NONLINEAR ERGODIC THEOREMS FOR ASYMPTOTICALLY ALMOST NONEXPANSIVE CURVES IN A HILBERT SPACE

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Received 3 May 2000

We introduce the notion of asymptotically almost nonexpansive curves which include almost-orbits of commutative semigroups of asymptotically nonexpansive type mappings and study the asymptotic behavior and prove nonlinear ergodic theorems for such curves. As applications of our main theorems, we obtain the results on the asymptotic behavior and ergodicity for a commutative semigroup of non-Lipschitzian mappings with nonconvex domains in a Hilbert space.

1. Introduction

Let H be a real Hilbert space with norm $\|\cdot\|$ and inner product $(\cdot,\cdot)$. Let C be a nonempty subset of H and G be a commutative semitopological semigroup with identity. In this case, $(G,\succcurlyeq)$ is a directed system when the binary relation "$\succcurlyeq$" on G is defined by $b \succcurlyeq a$ if and only if there is $c \in G$ such that $a+c=b$. Let $\Im = \{T(t) : t \in G\}$ be a semigroup acting on C, that is, $T(t+s)x = T(t)T(s)x$ for all $t,s \in G$ and $x \in C$. Recall that a semigroup $\Im$ on C is said to be

(a) nonexpansive if $\|T(t)x - T(t)y\| \le \|x-y\|$ for $x,y \in C$ and $t \in G$,

(b) asymptotically nonexpansive, [9], if there exists a function $k : G \mapsto [0,\infty)$ with $\limsup_{t\in G} k_t \le 1$ such that

$$\left\|T(t)x - T(t)y\right\| \le k_t \|x-y\| \tag{1.1}$$

for $x,y \in C$ and $t \in G$,

(c) of asymptotically nonexpansive type, [9], if for each $x \in C$, there is a function $r(\cdot,x) : G \mapsto [0,\infty)$ with $\lim_{t\in G} r(t,x) = 0$ such that

$$\left\|T(t)x - T(t)y\right\| \le \|x-y\| + r(t,x) \quad \forall y \in C,\ t \in G, \tag{1.2}$$

where $\lim_{t\in G}\alpha(t)$ denotes the limit of a net $\alpha(\cdot)$ on the directed system $(G,\succcurlyeq)$.

Abstract and Applied Analysis 5:3 (2000) 147–158
2000 Mathematics Subject Classification: 47H09, 47H10, 47H20
URL: http://aaa.hindawi.com/volume-5/S1085337500000312.html

It is easily seen that (a)$\Rightarrow$(b)$\Rightarrow$(c) and that both the inclusions are proper (cf. [9, page 112]).

In 1975, Baillon [1] proved the first nonlinear mean ergodic theorem for nonexpansive mappings in a Hilbert space: let C be a nonempty closed convex subset of a Hilbert space H and let T be a nonexpansive mapping of C into itself. If the set $F(T)$ of fixed points of T is nonempty, then the Cesáro means

$$S_n(x) = \frac{1}{n}\sum_{k=0}^{n-1} T^k x \tag{1.3}$$

converge weakly as $n \to \infty$ to a fixed point y of T for each $x \in C$. In this case, letting $y = Px$ for each $x \in C$, P is a nonexpansive retraction of C onto the fixed point set $F(T)$ of T such that $PT = TP = P$ and $Px \in \overline{\text{conv}}\{T^n x : n = 0, 1, 2, \dots\}$ for each $x \in C$, where $\overline{\text{conv}} A$ denotes the closure of the convex hull of A. The analogous results are given for nonexpansive semigroups by Baillon and Brézis [2] and Brézis and Browder [3]. In [13], Mizoguchi and Takahashi proved a nonlinear ergodic retraction theorem for Lipschitzian semigroups by using the notion of submean.

In this paper, we introduce the notion of asymptotically almost nonexpansive curves which include almost-orbits of commutative semigroups of asymptotically nonexpansive type mappings, and we prove nonlinear ergodic theorems for such curves. As applications of our main theorems, we obtain the results on the asymptotic behavior and ergodicity for a commutative semigroup of non-Lipschitzian mappings with nonconvex domains in a Hilbert space. Our results generalize and improve the previously known results of Baillon [1], Baillon and Brézis [2], Hirano and Takahashi [6], Ishihara and Takahashi [7], Lau, Nishiura, and Takahashi [10], Li and Ma [11, 12], Mizoguchi and Takahashi [13], Takahashi [14, 15], Takahashi and Zhang [16], and Tan and Xu [17] in many directions.

2. Preliminaries and notations

Throughout this paper, let H be a real Hilbert space with norm $\|\cdot\|$ and inner product $(\cdot,\cdot)$. Let G be a commutative semitopological semigroup with identity and let $m(G)$ be the Banach space of all bounded real-valued functions on G with the supremum norm. For each $s \in G$ and $f \in m(G)$, we define $r_s f$ in $m(G)$ given by

$$\big(r_s f\big)(t) = f(t+s) \quad \forall t \in G. \tag{2.1}$$

Let X be a subspace of $m(G)$ and μ be an element of X^* (the dual space of X). Then, we denote by $\mu(f)$ the value of μ at $f \in X$. To specify the variable t, we write the value $\mu(f)$ by $\mu(t)\langle f(t)\rangle$ or $\int f(t)d\mu(t)$. When X contains a constant 1, an element μ of X^* is called a mean on X if $\|\mu\| = \mu(1) = 1$. Further, let X be invariant under r_s for all $s \in G$. Then, a mean μ on X is said to be invariant if $\mu(r_s f) = \mu(f)$ for all $s \in G$ and $f \in X$. For $s \in G$, we can define a point evaluation δ_s by $\delta_s(f) = f(s)$ for every $f \in m(G)$. A convex combination of point evaluations is called a finite mean on G. Recently, the notion of the almost nonexpansive curve was introduced by Rouhani [5] and Kada and Takahashi [8].

Let $u(\cdot) : G \mapsto H$ be a function, in what follows we refer to such $u(\cdot)$ as a curve in H. A bounded function u is called an almost nonexpansive curve if there exists a function $\varepsilon : G \times G \to \mathbb{R}$ with $\lim_{s\cdot t\in G} \varepsilon(s,t) = 0$, such that

$$\left\|u(h+s)-u(h+t)\right\|^2 \le \left\|u(s)-u(t)\right\|^2 + \varepsilon(s,t) \quad \forall s,t,h \in G. \tag{2.2}$$

In the case $\varepsilon(s,t) = 0$ for all $s,t \in G$, u is called a nonexpansive curve.

Now, we define the concept of the asymptotically almost nonexpansive curve.

Definition 2.1. The curve $u(\cdot)$ is said to be asymptotically almost nonexpansive if the following conditions are satisfied:

(1) $\|u(h+t) - u(h+s)\|^2 \le \|u(t) - u(s)\|^2 + \varepsilon(t,s,h)$ for all $t,s,h \in G$, where $\varepsilon(t,s,h) \ge 0$ for all $t,s,h \in G$;

(2) for an arbitrary $\varepsilon > 0$ there exists $t_0 \in G$, and for each $t \succcurlyeq t_0$ there exists $h_t = h(\varepsilon,t) \in G$ such that

$$\varepsilon(t,s,h) < \varepsilon \quad \forall t \succcurlyeq t_0,\ s \succcurlyeq t_0,\ h \succcurlyeq h_t. \tag{2.3}$$

Note that, if $u(\cdot)$ is bounded then condition (1) is equivalent to

$$\|u(h+t)-u(h+s)\| \le \|u(t)-u(s)\| + \varepsilon_1(t,s,h) \quad \forall t,s,h \in G, \tag{2.4}$$

where $\varepsilon_1(t,s,h)$ satisfies the same condition (2) as $\varepsilon(t,s,h)$. We denote by $L(u)$ the following subset (possibly empty) of H:

$$L(u) = \left\{z \in H : \lim_{t\in G} \left\|u(t)-z\right\| \text{ exists}\right\}. \tag{2.5}$$

Throughout the rest of this paper, $u(\cdot)$ is a bounded asymptotically almost nonexpansive curve and X is a subspace of $m(G)$ containing constants invariant under r_s for each $s \in G$. Furthermore, suppose that for each $x \in H$, the function $t \mapsto \|u(t)-x\|^2$ is in X. Then by Riesz theorem, there exists a unique element u_μ in H such that

$$\mu_t\big(u(t),x\big) = \big(u_\mu,x\big) \quad \forall x \in H. \tag{2.6}$$

We denote u_μ by $\mu_t\langle u(t)\rangle$. If μ is a finite mean on G,

$$\mu = \sum_{i=1}^{n} a_i\delta_{t_i} \quad \left(t_i \in G,\ a_i \ge 0,\ 1 \le i \le n,\ \sum_{i=1}^{n} a_i = 1\right), \tag{2.7}$$

then

$$\mu_t\big\langle u(t)\big\rangle = \sum_{i=1}^{n} a_i u\big(t_i\big). \tag{2.8}$$

We denote by $\omega_w(u)$ the set of all weak limits of subnets of the net $\{u(t) : t \in G\}$.

3. Asymptotic behavior of curves

We begin with the following lemmas and proposition which play an important role in the proof of our main theorems.

LEMMA 3.1. *Let $u(\cdot)$ be a bounded asymptotically almost nonexpansive curve. Then the set $L(u)$ (possibly empty) is closed and convex.*

Proof. We can show the closedness from this inequality,

$$\begin{aligned}
&\big|\|u(t)-x\|-\|u(s)-x\|\big| \\
&\quad=\big|\|u(t)-x\|-\|u(t)-x_n\|+\|u(t)-x_n\|-\|u(s)-x_n\|+\|u(s)-x_n\|-\|u(s)-x\|\big| \\
&\quad\le\big|\|u(t)-x\|-\|u(t)-x_n\|\big|+\big|\|u(t)-x_n\|-\|u(s)-x_n\|\big|+\big|\|u(s)-x_n\|-\|u(s)-x\|\big| \\
&\quad\le 2\|x_n-x\|+\big|\|u(t)-x_n\|-\|u(s)-x_n\|\big|.
\end{aligned} \tag{3.1}$$

And also, the convexity follows from the equality

$$\begin{aligned}
\big\|u(t)-\big(\lambda q_1+(1-\lambda)q_2\big)\big\|^2=\lambda\big\|u(t)-q_1\big\|^2+(1-\lambda)\big\|u(t)-q_2\big\|^2 \\
-\lambda(1-\lambda)\big\|q_1-q_2\big\|^2.
\end{aligned} \tag{3.2}$$

□

PROPOSITION 3.2. *The set $\bigcap_{s\in G}\overline{\operatorname{conv}}\{u(t):t\succcurlyeq s\}\cap L(u)$ consists of at most one point.*

Proof. Suppose that $L(u)\neq\emptyset$. Let p be the unique asymptotic center of $\{u(t):t\in G\}$ in $L(u)$ and $x\in\bigcap_{s\in G}\overline{\operatorname{conv}}\{u(t):t\succcurlyeq s\}\cap L(u)$. We conclude the proof by showing that $x=p$. Since

$$\big\|u(t)-x\big\|^2=\big\|u(t)-p\big\|^2+\|x-p\|^2+2\big(u(t)-p,p-x\big), \tag{3.3}$$

we have

$$2\lim_{t\in G}\big(u(t)-p,p-x\big)+\|p-x\|^2\ge 0. \tag{3.4}$$

For any $\varepsilon>0$, there exists $t_0\in G$ such that

$$2\big(u(t)-p,p-x\big)+\|p-x\|^2\ge-\varepsilon\quad\forall t\succcurlyeq t_0. \tag{3.5}$$

Since $x\in\overline{\operatorname{conv}}\{u(t):t\succcurlyeq t_0\}$, it follows that

$$2(x-p,p-x)+\|p-x\|^2\ge-\varepsilon, \tag{3.6}$$

that is, $\|p-x\|^2\le\varepsilon$. Since $\varepsilon>0$ is arbitrary, we have $x=p$. This completes the proof.

□

Since G is commutative, there exists a net $\{\lambda_\alpha:\alpha\in I\}$ of finite means on G such that

$$\lim_{\alpha\in I}\big\|\lambda_\alpha-r_s^*\lambda_\alpha\big\|=0\quad\forall s\in G, \tag{3.7}$$

where I is a directed set and r_s^* is the conjugate of r_s (see [4]).

LEMMA 3.3. *$\lambda_\alpha(t)\langle u(t+h)\rangle$ converges weakly to an element p in $\bigcap_{s\in G}\overline{\text{conv}}\{u(t):t\succcurlyeq s\}$ $\cap L(u)$ uniformly in $h\in G$.*

Proof. For any $\{t_\alpha\}\in G$, let W be the set of all weak limit points of $\lambda_\alpha(t)\langle u(t+t_\alpha)\rangle$. In view of Proposition 3.2, it suffices to show that

$$W\subset\bigcap_{s\in G}\overline{\text{conv}}\{u(t):t\succcurlyeq s\}\cap L(u). \tag{3.8}$$

To show this, let $\{t_{\alpha_\beta}:\beta\in J\}$ be a subnet of $\{t_\alpha:\alpha\in B\}$ such that $\lambda_{\alpha_\beta}(t)\langle u(t+t_{\alpha_\beta})\rangle$ converges weakly to some z in H, where J is a directed set. For any $\varepsilon>0$, there exists $t_\varepsilon\in G$ such that for any $t\succcurlyeq t_\epsilon$, there exists $h_t\in G$ such that

$$\varepsilon(t,s,h)<\varepsilon\quad\forall t,s\succcurlyeq t_\epsilon,\ h\succcurlyeq h_t. \tag{3.9}$$

Then

$$\begin{aligned}
&\|u(h+t)-z\|^2-\|u(t)-z\|^2-2\Big(u(h+t)-u(t),\lambda_{\alpha_\beta}(s)\big\langle u\big(s+t_{\alpha_\beta}+t_\varepsilon\big)\big\rangle-z\Big)\\
&\quad=\Big(u(h+t)-u(t),u(h+t)+u(t)-2\lambda_{\alpha_\beta}(s)\big\langle u\big(s+t_{\alpha_\beta}+t_\varepsilon\big)\big\rangle\Big)\\
&\quad=\lambda_{\alpha_\beta}(s)\Big(\big\|u(h+t)-u\big(s+t_{\alpha_\beta}+t_\varepsilon\big)\big\|^2-\big\|u(t)-u\big(s+t_{\alpha_\beta}+t_\varepsilon\big)\big\|^2\Big)\\
&\quad\le\lambda_{\alpha_\beta}(s)\Big(\big\|u(h+t)-u\big(h+s+t_{\alpha_\beta}+t_\varepsilon\big)\big\|^2-\big\|u(t)-u\big(s+t_{\alpha_\beta}+t_\varepsilon\big)\big\|^2\Big)\\
&\qquad+4M^2\big\|\lambda_{\alpha_\beta}-r_h^*\lambda_{\alpha_\beta}\big\|\\
&\quad<\varepsilon+4M^2\big\|\lambda_{\alpha_\beta}-l_h^*\lambda_{\alpha_\beta}\big\|
\end{aligned} \tag{3.10}$$

for all $t\succcurlyeq t_\epsilon$ and $h\succcurlyeq h_t$, where $M=\sup_{t\in G}\|u(t)\|$. Note that $\lambda_{\alpha_\beta}(t)\langle u(t+t_{\alpha_\beta}+t_\varepsilon)\rangle$ converges weakly to z. For fixed $t\succcurlyeq t_\varepsilon$ and $h\succcurlyeq h_t$, taking the limit for $\beta\in J$, we have

$$\|u(h+t)-z\|^2-\|u(t)-z\|^2\le\varepsilon\quad\forall t\succcurlyeq t_\varepsilon,\ h\succcurlyeq h_t. \tag{3.11}$$

Therefore,

$$\inf_{s\in G}\sup_{\tau\succcurlyeq s}\|u(\tau)-z\|^2\le\|u(t)-z\|^2+\varepsilon\quad\forall t\succcurlyeq t_\varepsilon, \tag{3.12}$$

and hence

$$\inf_{s\in G}\sup_{\tau\succcurlyeq s}\|u(\tau)-z\|^2\le\sup_{s\in G}\inf_{\tau\succcurlyeq s}\|u(\tau)-z\|^2+\varepsilon. \tag{3.13}$$

Since $\varepsilon>0$ is arbitrary, we have $z\in L(u)$.

Now, we show that $z\in\bigcap_{s\in G}\overline{\text{conv}}\{u(t):t\succcurlyeq s\}$. For each $s\in G$, since $\lambda_{\alpha_\beta}(t)\langle u(t+t_{\alpha_\beta}+s)\rangle\in\overline{\text{conv}}\{u(t):t\succcurlyeq s\}$, we get $z\in\bigcap_{s\in G}\overline{\text{conv}}\{u(t):t\succcurlyeq s\}$. This completes the proof. □

Now, we can prove the ergodic convergence theorem for asymptotically almost nonexpansive curves.

A net $\{\mu_\alpha:\alpha\in A\}$ of continuous linear functionals on X is called strongly regular if it satisfies the following conditions:

(a) $\sup_{\alpha \in A} \|\mu_\alpha\| < +\infty$;
(b) $\lim_{\alpha \in A} \mu_\alpha(1) = 1$;
(c) $\lim_{\alpha \in A} \|\mu_\alpha - r_s^* \mu_\alpha\| = 0$ for every $s \in G$.

THEOREM 3.4. *Let $\{\mu_\alpha : \alpha \in A\}$ be a strongly regular net of continuous linear functional on X. Then there exists $p \in \bigcap_{s \in G} \overline{\text{conv}}\{u(t) : t \succcurlyeq s\} \cap L(u)$ such that*

$$w - \lim_{\alpha \in A} \int u(t+h)\, d\mu_\alpha(t) = p \quad \textit{uniformly in } h \in G. \tag{3.14}$$

Moreover, $u_\mu = p$ for each invariant mean μ.

Proof. By Lemma 3.3, there exists $p \in \bigcap_{s \in G} \overline{\text{conv}}\{u(t) : t \succcurlyeq s\} \bigcap L(u)$ and for any $\varepsilon > 0$ and $y_0 \in H$ with $\|y_0\| = 1$, there exists $\alpha_0 \in B$ such that

$$\left|\left(\lambda_{\alpha_0}(t)\langle u(t+h)\rangle - p, y_0\right)\right| < \frac{\varepsilon}{\sup_{\alpha \in A} \|\mu_\alpha\|} \quad \forall h \in G. \tag{3.15}$$

Suppose that

$$\lambda_{\alpha_0} = \sum_{i=1}^{n} a_i \delta_{t_i}, \quad t_i \in G,\ a_i \geq 0,\ i = 1, 2, \ldots, n,\ \sum_{i=1}^{n} a_i = 1. \tag{3.16}$$

Since $\{\mu_\alpha : \alpha \in A\}$ is strongly regular, there exists $\alpha_1 \in A$ such that

$$\begin{gathered} \left|\mu_\alpha(1) - 1\right| < \frac{\varepsilon}{(\|p\| + 1)}, \\ \left\|\mu_\alpha - r_{s_i}^* \mu_\alpha\right\| < \frac{\varepsilon}{M}, \quad 1 \leq i \leq n,\ \forall \alpha \succcurlyeq \alpha_1, \end{gathered} \tag{3.17}$$

where $M = \sup\{\|u(t)\| : t \in G\}$. Since for all $\alpha \succcurlyeq \alpha_1$, $h \in G$,

$$\begin{aligned} &\left|\left(\int \lambda_{\alpha_0}(t)\big\langle u(t+s+h)\big\rangle d\mu_\alpha(s) - p, y_0\right)\right| \\ &\quad = \left|\int \left(\lambda_{\alpha_0}(t)\big\langle u(t+s+h)\big\rangle - p, y_0\right) d\mu_\alpha(s) - \big(p, y_0\big)\big(\mu_\alpha(1) - 1\big)\right| \\ &\quad \leq \sup_{\alpha \in A} \left\|\mu_\alpha\right\| \sup_{s \in G} \left|\left(\lambda_{\alpha_0}(t)\big\langle u(t+s+h)\big\rangle - p, y_0\right)\right| + \varepsilon \leq 2\varepsilon, \\ &\left|\left(\int u(s+h)\, d\mu_\alpha(s), y_0\right) - \left(\int \lambda_{\alpha_0}(t)\big\langle u(t+s+h)\big\rangle d\mu_\alpha(s), y_0\right)\right| \\ &\quad = \left|\left(\int \left(u(s+h) - \sum_{i=1}^{n} a_i u\big(t_i + s + h\big)\right) d\mu_\alpha(s), y_0\right)\right| \\ &\quad \leq \sum_{i=1}^{n} a_i M \left\|\mu_\alpha - r_{t_i}^* \mu_\alpha\right\| < \varepsilon. \end{aligned} \tag{3.18}$$

Thus, we obtain, for all $\alpha \succcurlyeq \alpha_1, h \in G$,

$$\left|\left(\int u(s+h)\, d\mu_\alpha(s) - p, y_0\right)\right| < 3\varepsilon. \tag{3.19}$$

This completes the proof. □

Theorem 3.5. *Let $u(\cdot)$ be a bounded asymptotically almost nonexpansive curve. Then the following conditions are equivalent:*

(1) $w-\lim_{t\in G} u(t)$ *exists;*
(2) $\omega_w(u)\subset L(u)$*;*
(3) $w-\lim_{t\in G}(u(h+t)-u(t))=0$ *for every* $h\in G$.

Proof. (3)⇒(2). Let $\varepsilon>0$. Then there exists $t_\varepsilon\in G$ and for each $t\succcurlyeq t_\varepsilon$ there exists $h_t\in G$ such that

$$\varepsilon(t,s,h)<\varepsilon \quad \forall t,s\succcurlyeq t_\varepsilon,\ h\succcurlyeq h_t. \tag{3.20}$$

Let $z\in\omega_w(u)$. Then we can take a subnet $\{u(t_\alpha):\alpha\in J\}$ with $t_\alpha\succcurlyeq t_\varepsilon$ for each $\alpha\in J$ and

$$w-\lim_{\alpha\in J} u(t_\alpha)=z. \tag{3.21}$$

Since $L(u)$ is nonempty by Lemma 3.3, let $p\in L(u)$. Since for each $t\succcurlyeq t_\varepsilon$ and $h\succcurlyeq h_t$,

$$\begin{aligned}
&\|u(h+t_\alpha)-p\|^2-\|u(t_\alpha)-p\|^2+2(u(h+t_\alpha)-u(t_\alpha),p-z)\\
&\quad=\|u(h+t_\alpha)-z\|^2-\|u(t_\alpha)-z\|^2\\
&\quad=\|u(h+t_\alpha)-u(h+t)\|^2+\|u(h+t)-z\|^2\\
&\qquad+2(u(h+t_\alpha)-u(h+t),u(h+t)-z)-\|u(t_\alpha)-z\|^2\\
&\quad\le\|u(t_\alpha)-u(t)\|^2+\varepsilon+\|u(h+t)-z\|^2\\
&\qquad+2(u(h+t_\alpha)-u(h+t),u(h+t)-z)-\|u(t_\alpha)-z\|^2\\
&\quad=\|u(h+t)-z\|^2+\|u(t)-z\|^2+2(u(t_\alpha)-z,z-u(t))\\
&\qquad+2(u(h+t_\alpha)-u(h+t),u(h+t)-z)+\varepsilon,
\end{aligned} \tag{3.22}$$

for fixed $t\succcurlyeq t_\varepsilon$ and $h\succcurlyeq h_t$. Taking the limit for $\alpha\in J$, we have

$$\|u(h+t)-z\|^2\le\|u(t)-z\|^2+\varepsilon. \tag{3.23}$$

This implies $z\in L(u)$ in the same way as in Lemma 3.3.

(2)⇒(1). Since $\omega_w(u)\subset\bigcap_{s\in G}\overline{\text{conv}}\{u(t):t\ge s\}$, $\omega_w(u)$ is a singleton from Proposition 3.2. This implies (1) holds.

(1)⇒(3). It is clear. □

4. Asymptotic behavior of almost-orbits

In this section, using the main results in Section 3, we prove the ergodic theorems and weak convergence theorems for almost-orbits of commutative semigroups of asymptotically nonexpansive type mappings with nonconvex domains.

Let C be a nonempty subset of a Hilbert space H and $\Im = \{T(t) : t \in G\}$ be a family of mappings from C into itself. Recall that $\Im$ is said to be a commutative semigroup of asymptotically nonexpansive type mappings on C if the following conditions are satisfied:

(a) $T(t+s)x = T(t)T(s)x$ for all $t, s \in G$ and $x \in C$;

(b) for each $x \in C$ and $t \in G$, there exists $\alpha(t,x) \geq 0$ such that

$$\left\|T(t)x - T(t)y\right\| \leq \|x-y\| + \alpha(t,x) \quad \forall y \in C, \tag{4.1}$$

with

$$\lim_{t\in G} \alpha(t,x) = 0 \quad \forall x \in C. \tag{4.2}$$

A function $u(\cdot) : G \mapsto C$ is said to be an almost-orbit of $\Im = \{T(t) : t \in G\}$ if

$$\lim_{t\in G}\left[\sup_{h\in G}\left\|u(h+t) - T(h)u(t)\right\|\right] = 0. \tag{4.3}$$

Throughout the rest of this section, $\Im = \{T(t) : t \in G\}$ is a commutative semigroup of asymptotically nonexpansive type mappings on C, $u(\cdot) : G \mapsto C$ is a bounded almost-orbit of $\Im = \{T(t) : t \in G\}$, and X is a subspace of $m(G)$ containing constants invariant under r_s for each $s \in G$. Furthermore, suppose that for each $x \in H$, the function $t \mapsto \|u(t) - x\|^2$ is in X. Denote by $F(\Im)$ the set of common fixed points of $\Im = \{T(t) : t \in G\}$.

We begin with the following lemmas.

LEMMA 4.1. *Let $u(\cdot)$ be a bounded almost-orbit of the commutative semigroup $\Im = \{T(t) : t \in G\}$ of asymptotically nonexpansive type mappings on C. Then it is an asymptotically almost nonexpansive curve.*

Proof. Put $\varphi(t) = \sup_{h\in G}\|u(h+t) - T(h)u(t)\|$. Then $\lim_{t\in G}\varphi(t) = 0$. Since

$$\begin{aligned}\left\|u(h+t) - u(h+s)\right\| &\leq \left\|u(h+t) - T(h)u(t)\right\| + \left\|T(h)u(t) - T(h)u(s)\right\| \\ &\quad + \left\|u(h+s) - T(h)u(s)\right\| \\ &\leq \varphi(t) + \varphi(s) + \alpha\big(h, u(t)\big) + \left\|u(t) - u(s)\right\|,\end{aligned} \tag{4.4}$$

for every $h, t, s \in G$. It is easily seen that $u(\cdot)$ is an asymptotically almost nonexpansive curve. □

LEMMA 4.2. *If $u(\cdot)$ and $v(\cdot)$ are almost-orbits of $\Im$, then $\lim_{t\in G}\|u(t) - v(t)\|$ exists. Furthermore, we have $F(\Im) \subseteq L(u)$.*

Proof. Set

$$\begin{aligned}\varphi(t) &= \sup_{s\in G}\left\|u(s+t) - T(s)u(t)\right\|, \\ \psi(t) &= \sup_{s\in G}\left\|v(s+t) - T(s)v(t)\right\|.\end{aligned} \tag{4.5}$$

Then, $\lim_{t\in G}\varphi(t)=\lim_{t\in G}\psi(t)=0$. Since for each $t,s\in G$,

$$\begin{aligned}\|u(s+t)-v(s+t)\| &\le \|u(s+t)-T(s)u(t)\|+\|T(s)u(t)-T(s)v(t)\|\\ &\quad+\|v(s+t)-T(s)v(t)\|\\ &\le \varphi(t)+\psi(t)+\alpha(s,u(t))+\|u(t)-v(t)\|,\end{aligned} \tag{4.6}$$

$$\inf_{s\in G}\sup_{\tau\succcurlyeq s}\|u(\tau)-v(\tau)\|\le\varphi(t)+\psi(t)+\|u(t)-v(t)\|.$$

It follows that

$$\inf_{s\in G}\sup_{\tau\succcurlyeq s}\|u(\tau)-v(\tau)\|\le\sup_{s\in G}\inf_{\tau\succcurlyeq s}\|u(\tau)-v(\tau)\|, \tag{4.7}$$

which complete the proof of the first part. The second part is obvious. □

We can prove the following proposition from Lemma 4.2 and Proposition 3.2.

Proposition 4.3. *The set* $\bigcap_{s\in G}\overline{\operatorname{conv}}\{u(t):t\succcurlyeq s\}\cap F(\Im)$ *consists of at most one point.*

From Theorem 3.4, we can prove the following theorem which is an extension of the result of Tan and Xu [17] in many directions.

Theorem 4.4. *Let C be a nonempty subset of H, $\Im=\{T(t):t\in G\}$ a commutative semigroup of asymptotically nonexpansive type mappings on C, and $u(\cdot)$ be a bounded almost-orbit of $\Im$. If $\{\mu_\alpha:\alpha\in A\}$ is a strongly regular net of continuous linear functional on X, then*

$$w-\lim_{\alpha\in A}\int u(t+h)\,d\mu_\alpha(t)=p\in\bigcap_{s\in G}\overline{\operatorname{conv}}\{u(t):t\succcurlyeq s\}\bigcap L(u) \tag{4.8}$$

uniformly in $h\in G$. Further, if each $T(t)$ is continuous and $\bigcap_{s\in G}\overline{\operatorname{conv}}\{u(t):t\succcurlyeq s\}\subset C$, then $p\in F(\Im)$.

Proof. By Lemma 4.1 and Theorem 3.4, we need only to prove that if each $T(t)$ is continuous and $\bigcap_{s\in G}\overline{\operatorname{conv}}\{u(t):t\succcurlyeq s\}\subset C$, then $p\in F(\Im)$. By assumption, we have $p\in C$. Let $0<\varepsilon\le 1$. Then there exists $t_1\in G$ such that

$$\begin{gathered}\varphi(t)=\sup_{h\in G}\|T(h)u(t)-u(h+t)\|<\frac{\varepsilon}{8d},\\ \alpha(t,p)<\frac{\varepsilon}{8d},\end{gathered} \tag{4.9}$$

for each $t\succcurlyeq t_1$, where $d=1+\sup\{\|u(t)-p\|:t\in G\}$. Since

$$\begin{aligned}&\|T(s)p-p\|^2+2\big(u(s+t+t_1)-p,\,p-T(s)p\big)+\|u(s+t+t_1)-p\|^2\\ &\quad=\|u(s+t+t_1)-T(s)u(t+t_1)\|^2+\|T(s)u(t+t_1)-T(s)p\|^2\\ &\qquad+2\big(u(s+t+t_1)-T(s)u(t+t_1),\,T(s)u(t+t_1)-T(s)p\big)\\ &\quad\le\|u(t+t_1)-p\|^2+\varepsilon\end{aligned} \tag{4.10}$$

for $s \succcurlyeq t_1$, this implies that

$$\begin{aligned} &\|T(s)p-p\|^2\mu_\alpha(1)+2\mu_\alpha(t)\big(u\big(s+t+t_1\big)-p,\, p-T(s)p\big) \\ &\quad \leq \big(\mu_\alpha - r_s^*\mu_\alpha\big)(t)\big\|u\big(t+t_1\big)-p\big\|^2+\mu_\alpha(1)\varepsilon. \end{aligned} \tag{4.11}$$

Taking the limsup for $\alpha \in A$, we get

$$\big\|T(s)p-p\big\|^2 \leq \varepsilon \quad \forall s \succcurlyeq t_1. \tag{4.12}$$

It follows that $T(t)p$ is convergent strongly to p, therefore, $p \in F(\Im)$ by the continuity of $\{T(t) : t \in G\}$. This completes the proof. □

Let $\mathrm{AO}(\Im)$ be the set of all almost-orbits of $\Im$. Then for each $h \in G$ and $u \in \mathrm{AO}(\Im)$, the function $v : G \mapsto C$, defined by $v(t) = T(h)u(t)$, is also an almost-orbit of $\Im$. In fact, as before, we set $\varphi(t) = \sup_{s\in G}\|u(s+t)-T(s)u(t)\|$. Since

$$\begin{aligned} \big\|v(s+t)-T(s)v(t)\big\| &= \big\|T(h)u(s+t)-T(s)T(h)u(t)\big\| \\ &\leq \big\|T(h)u(s+t)-u(h+s+t)\big\| \\ &\quad + \big\|u(h+s+t)-T(s+h)u(t)\big\| \\ &\leq \varphi(s+t)+\varphi(t), \end{aligned} \tag{4.13}$$

the result follows.

Using Theorem 4.4, we have the following ergodic retraction theorem.

THEOREM 4.5. *Let C be a nonempty bounded subset of a Hilbert space H and let $\Im$ be a commutative semigroup of asymptotically nonexpansive type mappings on C such that each $T(t)$ is continuous. Then for an invariant mean μ, the mapping $P : u \mapsto u_\mu$ is a unique retraction from the set* $\mathrm{AO}(\Im)$ *onto $F(\Im)$ such that*

(1) *P is nonexpansive in the sense that*

$$\|Pu-Pv\| \leq \lim_{t\in G}\|u(t)-v(t)\|; \tag{4.14}$$

(2) *$PT(h)u = T(h)Pu = Pu$ for $u \in \mathrm{AO}(\Im)$ and $h \in G$;*

(3) *$Pu \in \bigcap_{s\in G}\overline{\mathrm{conv}}\{u(t) : t \succcurlyeq s\}$ for $u \in \mathrm{AO}(\Im)$.*

As a direct consequence of Theorem 3.5, we can prove the following theorem which is an extension of the Takahashi and Zhang [16]. Note that we do not assume $F(\Im)$ to be nonempty.

THEOREM 4.6. *Let C be a nonempty subset of a Hilbert space H and let $\Im$ be a commutative semigroup of asymptotically nonexpansive type mappings on C, and let*

$u(\cdot)$ be a bounded almost-orbit of $\Im$. Then $w-\lim_{t\in G} u(t)$ exists (in $L(u)$) if and only if $w-\lim_{t\in G}(u(h+t)-u(t))=0$ for all $h\in G$.

Acknowledgement

This work was supported by Korea Research Foundation Grant (KRF-99-041-D00025).

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