, hence for disjoint Borel subsets E_i of K and $a_i, b_i \in \mathcal{P}$ such that $a_i \leq b_i + v$, we have $\sum_{i=1}^n \chi_{E_i} \otimes a_i \leq \sum_{i=1}^n \chi_{E_i} \otimes b_i + v$, hence $\sum_{i=1}^n \vartheta^{a_i}(E_i) \leq \sum_{i=1}^n \vartheta^{b_i}(E_i) + 1$. This shows $\vartheta^v(K) \leq 1$. Every $\mathcal{K}$-bounded measure, on the other hand, is continuous relative to some inductive limit neighborhood, as shown in Theorem 3.5(c).

LEMMA 4.1. *Let ϑ be continuous relative to the neighborhood $\mathfrak{v} \in \mathfrak{V}$.*

(a) *If for $E \in \mathcal{B}_{\mathcal{K}}$ there is $s \in \mathfrak{v}$ such that $s(x) = \infty$ for all $x \in E$, then $\vartheta(E) = 0$.*

(b) *If for $E \in \mathcal{B}_{\mathcal{K}}$ and $v \in \mathcal{V}$ there is $s \in \mathfrak{v}$ such that $s(x) \geq v_\infty$ for all $x \in E$, then $\vartheta^v(E) = 0$.*

(c) *If $\mathfrak{v}$ is lower semicontinuous, then every $f \in \mathcal{C}^r(X, \mathcal{P}) \cap \mathcal{F}_{\mathfrak{V}}(X, \mathcal{P})$ is integrable.*

Proof. Suppose that ϑ is continuous relative to $\mathfrak{v} \in \mathfrak{V}$. For $s \in \mathfrak{v}$, a relatively compact Borel set $E \subset X$ such that $s(x) = \infty$ for all $x \in E$, and $a \in \mathcal{P}$ we have $\chi_E \otimes a \leq \varepsilon s$ and $0 \leq \chi_E \otimes a + \varepsilon s$ for all $\varepsilon \geq 0$. Thus $\int_X \chi_E \otimes a\,d\vartheta \leq \varepsilon$ and $0 \leq \int_X \chi_E \otimes a\,d\vartheta + \varepsilon$. This shows that $\vartheta^a(E) = \int_X \chi_E \otimes a\,d\vartheta = 0$, as claimed in (a). Part (b) follows in similar fashion. For (c), let $\mathfrak{v}$ be lower semicontinuous and let $f \in \mathcal{C}^r(X, \mathcal{P}) \cap \mathcal{F}_{\mathfrak{V}}(X, \mathcal{P})$. Then $f + \rho\mathfrak{v} \geq 0$ for some $\rho \geq 0$, that is, $f + s \geq 0$ for some $s \in \rho\mathfrak{v}$. We have $s(x) = \sum_{i=1}^n \varphi_i(x) v_i$ for all $x \in X \setminus O_s$ and $s(x) = \infty$ on O_s. By our definition of r-continuity there are continuous positive real-valued functions ψ_i such that $f + \psi_i \otimes v_i \geq 0$. Set $\phi_i(x) = \min\{\varphi_i(x), \psi_i(x)\}$ for $x \in X \setminus O_s$ and $\phi_i(x) = \psi_i(x)$ on O_s. The function $t(x) = \sum_{i=1}^n \phi_i(x) v_i$ is measurable, $t \leq s$ implies $\int_X t\,d\vartheta \leq \rho$, and $f + t \geq 0$. This demonstrates that f is indeed integrable. □

In this way, a $\mathfrak{V}$-continuous Borel measure ϑ defines a continuous linear functional on the subcone of integrable functions in $\mathcal{F}(X, \mathcal{P})$. If ϑ is continuous relative to $\mathfrak{v} \in \mathfrak{V}$, then this functional is in the polar of $\mathfrak{v}$. Conversely, we demonstrate in our main result that for a lower semicontinuous topology, every continuous linear functional on the locally convex cone $(\mathcal{C}^r_{\mathfrak{V}}(X, \mathcal{P}), \mathfrak{V})$ is of this type.

THEOREM 4.2. *Every continuous linear functional on $(\mathcal{C}^r_{\mathfrak{V}}(X, \mathcal{P}), \mathfrak{V})$ may be represented as an integral with respect to a $\mathfrak{V}$-continuous quasiregular $\mathcal{P}^*$-valued Borel measure on X. More precisely, for a lower semicontinuous neighborhood $\mathfrak{v} \in \mathfrak{V}$ and $\mu \in \mathfrak{v}^\circ$ there is a quasiregular $\mathcal{P}^*$-valued Borel measure ϑ, which is continuous relative to $\mathfrak{v}$ and*

$$\int_X f\,d\vartheta = \mu(f) \quad \forall f \in \mathcal{C}^r_{\mathfrak{V}}(X, \mathcal{P}). \tag{4.4}$$

Proof. As every locally convex cone may be embedded into a full cone, and as continuous linear functionals on locally convex cones (hence on cone-valued functions) may be suitably extended, we may assume that $(\mathcal{P}, \mathcal{V})$ is a full locally convex cone. Now, under the assumptions of the theorem, let μ be a continuous linear functional on $(\mathcal{C}^r_{\mathfrak{V}}(X,\mathcal{P}), \mathfrak{V})$, contained in the polar of some lower semicontinuous neighborhood $\mathfrak{v} \in \mathfrak{V}$. The functional μ may be extended (see [3, Theorem II.2.9]) to a linear functional on the larger cone $(\mathcal{F}_{\mathfrak{V}}(X,\mathcal{P}), \mathfrak{V})$, also denoted by μ and contained in the polar of $\mathfrak{v}$; that is, $\mu(f) \le \mu(g) + 1$ holds whenever $f \le g + \mathfrak{v}$ for $f, g \in \mathcal{F}_{\mathfrak{V}}(X,\mathcal{P})$. Using this, we proceed to construct a $\mathfrak{V}$-continuous quasiregular $\mathcal{P}^*$-valued Borel measure ϑ on X. We will follow some of the main lines of the standard proof for the Riesz representation theorem (cf. [7]), though the presence of unbounded elements in $\mathcal{P}$, hence nonfinite measures, will complicate matters. For a fixed element $a \in \mathcal{P}_+$ we define an $\overline{\mathbb{R}}_+$-valued set function ϑ^a on $\mathcal{B}_{\mathcal{K}}$ as follows: for an open set $O \in \mathcal{B}_{\mathcal{K}}$ we set

$$\vartheta^a(O) = \sup\left\{\mu\left(\chi_U \otimes a\right) \mid \overline{U} \subset O,\ U \text{ open}\right\}. \tag{4.5}$$

(As usual, $\overline{U}$ denotes the topological closure of U.) Then

$$\vartheta^a(E) = \inf\left\{\vartheta^a(O) \mid E \subset O,\ O \in \mathcal{B}_{\mathcal{K}} \text{ open}\right\} \tag{4.6}$$

for $E \in \mathcal{B}_{\mathcal{K}}$, defines a σ-additive set function on $\mathcal{B}_{\mathcal{K}}$. We omit the details of this procedure which may be easily checked. We shall however list a few observations. Let us denote by $\mathcal{F}_{\mathcal{K}}(X)$ the cone of all real-valued functions on X with compact support, and by $\mathcal{C}_{\mathcal{K}}(X)$ the subcone of all continuous functions in $\mathcal{F}_{\mathcal{K}}(X)$. For a subset $E \subset X$ and $\varphi \in \mathcal{F}_{\mathcal{K}}(X)$ we write $E \prec \varphi$ if $\chi_E \le \varphi$, and $\varphi \prec E$ if $0 \le \varphi \le \chi_E$ and $\operatorname{supp}(\varphi) \subset E$. Then

(i) $\vartheta^a(O) = \sup\{\mu(\varphi \otimes a) \mid \varphi \in \mathcal{C}_{\mathcal{K}}(X),\ \varphi \prec O\} \le \mu(\chi_O \otimes a)$ for all open $O \in \mathcal{B}_{\mathcal{K}}$.

(ii) $\vartheta^a(K) = \inf\{\mu(\varphi \otimes a) \mid \varphi \in \mathcal{C}_{\mathcal{K}}(X),\ K \prec \varphi\} \ge \mu(\chi_K \otimes a)$ for all $K \in \mathcal{K}$.

(iii) $\vartheta^a(E) = \inf\{\vartheta^a(O) \mid E \subset O,\ O \in \mathcal{B}_{\mathcal{K}} \text{ open}\}$ for all $E \in \mathcal{B}_{\mathcal{K}}$.

(iv) $\vartheta^a(O) = \sup\{\vartheta^a(K) \mid K \subset O,\ K \in \mathcal{K}\}$ for all open $O \in \mathcal{B}_{\mathcal{K}}$.

Conditions (iii) and (vi) represent quasiregularity. Furthermore,

(v) $\vartheta^{a+b}(E) = \vartheta^a(E) + \vartheta^b(E)$ and $\vartheta^{\alpha a}(E) = \alpha\vartheta^a(E)$ for all $a, b \in \mathcal{P}_+$, $\alpha \ge 0$, and $E \in \mathcal{B}_{\mathcal{K}}$.

(vi) $\vartheta^a(E) \le \vartheta^b(E)$ whenever $a \le b$ for $a, b \in \mathcal{P}_+$ and $E \in \mathcal{B}_{\mathcal{K}}$.

Statements (v) and (vi) show that for a fixed $E \in \mathcal{B}_{\mathcal{K}}$, the mapping $a \mapsto \vartheta_E(a) = \vartheta^a(E)$ defines a monotone linear functional on $\mathcal{P}_+$. Moreover, for any $E \in \mathcal{B}_{\mathcal{K}}$ we may choose an open set $O \in \mathcal{B}_{\mathcal{K}}$ such that $E \subset O$. There is $v_E \in \mathcal{V}$ such that $\chi_O \otimes v_E \le \mathfrak{v}$, hence $\mu(\chi_O \otimes v_E) \le 1$. Thus $\vartheta^{v_E}(E) \le \vartheta^{v_E}(O) \le \mu(\chi_O \otimes v_E) \le 1$ by (i). For an arbitrary (not necessarily positive) element $a \in \mathcal{P}$ we choose $\rho \ge 0$ such that $0 \le a + \rho v_E$ and set

$$\vartheta_E(a) = \vartheta^a(E) = \vartheta^{(a+\rho v_E)}(E) - \rho\vartheta^{v_E}(E). \tag{4.7}$$

In this way, ϑ_E becomes a continuous linear functional on $\mathcal{P}$, that is, to say an element of v_E°, and ϑ is indeed a $\mathcal{K}$-bounded quasiregular $\mathcal{P}^*$-valued Borel measure on X.

(vii) $\int_X \varphi \otimes a\, d\vartheta \le \mu(\varphi \otimes a)$ for every positive real-valued lower semicontinuous function φ on X and $a \in \mathcal{P}_+$.

We may assume that $\mu(\varphi \otimes a) < +\infty$, as there is nothing to prove otherwise. Measurability, hence integrability for the positive function $\varphi \otimes a$ was shown in Theorem 2.6. In a first step, assume that the support of φ is compact and that $0 \le \varphi \le 1$. We fix $n \ge 2$ in $\mathbb{N}$, and for $i \ge 1$ define relatively compact open sets

$$O_i = \{x \in X \mid n\varphi(x) > i\}. \tag{4.8}$$

Then $O_{i+1} \subset O_i$ and $O_n = \emptyset$. As $\chi_{O_1} \otimes a \le n(\varphi \otimes a)$, we conclude, using (i), that $\vartheta^a(O_1) \le \mu(\chi_{O_1} \otimes a) \le n\mu(\varphi \otimes a) < +\infty$. We define lower semicontinuous functions $\psi_i \in \mathcal{C}_{\mathcal{K}}(X)$ by

$$\psi_i(x) = \begin{cases} 1, & \text{if } x \in O_{i+1} \\ n\varphi(x) - i, & \text{if } x \in O_i \setminus O_{i+1} \\ 0, & \text{if } x \notin O_i. \end{cases} \tag{4.9}$$

Thus $O_{i+1} \prec \psi_i$. We set $\varphi_n = (1/n)\sum_{i=1}^n \psi_i$. Then $\varphi_n(x) = 0$ if $\varphi(x) \le 1/n$ and $\varphi_n(x) = \varphi(x) - (1/n)$ if $\varphi(x) > 1/n$. Furthermore

$$\chi_{O_{i+1}} \otimes a \le \psi_i \otimes a \le \chi_{O_i} \otimes a, \quad \text{hence } \vartheta^a\big(O_{i+1}\big) \le \mu\big(\chi_{O_{i+1}} \otimes a\big) \le \mu\big(\psi_i \otimes a\big). \tag{4.10}$$

We choose the step function $h = (1/n)\sum_{i=1}^n \chi_{O_i} \otimes a \in \mathcal{S}_{\mathcal{K}}(X, \mathcal{P}_+)$. Then $\varphi_n \otimes a \le h$ by the above, and

$$\begin{aligned} \int_X \varphi_n \otimes a\, d\vartheta &\le \int_X h\, d\vartheta = \frac{1}{n}\sum_{i=1}^n \vartheta^a\big(O_i\big) = \frac{1}{n}\left(\sum_{i=1}^n \vartheta^a\big(O_{i+1}\big) + \vartheta^a\big(O_1\big)\right) \\ &\le \frac{1}{n}\left(\sum_{i=1}^n \mu\big(\psi_i \otimes a\big) + \vartheta^a\big(O_1\big)\right) = \mu\big(\varphi_n \otimes a\big) + \frac{1}{n}\vartheta^a\big(O_1\big). \end{aligned} \tag{4.11}$$

As $\vartheta^a(O_1)$ is finite, this yields $\int_X \varphi_n \otimes a\, d\vartheta \le \mu(\varphi_n \otimes a) \le \mu(\varphi \otimes a)$. Now, as $\varphi_n \le \varphi$ and $\varphi_n \nearrow \varphi$, Theorem 3.3 yields

$$\int_X \varphi\, d\vartheta \le \underline{\lim}_n \int_X \varphi_n\, d\vartheta \le \mu(\varphi \otimes a). \tag{4.12}$$

Now, in a second step we still assume that φ has compact support, but is not necessarily bounded. Then we set $\varphi_n(x) = \varphi(x)$ if $\varphi(x) \le n$ and $\varphi_n(x) = n$ else. The preceding yields $\int_X \varphi\, d\vartheta \le \underline{\lim}_n \int_X \varphi_n\, d\vartheta \le \mu(\varphi \otimes a)$. Again using Theorem 3.3, we infer that $\int_X \varphi\, d\vartheta \le \underline{\lim}_n \int_X \varphi_n\, d\vartheta \le \mu(\varphi \otimes a)$ holds in this case as well. Finally, if the support of φ is not compact, we consider the lower semicontinuous functions $\varphi\chi_O$ for relatively compact open subsets O. We have $\int_X \varphi\chi_O \otimes a\, d\vartheta \le \mu(\varphi\chi_O \otimes a) \le \mu(\varphi \otimes a)$ by the above. Using Lemma 3.1, this yields

$$\int_X \varphi \otimes a\, d\vartheta = \sup_{K \in \mathcal{K}} \int_X \varphi\chi_K \otimes a\, d\vartheta = \sup_{O \in \mathcal{B}_{\mathcal{K}}} \int_X \varphi\chi_O \otimes a\, d\vartheta \le \mu(\varphi \otimes a) \tag{4.13}$$

as well. A similar but slightly simpler argument leads to a dual statement for upper semicontinuous functions. We omit the details.

(viii) $\int_X \varphi \otimes a \, d\vartheta \geq \mu(\varphi \otimes a)$ for every positive real-valued upper semicontinuous function φ with compact support $K \subset X$ and $a \in \mathcal{P}_+$ such that $\vartheta^a(K) < +\infty$.

(ix) $\int_X f \, d\vartheta = \mu(f)$ for all $f \in \mathcal{C}^r_{\mathcal{K}}(X, \mathcal{P})$.

Let $f \in \mathcal{C}^r_{\mathcal{K}}(X, \mathcal{P})$ be supported by $K \in \mathcal{K}$. Let U be a relatively compact open set containing K and choose $v \in \mathcal{V}$ such that $\vartheta^v(U) \leq \mu(\chi_U \otimes v) \leq 1$. By our definition of r-continuity there is a positive function $\psi \in \mathcal{C}_{\mathcal{K}}(X)$ such that $g = f + \psi \otimes v$ satisfies the definition of a positive r-continuous function in Section 2.3. We may assume that the support of ψ is contained in U and that ψ is bounded above by some $\rho \geq 0$. Statements (vii) and (viii) yield

$$\int_X \psi \otimes v \, d\vartheta = \mu(\psi \otimes v) \leq \rho \mu\big(\chi_U \otimes v\big) \leq \rho. \tag{4.14}$$

The function g is therefore integrable, $\operatorname{supp}(g) \subset U$, and we proceed to verify that $\int_X g \, d\vartheta = \mu(g)$. For this, first suppose that there is an open subset O of U and $\gamma > 1$ such that

$$g(x) \leq \gamma g(y) + v \quad \text{whenever } x, y \in O, \tag{4.15}$$

and that $\vartheta^a(O) = +\infty$ for $a = g(x)$ for some $x \in O$. Then $h = \chi_O \otimes a \leq \gamma g + \chi_U \otimes v$, and both

$$\vartheta^a(O) = \int_X h \, d\vartheta \leq \gamma \int_X g \, d\vartheta + 1, \qquad \vartheta^a(O) \leq \mu\big(\chi_O \otimes a\big) \leq \gamma \mu(g) + 1, \tag{4.16}$$

hence $\int_X g \, d\vartheta = \mu(g) = +\infty$. If, on the other hand, there is no such open set $O \subset U$, then for every choice of $\gamma > 1$ and $0 \leq \varepsilon \leq 1$ there are open sets $O_1, \ldots, O_n$ in U whose union contains $\operatorname{supp}(g)$ such that $g(y) \leq \gamma g(x) + \varepsilon v$ whenever $x, y \in O_i$ for any $i = 1, \ldots, n$. There is a corresponding set $\varphi_1, \ldots, \varphi_n$ of positive functions in $\mathcal{C}_{\mathcal{K}}(X)$ such that $\varphi_i \prec O_i$ and $\sum_{i=1}^n \varphi_i(x) = 1$ for all $x \in \operatorname{supp}(g)$. We choose $a_i = g(x_i)$ for some $x_i \in O_i$ and set $h = \sum_{i=1}^n \varphi_i \otimes a_i \in \mathcal{C}_{\mathcal{K}}(X, \mathcal{P})$. As $\vartheta^{a_i}(O_i) < +\infty$ for all $i = 1, \ldots, n$, we know from (vii) and (viii) that $\int_X h \, d\vartheta = \mu(h)$. We have

$$\varphi_i(x) a_i \leq \gamma \varphi_i(x) g(x) + \varepsilon v, \quad \varphi_i(x) g(x) \leq \gamma \varphi_i(x) a_i + \varepsilon v \quad \forall x \in U. \tag{4.17}$$

This yields

$$h \leq \gamma g + \varepsilon\big(\chi_U \otimes v\big), \qquad g \leq \gamma h + \varepsilon\big(\chi_U \otimes v\big). \tag{4.18}$$

Therefore

$$\int_X h \, d\vartheta \leq \gamma \int_X g \, d\vartheta + \varepsilon, \qquad \int_X g \, d\vartheta \leq \gamma \int_X h \, d\vartheta + \varepsilon, \tag{4.19}$$

and likewise

$$\mu(h) \leq \gamma \mu(g) + \varepsilon, \qquad \mu(g) \leq \gamma \mu(h) + \varepsilon. \tag{4.20}$$

Thus $\int_X g \, d\vartheta = \mu(g)$, indeed. Now from

$$\int_X f \, d\vartheta + \int_X \psi \otimes v \, d\vartheta = \mu(f) + \mu(\psi \otimes v), \tag{4.21}$$

we conclude that $\int_X f \, d\vartheta = \mu(f)$.

(x) ϑ is continuous relative to $\mathfrak{v}$.

Suppose that $f \le g + \mathfrak{v}$ holds for integrable functions $f, g \in \mathcal{F}(X,\mathcal{P})$. Then $f \le g + s$ for some $s \in \mathfrak{v}$. The set $O_s = \{x \in X \mid s(x) = \infty\}$ is open in X and there are lower semicontinuous $\overline{\mathbb{R}}_+$-valued functions φ_i such that $s(x) = \sum_{i=1}^n \varphi_i(x) v_i$ for all $x \in X \setminus O_s$. For every relatively compact subset E of O_s there is a relatively compact open set U such that $E \subset U \subset O_s$, and for every $a \in \mathcal{P}_+$ and $\varepsilon > 0$ we have $0 \le \vartheta^a(E) \le \vartheta^a(U) \le \mu(\chi_U \otimes a) \le \varepsilon$ as $\chi_U \otimes a \le \varepsilon s$. Thus $\vartheta^a(E) = 0$. This implies $\int_{O_s} h\, d\vartheta = 0$, hence $\int_X h\, d\vartheta = \int_{(X\setminus O_s)} h\, d\vartheta$ for every integrable function $h \in \mathcal{F}(X,\mathcal{P})$. Next, for every $m \in \mathbb{N}$ we set $s_m = \sum_{i=1}^n \varphi_i^m \otimes v_i$, where $\varphi_i^m(x) = m$ for $x \in O_s$ and $\varphi_i^m(x) = \inf\{\varphi_i(x), m\}$ for $x \in X \setminus O_s$. The functions φ_i^m are positive real-valued, bounded and lower semicontinuous, and (vii) yields $\int_X s_m\, d\vartheta \le \mu(s_m) \le 1$. Furthermore, we observe that $(g + s_m) \nearrow f$ on $X \setminus O_s$ as a consequence of our definition of the infinite neighborhoods v_∞ in Subsection 2.1. Now Theorem 3.3 yields $\int_X f\, d\vartheta \le \underline{\lim}_m \int_X (g + s_m)\, d\vartheta \le \int_X g\, d\vartheta + 1$, as claimed.

Thus finally, both ϑ and μ represent continuous linear functionals on the subcone of integrable functions in $\mathcal{F}_{\mathfrak{V}}(X,\mathcal{P})$. Both functionals coincide on $\mathcal{C}^r_{\mathcal{K}}(X,\mathcal{P})$, hence on its closure:

(ix) $\int_X f\, d\vartheta = \mu(f)$ for all $f \in \mathcal{C}^r_{\mathfrak{V}}(X,\mathcal{P})$. □

4.2. Examples. (a) *The case* $\mathcal{P} = \overline{\mathbb{R}}$. We already observed in Example 3.9(a) that the concepts of continuity and r-continuity coincide in this case and that an $\overline{\mathbb{R}}^*$-valued Borel measure ϑ on X is the sum of an ordinary $\overline{\mathbb{R}}_+$-valued Borel measure ϑ_1 and a measure ϑ_2 that takes only the values 0 and $\bar{0}$. ϑ_1 is finite on compact subsets of X. Lower semicontinuous neighborhoods $\mathfrak{v}$ for $\mathcal{F}(X,\overline{\mathbb{R}})$ contain functions s of the following type: $s(x) = \infty$ on an open set $O_s \subset X$, and $s(x) = \varphi(x) \cdot 1$ (the latter represents the neighborhood $1 \in \mathcal{V}$) for all $x \in X \setminus O_s$ for a lower semicontinuous $\overline{\mathbb{R}}_+$-valued function on X. Recall that $a \le b + \infty$ for all $a, b \in \overline{\mathbb{R}}$, whereas $a \le b + 1_\infty$ means that a is finite if b is finite. If the measure ϑ is continuous relative to a lower semicontinuous neighborhood $\mathfrak{v} \in \mathfrak{V}$, then $\vartheta_1(E) = \vartheta_2(E) = 0$ for every relatively compact Borel set $E \subset O_s$ for some $s \in \mathfrak{v}$. Likewise, $\vartheta_1(E) = 0$ whenever $s(x) \ge 1_\infty$ for all $x \in E$ for a function $s \in \mathfrak{v}$ (see Lemma 4.1(a) and (b)). Thus, following Theorem 4.2, a continuous linear functional μ on $\mathcal{C}^r_{\mathfrak{V}}(X,\overline{\mathbb{R}})$ may be represented as an integral

$$\mu(f) = \int_X f\, d\vartheta_1 + \int_X f\, d\vartheta_2. \tag{4.22}$$

Moreover, $\int_X f\, d\vartheta_2 = +\infty$, if $\vartheta_2(O) = \bar{0}$, where $O = \{x \in X \mid f(x) = +\infty\}$, and $\int_X f\, d\vartheta_2 = 0$ else.

(b) *Weighted spaces of functions.* For locally convex vector spaces $\mathcal{P}$ the following is due to Nachbin [4] and Prolla [5]: let X be a locally compact Hausdorff space, $(\mathcal{P},\mathcal{V})$ a locally convex cone. A family Ω of nonnegative real-valued upper semicontinuous functions on X is called a *family of weights* (see [4, 5]) if for all $\omega_1, \omega_2 \in \Omega$ there are $\omega_3 \in \Omega$ and $\rho > 0$ such that $\omega_1 \le \rho\omega_3$ and $\omega_2 \le \rho\omega_3$. With any family of weights Ω we associate the cone of continuous $\mathcal{P}$-valued functions on X

$$\mathcal{C}_\Omega(X,\mathcal{P}) = \big\{f \in \mathcal{C}(X,\mathcal{P}) \mid \omega f \text{ vanishes at infinity for all } \omega \in \Omega\big\}. \tag{4.23}$$

The *weighted topology* on $\mathcal{C}_\Omega(X,\mathcal{P})$ is given by the neighborhoods v_ω, for $v \in \mathcal{V}$ and $\omega \in \Omega$, such that

$$f \le g + v_\omega \quad \text{if } \omega(x) f(x) \le \omega(x) g(x) + v \ \forall x \in X. \tag{4.24}$$

A simple compactness argument shows that every $f \in \mathcal{C}_\Omega(X,\mathcal{P})$ is bounded below with respect to these neighborhoods. With $\mathcal{V}_\Omega = \{v_\omega \mid v \in \mathcal{V}, \omega \in \Omega\}$, then $(\mathcal{C}_\Omega(X,\mathcal{P}), \mathcal{V}_\Omega)$ forms a locally convex cone of *weighted $\mathcal{P}$-valued functions.* Examples for such cones may be found in [3, Chapter V.1]. We may reformulate this setting as a lower semicontinuous inductive limit topology in the sense of Section 4.1: for $v \in \mathcal{V}$ and $\omega \in \Omega$ let O_ω be the complement of the support of ω and set $s_{\omega,v}(x) = \infty$ for $x \in O$ and $s_{\omega,v}(x) = (1/\omega(x))v$ else; that is, in particular, $s_{\omega,v}(x) = v_\infty$ if $\omega(x) = 0$ for $x \in \operatorname{supp}(\omega)$. Then $\mathfrak{v}_{\omega,v} = \{s_{\omega,v}\}$ is a lower semicontinuous neighborhood, as for every compact subset K of X there is $\rho > 0$ such that $\omega(x) \le \rho$ for all $x \in K$, hence $(1/\rho)(\chi_K \otimes v) \le s$. Thus $\mathfrak{V}_\Omega = \{\mathfrak{v}_{\omega,v} \mid v \in \mathcal{V},\ \omega \in \Omega\}$ forms a basis for a lower semicontinuous inductive limit topology. Clearly $f \le g + \mathfrak{v}_{\omega,v}$ implies $f \le g + v_\omega$ for $f, g \in \mathcal{F}(X,\mathcal{P})$. The reverse holds true if both functions f and g are continuous. Indeed, for $f \le g + v_\omega$ we have $f(x) \le g(x) + s(x)$ for all $x \in O$ and all x such that $\omega(x) > 0$. If, on the other hand, $x \in \operatorname{supp}(\omega)$ and $\omega(x) = 0$, we choose a neighborhood $U(x)$ of x such that $f(x) \le f(y) + v$ and $g(y) \le g(x) + v$ for all $y \in U(x)$. There is such an element y with $\omega(y) > 0$. Then

$$\begin{aligned} f(x) \le f(y) + v &\le g(y) + \left(\frac{1}{\omega(y)} + 1\right) v \\ &\le g(x) + \left(\frac{1}{\omega(y)} + 2\right) v \le g(x) + s(x) \end{aligned} \tag{4.25}$$

as $s(x) = v_\infty$. This shows that $f \le g + \mathfrak{v}_{\omega,v}$. The inductive limit and the weighted topologies therefore coincide for continuous functions, and the locally convex cone $(\mathcal{C}_\Omega(X,\mathcal{P}), \mathcal{V}_\Omega)$ is a subcone of $(\mathcal{C}^r_{\mathfrak{V}_\Omega}(X,\mathcal{P}), \mathfrak{V}_\Omega)$. Following Theorem 4.2, every continuous linear functional μ on $\mathcal{C}_\Omega(X,\mathcal{P})$ may be represented as a $\mathfrak{V}$-continuous quasiregular functional-valued Borel measure ϑ on X, that is,

$$\mu(f) = \int_X f \, d\vartheta \quad \forall f \in \mathcal{C}_\Omega(X,\mathcal{P}). \tag{4.26}$$

If $\mu \in \mathfrak{v}^\circ_{\omega,v}$, then ϑ is continuous relative to $\mathfrak{v}_{\omega,v}$. Prolla's result in [5], on the other hand, yields a functional-valued Borel measure $\tilde{\vartheta}$, such that $\tilde{\vartheta}^v(X) \le 1$ and

$$\mu(f) = \int_X \omega f \, d\tilde{\vartheta} \quad \forall f \in \mathcal{C}_\Omega(X,\mathcal{P}). \tag{4.27}$$

This measure $\tilde{\vartheta}$ may be constructed from ϑ in an obvious way: let $\varphi_n(x) = \inf\{1/\omega(x), n\}$, and for $E \in \mathcal{B}$ set

$$\tilde{\vartheta}(E) = \lim_{n\to\infty} \int_X \varphi_n \chi_E \, d\vartheta, \quad \text{that is, } \tilde{\vartheta}^a(E) = \lim_{n\to\infty} \int_X \varphi_n \chi_E \otimes a \, d\vartheta \tag{4.28}$$

for every $a \in \mathcal{P}$. It is straightforward to check that this limit exists and that $\tilde{\vartheta}$ represents the functional μ as claimed.

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