THE EFFECT OF THE GRAPH TOPOLOGY ON THE EXISTENCE OF MULTIPEAK SOLUTIONS FOR NONLINEAR SCHRÖDINGER EQUATIONS

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1. Introduction

Consider the problem

$$\begin{aligned} -\varepsilon^2 \Delta u + V(y)u &= u^{p-1}, \quad y \in \mathbb{R}^N, \\ u &> 0, \quad y \in \mathbb{R}^N, \\ u &\longrightarrow 0, \quad \text{as } |y| \longrightarrow +\infty, \end{aligned} \tag{1.1}$$

where $V(y)$ is a smooth bounded function with positive lower bound, $\varepsilon > 0$ is a small number, $2 < p < 2N/(N-2)$ if $N > 2$ and $2 < p < +\infty$ if $N = 2$.

Many works have been done on problem (1.1) recently (cf. [6, 7, 8, 16, 21, 22, 23]). One of the results in the papers just mentioned is that if $x_1, x_2, \ldots, x_k$ are k different strictly local minimum points of $V(y)$, then (1.1) has a k-peak solution u_ε, that is, solution with exactly k local maximum points, such that u_ε has exactly one local maximum point in a neighborhood of x_j, $j = 1, \ldots, k$. The same conclusion is also true if $x_1, x_2, \ldots, x_k$ are k different strictly local maximum points of $V(y)$. Actually, it is proved in [23] that (1.1) has a multipeak solution with all its peaks near an isolated maximum point of $V(y)$. Thus a natural question is what will happen if $V(y)$ attains its local minimum or local maximum on a connected set. Especially, if $V(y)$ attains its local minimum on a connected set which contains infinitely many points, it is interesting to study whether (1.1) has multipeak solution concentrating on this set. Generally, this is not true as shown in Example 1.6.

The main results of this paper consist of three parts. First, we study how the topological structure of the local minimum set of the potential $V(y)$ affects the existence of multipeak solutions for (1.1). We show that if the minimum set of $V(y)$ has nontrivial reduced homology, then for each $k \geq 1$, (1.1) has at least

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one k-peak solution such that each local maximum point of this solution tends to a point in this minimum set as $\varepsilon \to 0$.

Second, we construct solutions with their peaks near a connected maximum set of $V(y)$. Unlike the case of minimum sets, we show that for any connected maximum set of $V(y)$ and for any positive integer $k \geq 2$, (1.1) always has at least N different solutions with all their peaks tending to this maximum set as $\varepsilon \to 0$.

Suppose that M_1 and M_2 are two disjoint connected sets such that $V(y)$ attains a local minimum or a local maximum on M_1 and on M_2. The third problem studied in this paper is to construct a (k_1+k_2)-peak solution u_ε such that u_ε has exactly k_i local maximum points near M_i, $i = 1, 2$.

Before we state our results precisely, we give some notations first.

For any constant $\bar{V} > 0$, let $U_{\bar{V}}(y)$ be the unique solution of

$$\begin{aligned} -\Delta u + \bar{V}u &= u^{p-1}, \quad y \in \mathbb{R}^N, \\ u &> 0, \quad y \in \mathbb{R}^N, \\ u \in H^1\big(\mathbb{R}^N\big), &\qquad u(0) = \max_{y\in\mathbb{R}^N} u(y). \end{aligned} \tag{1.2}$$

Let w be the unique solution of

$$\begin{aligned} -\Delta u + u &= u^{p-1}, \quad y \in \mathbb{R}^N, \\ u &> 0, \quad y \in \mathbb{R}^N, \\ u \in H^1\big(\mathbb{R}^N\big), &\qquad u(0) = \max_{y\in\mathbb{R}^N} u(y). \end{aligned} \tag{1.3}$$

Then $U_{\bar{V}}(y) = \bar{V}^{1/(p-2)} w(\bar{V}y)$. Denote $U_{\varepsilon,z,\bar{V}}(y) = U_{\bar{V}}((y-z)/\varepsilon)$.

For any fixed integer $k > 0$ and $V_j > 0$, $j = 1, \ldots, k$, we denote

$$\begin{aligned} E_{\varepsilon,x,k} = \Bigg\{ v \in H^1\big(\mathbb{R}^N\big) : \big\langle U_{\varepsilon,x_j,V_j}, v\big\rangle_\varepsilon = \left\langle \frac{\partial U_{\varepsilon,x_j,V_j}}{\partial x_{ji}}, v \right\rangle_\varepsilon = 0, \\ j = 1, \ldots, k, i = 1, \ldots, N \Bigg\}, \end{aligned} \tag{1.4}$$

where $\langle u, v\rangle_\varepsilon = \int_{\mathbb{R}^N} \varepsilon^2 Du Dv + V(y)uv$. We also denote $\|v\|_\varepsilon^2 = \int_{\mathbb{R}^N} \varepsilon^2 |Du|^2 + V(y)v^2$.

Definition 1.1. Let M be a connected compact set in $\mathbb{R}^N$. M is said to be a local minimum (maximum) set of $V(y)$ if there are constants $\gamma > 0$ and V_M, such that $V(y) = V_M$ for $y \in M$, $V(y) > V_M (V(y) < V_M)$ for $y \in M_\gamma \setminus M$, where $M_\gamma = \{z : z \in \mathbb{R}^N, d(z, M) \leq \gamma\}$.

Throughout this paper, all the homologies are with $\mathbb{Z}_2$-coefficients. Now we are ready to state our main results.

THEOREM 1.2. *Suppose that M is a connected compact local minimum set of $V(y)$ such that ∂M_γ is a C^1-manifold of dimension $N-1$ for each $\gamma > 0$ small. Moreover, we assume that for any $\gamma > 0$ small, the following conditions hold:*

$$V(y) \le V_M + ad(y,M)^h, \qquad \left|D^m V(y)\right| = O\left(d(y,M)^{h-m}\right), \tag{1.5}$$

for $m = 1, \dots, [h]$, $y \in M_\gamma$,

$$\langle DV(y), n\rangle \ge c_0 \gamma^{h-1}, \quad \forall y \in \partial M_\gamma, \tag{1.6}$$

where a, c_0, and $h \ge 2$ are some positive constants, n is the outward unit normal of ∂M_γ at y. If the reduced homology of M is nontrivial, then for each integer $k \ge 2$, there is an $\varepsilon_0 > 0$, such that for every $\varepsilon \in (0, \varepsilon_0]$, (1.1) has at least one k-peak solution of the form

$$u_\varepsilon = \sum_{j=1}^{k} \alpha_{\varepsilon,j} U_{\varepsilon,x_{\varepsilon,j},V_M} + v_\varepsilon, \tag{1.7}$$

where $v_\varepsilon \in E_{\varepsilon,x,k}$ and as $\varepsilon \to 0$,

$$\alpha_{\varepsilon,j} \longrightarrow 1, \qquad \frac{\left|x_{\varepsilon,i} - x_{\varepsilon,j}\right|}{\varepsilon} \longrightarrow \infty, \quad x_{\varepsilon,j} \longrightarrow x_j \in M, \qquad \left\|v_\varepsilon\right\|_\varepsilon^2 = o\left(\varepsilon^N\right), \tag{1.8}$$

for $i, j = 1, \dots, k$ and $i \ne j$. Moreover, if $k = 2$, (1.1) has at least $\operatorname{cuplength}(M)$ *distinct solutions of the form (1.7) satisfying (1.8).*

For any set M, denote

$$A_k = \left(\underbrace{M_\gamma \times \cdots \times M_\gamma}_{k} \setminus \bigcup_{i \ne j} \left\{ \left|x_i - x_j\right| < d \right\} \right) / \sigma_k, \tag{1.9}$$

where $d > 0$ and $\gamma > 0$ are small constants, σ_k is the group of permutations of k letters acting on $\underbrace{M_\gamma \times \cdots \times M_\gamma}_{k} \setminus \bigcup_{i \ne j} \{|x_i - x_j| < d\}$.

THEOREM 1.3. *Suppose that M is a connected compact local maximum set of $V(x)$. Then for each positive integer $k \ge 2$, there is an $\varepsilon_0 > 0$, such that for every $\varepsilon \in (0, \varepsilon_0]$, (1.1) has at least* $\operatorname{Cat}_{A_k}(A_k)$ *solutions of the form*

$$u_\varepsilon = \sum_{j=1}^{k} \alpha_{\varepsilon,j} U_{\varepsilon,x_{\varepsilon,j},V_M} + v_\varepsilon, \tag{1.10}$$

where $v_\varepsilon \in E_{\varepsilon,x,k}$ *and as* $\varepsilon \to 0$,

$$\alpha_{\varepsilon,j} \longrightarrow 1, \qquad \frac{|x_{\varepsilon,i} - x_{\varepsilon,j}|}{\varepsilon} \longrightarrow \infty, \quad x_{\varepsilon,j} \longrightarrow x_j \in M, \qquad \|v_\varepsilon\|_\varepsilon^2 = o(\varepsilon^N), \tag{1.11}$$

for $i, j = 1, \dots, k$ *and* $i \neq j$.

THEOREM 1.4. *Let* $M_1, \dots, M_l$ *be disjoint connected compact sets such that* $\partial M_{i,\gamma}$ *is a* C^1*-manifold of dimension* $N-1$ *for each* $\gamma > 0$ *small and* $i = 1, \dots, l$. *Suppose that the following conditions hold: there are constants* $h \geq 2$, $c_1 \geq c_0 > 0$ *such that*

$$\left.\begin{aligned} V_{M_i} \leq V(y) \leq V_{M_i} + c_1 d(y, M_i)^h, \\ |D^m V(y)| = O\big(d(y, M_i)^{h-m}\big), \end{aligned}\right\} \quad \forall y \in M_{i,\gamma},\ m = 1, \dots, [h],\ i = 1, \dots, l_1,$$

$$\langle DV(y), n\rangle \geq c_0 \gamma^{h-1}, \quad \forall y \in \partial M_{i,\gamma},\ i = 1, \dots, l_1,$$

$$\left.\begin{aligned} c_0 d(y, M_i)^h \leq V_{M_i} - V(y) \leq c_1 d(y, M_i)^h, \\ |D^m V(y)| = O\big(d(y, M_i)^{h-m}\big), \end{aligned}\right\} \quad \begin{aligned} &\forall y \in M_{i,\gamma},\ m = 1, \dots, [h], \\ &i = l_1 + 1, \dots, l. \end{aligned} \tag{1.12}$$

(i) *If each* M_i, $i = 1, \dots, l_1$, *has nontrivial reduced homology, then for any positive integer* k_i, $i = 1, \dots, l$, *there is an* $\varepsilon_0 > 0$ *such that for each* $\varepsilon \in (0, \varepsilon_0]$, *(1.1) has at least one solution of the form*

$$u_\varepsilon = \sum_{i=1}^{l} \sum_{j=1}^{k_i} \alpha_{\varepsilon,i,j} U_{\varepsilon, x_{\varepsilon,i,j}, V_{M_i}} + v_\varepsilon, \tag{1.13}$$

where $v_\varepsilon \in E_{\varepsilon,x,\sum_{i=1}^{l} k_i}$ *and as* $\varepsilon \to 0$,

$$\begin{gathered} \alpha_{\varepsilon,i,j} \longrightarrow 1, \qquad \|v_\varepsilon\|_\varepsilon^2 = o(\varepsilon^N), \\ \frac{|x_{\varepsilon,i,j} - x_{\varepsilon,i,m}|}{\varepsilon} \longrightarrow \infty, \quad x_{\varepsilon,i,j} \longrightarrow x_{i,j} \in M_i, \end{gathered} \tag{1.14}$$

for $i = 1, \dots, l,\ j, m = 1, \dots, k_i$, *and* $j \neq m$.

(ii) *If some of the minimum sets* $M_{i_1}, \dots, M_{i_t}$ *have trivial reduced homology, then the conclusion in (i) holds for* $k_{i_j} = 1,\ j = 1, \dots, t$.

The basic idea to prove Theorem 1.2 can also be used to obtain the following result.

THEOREM 1.5. *Let M be a connected compact local minimum or maximum set. Then there is an $\varepsilon_0 > 0$, such that for each $\varepsilon \in (0, \varepsilon_0]$, (1.1) has at least $\mathrm{Cat}_M(M)$ single peak positive solutions of the form*

$$u_\varepsilon = \alpha_\varepsilon U_{\varepsilon, x_\varepsilon, V_M} + v_\varepsilon, \tag{1.15}$$

where $v_\varepsilon \in E_{\varepsilon, x_\varepsilon, 1}$, and as $\varepsilon \to 0$, $\alpha_\varepsilon \to 1$, $x_\varepsilon \to x_0 \in M$, and $\|v_\varepsilon\|_\varepsilon^2 = o(\varepsilon^N)$.

The assumption that M has nontrivial topology is essential in Theorem 1.2, as shown by the following example.

Example 1.6. Let $V(y) = V(|y|)$ be a smooth function satisfying $V(y) = 1$ for $|y| \le 1$, and $V(|y|)$ is strictly increasing in $|y| > 1$. Then we see that the minimum set of $V(y)$ is the unit ball in $\mathbb{R}^N$. By the moving plane method of Gidas, Ni, and Nirenberg [19], we know that every solution of (1.1) is radially symmetric and strictly decreasing. Therefore, the solution has a unique maximum point, and thus is a single peak solution.

Example 1.7. A typical example of $V(y)$ satisfying the conditions in Theorem 1.4 is that $V(y) = V(|y|)$ and $V(r)$ attains its local minimum or maximum at $r_l > \cdots > r_1 > 0$ with $V''(r_i) \neq 0$, $i = 1, \ldots, l$. In this case, $M_i = \{y : |y| = r_i\}$, $h = 2$.

Remark 1.8. If M is an N-dimensional smooth manifold without boundary, tubular neighborhoods of M are useful for verifying the conditions in Theorem 1.4.

Remark 1.9. By Proposition C.3, we know that a lower bound for $\mathrm{Cat}_{A_k} A_k$ is N. So (1.1) has at least N different k-peak solutions concentrating on the connected compact local maximum set of $V(y)$.

Since the work by Bahri and Coron [2], the effect of the domain topology on the existence and multiplicity of the solutions is one of the subjects which attract much attention. See, for example, [2, 3, 4, 10, 11, 12, 14, 15]. In [3], the category of the domain was used to estimate the number of the single peak solutions, while in [12, 14, 15], the effect of the domain topology on the existence of multipeak solutions was studied. The domain in problem (1.1) has trivial topology, so our results here emphasize the effect of the topology of the level set of the potential $V(y)$ on the existence and multiplicity of multipeak solutions for (1.1).

Finally, we point out that the idea in this paper works for the singularly perturbed Neumann problem

$$-\varepsilon^2 \Delta u + u = u^{p-1} \quad \text{in } \Omega, \qquad u > 0 \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial n} = 0 \quad \text{in } \partial\Omega, \tag{1.16}$$

where Ω is a bounded domain in $\mathbb{R}^N$. The role of the mean curvature function of the boundary $\partial\Omega$ in (1.16) is similar to that of the potential $V(x)$ in (1.1).

The estimates in Appendix C can be used to improve the multiplicity results in [13, 25]. For example, as a direct corollary of Proposition C.3 and the results in [13, 25], we have the following corollary.

COROLLARY 1.10. *For each integer $k \geq 2$, there is $\varepsilon_0 > 0$ such that for each $\varepsilon \in (0, \varepsilon_0]$,*

(i) *equation (1.16) has at least N boundary k-peak solutions with all their local maximum points near the global minimum set of the mean curvature function of $\partial\Omega$;*
(ii) *equation (1.16) has at least N interior k-peak solutions.*

This paper is arranged as follows. In Section 2, we reduce the problem of finding a multipeak solution for (1.1) to a finite-dimensional problem. Theorem 1.2 is proved in Section 3. Section 4 is devoted to the proof of Theorems 1.3 and 1.5. In Section 5, we prove Theorem 1.4. Some basic estimates and the topological results needed in the proof of the main results are presented in the appendices.

2. Reduction to finite-dimensional problem

First, we define

$$D_\varepsilon = \Big\{(\alpha, x, v) : \big|\alpha_j - 1\big| \leq \delta,\ v \in E_{\varepsilon,x,k},\ \|v\|_\varepsilon \leq \delta\varepsilon^{N/2},\ x = \big(x_1, \ldots, x_k\big),\ x_j \in \mathbb{R}^N,\ \frac{\big|x_i - x_j\big|}{\varepsilon} \geq R,\ i, j = 1, \ldots, k,\ i \neq j\Big\}, \tag{2.1}$$

where $\delta > 0$ is a fixed small constant and $R > 0$ is a fixed large constant.

We also define

$$J(\alpha, x, v) = I\left(\sum_{j=1}^{k} \alpha_j U_{\varepsilon,x_j,V_j} + v\right), \quad \forall (\alpha, x, v) \in D_\varepsilon, \tag{2.2}$$

where

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} \big(\varepsilon^2 |Du|^2 + V(y)u^2\big) - \frac{1}{p}\int_{\mathbb{R}^N} |u|^p. \tag{2.3}$$

It is well known now (see [1, 24]) that, if $\delta > 0$ is small enough and $R > 0$ is large enough,

$$u = \sum_{j=1}^{k} \alpha_j U_{\varepsilon,x_j,V_j} + v \tag{2.4}$$

is a positive critical point of $I(u)$ if and only if (α, x, v) is a critical point of $J(\alpha, x, v)$ in D_ε. So we need to solve the following system:

$$\frac{\partial J}{\partial \alpha_j} = 0, \quad j = 1, \dots, k, \tag{2.5}$$

$$\frac{\partial J}{\partial x_{jl}} = \sum_{h=1}^{N} G_{hj} \left\langle \frac{\partial^2 U_{\varepsilon, x_j, V_j}}{\partial x_{jh} \partial x_{jl}}, v \right\rangle_\varepsilon, \quad j = 1, \dots, k,\ l = 1, \dots, N, \tag{2.6}$$

$$\frac{\partial J}{\partial v} = \sum_{j=1}^{k} B_j U_{\varepsilon, x_j, V_j} + \sum_{j=1}^{k} \sum_{l=1}^{N} G_{jl} \frac{\partial U_{\varepsilon, x_j, V_j}}{\partial x_{jl}}, \tag{2.7}$$

for some constants $B_j, G_{jl} \in \mathbb{R}$, $j = 1, \dots, k$, $l = 1, \dots, N$.

In this section, we reduce the problem of solving the system (2.5), (2.6), and (2.7) to a finite-dimensional problem. We need the following proposition.

PROPOSITION 2.1. *There is an $\varepsilon_0 > 0$, such that for each $\varepsilon \in (0, \varepsilon_0]$, there exists a unique C^1-map $(\alpha_\varepsilon(x), v_\varepsilon(x)) : \mathbb{R}^{kN} \setminus \cup_{i \neq j} \{|x_i - x_j| \le \varepsilon R\} \to \mathbb{R}^k_+ \times H^1(\mathbb{R}^N)$ such that $v_\varepsilon \in E_{\varepsilon, x, k}$, (2.5) and (2.7) hold for some constants B_j and G_{jl}. Moreover,*

$$v_\varepsilon(\sigma_k x) = v_\varepsilon(x), \qquad \sigma_k \alpha_\varepsilon(\sigma_k x) = \alpha_\varepsilon(x), \tag{2.8}$$

$$\begin{aligned} |\alpha - 1|\varepsilon^{N/2} + \|v\|_\varepsilon = {} & O\left(\sum_{j=1}^{k} \left(\varepsilon^{N/2} \left| V(x_j) - V_j \right| + \sum_{m=1}^{[h]} \varepsilon^{N/2+m} \left| D^m V(x_j) \right| \right) \right) \\ & + O\left(\varepsilon^{N/2} \sum_{i \neq j} w^{(1+\tau)/2} \left(\frac{|x_i - x_j|}{\varepsilon} \right) \right), \end{aligned} \tag{2.9}$$

$$\begin{aligned} \varepsilon B_j,\ G_{jl} = {} & O\left(\varepsilon \sum_{j=1}^{k} \left(\left| V(x_j) - V_j \right| + \sum_{m=1}^{[h]} \varepsilon^m \left| D^m V(x_j) \right| \right) \right) \\ & + O\left(\varepsilon \sum_{i \neq j} w^{(1+\tau)/2} \left(\frac{|x_i - x_j|}{\varepsilon} \right) \right), \end{aligned} \tag{2.10}$$

where $\tau > 0$ is a fixed small constant.

Proof. We can follow the same procedure as in [5] to prove the existence part. Equation (2.8) is a direct consequence of the fact $J(\alpha, x, v) = J(\sigma_k \alpha, \sigma_k x, v)$ and the uniqueness of $(\alpha_\varepsilon(x), v_\varepsilon(x))$ satisfying (2.5) and (2.7). To get the estimate (2.9), we just need to use Lemma A.3. We can solve a system as in [24, pages 22–23] and use Lemma A.4 to get the estimate (2.10). Since the procedure is quite standard, we omit the details. □

Let $(\alpha_\varepsilon(x), v_\varepsilon(x))$ be the map obtained in Proposition 2.2. Define

$$K(x) = J\big(\alpha_\varepsilon(x), x, v_\varepsilon(x)\big). \tag{2.11}$$

In order to solve (2.5), (2.6), and (2.7), we only need to find a critical point for $K(x)$ in a suitable domain. So we need the following propositions.

PROPOSITION 2.2. *Suppose that $F(x)$ is a C^2-function defined in a bounded domain Ω of $\mathbb{R}^{kN}$. If F satisfies either $F(x) > c$ or $\partial F(x)/\partial n > 0$ at each $x \in \partial\Omega$, where n is the outward unit normal of $\partial\Omega$ at x, then*

$$\#\big\{x : DF(x) = 0, x \in F^c\big\} \geq \mathrm{Cat}_{F^c}\big(F^c\big), \tag{2.12}$$

where $F^c = \{x : x \in \Omega, F(x) \leq c\}$. In particular, $F(x)$ has at least one critical point in F^c.

Proof. Notice that our assumption implies that the flow

$$\frac{dx(t)}{dt} = -DF\big(x(t)\big), \quad x(0) = x_0 \in F^c, \tag{2.13}$$

does not leave Ω. In fact, suppose that $x(t)$ touches the boundary at some time t_0. Since F is decreasing along $x(t)$, then $F(x(t_0)) \leq c$. Thus, by assumption, $\partial F(x(t_0))/\partial n > 0$, which implies that $-DF(x(t_0))$ points into Ω. So $x(t)$ moves into Ω. Then Proposition 2.2 follows directly from the Ljusternik-Schnirelman theory. □

PROPOSITION 2.3. *Suppose that $F(x)$ is a C^2-function defined in a bounded domain Ω of $\mathbb{R}^{kN}$. Let $c_2 > c_1$ be two constants such that neither c_2 nor c_1 is a critical value of $F(x)$. If F satisfies either $F(x) < c_1$ or $\partial F(x)/\partial n > 0$ for each $x \in \partial\Omega$, then*

$$\#\big\{x : DF(x) = 0,\ x \in F^{c_2} \setminus F^{c_1}\big\} \geq \mathrm{Cat}_{F^{c_2}}\big(F^{c_2}, F^{c_1}\big). \tag{2.14}$$

In particular, if F^{c_2} cannot be deformed into F^{c_1}, F has at least one critical point in $F^{c_2} \setminus F^{c_1}$.

Proof. Similar to Proposition 2.2, our assumption implies that the

$$\frac{dx(t)}{dt} = -DF\big(x(t)\big), \quad x(0) = x_0 \in F^{c_2}, \tag{2.15}$$

does not leave Ω before it reaches F^{c_1}. So Proposition 2.2 follows directly from the Ljusternik-Schnirelman theory. □

3. Multipeak solutions concentrating on the minimum set

Suppose that M is a connected compact local minimum set of $V(y)$ and $M_\gamma = \{y : d(y, M) \le \gamma\}$. For $k \ge 2$, let

$$\Omega_\gamma = \left\{ x = (x_1, \dots, x_k) : x_j \in M_\gamma,\ j = 1, \dots, k,\ \frac{|x_i - x_j|}{\varepsilon} \ge R, \forall i \ne j \right\}. \quad (3.1)$$

Define

$$c_{\varepsilon,1} = \varepsilon^N \left(k V_M^{p/(p-2)-N/2} A - T\varepsilon^{\alpha h} \right), \qquad c_{\varepsilon,2} = \varepsilon^N \left(k V_M^{p/(p-2)-N/2} A + \eta \right), \quad (3.2)$$

where $T > 0$ is a large constant and $\alpha \in (0, 1)$ is a fixed constant close to 1. We also let

$$K^c = \left\{ x : x \in \Omega_{\varepsilon^\alpha}, K(x) \le c \right\}. \quad (3.3)$$

In this section, we apply Proposition 2.3 to prove that for $\varepsilon > 0$ small, $K(x)$ has a critical point in $K^{c_{\varepsilon,2}} \setminus K^{c_{\varepsilon,1}}$. First, we prove the following lemma.

LEMMA 3.1. *For each $x \in \partial\Omega_{\varepsilon^\alpha}$, either $K(x) < c_{\varepsilon,1}$, or $\partial K(x)/\partial n > 0$.*

Proof. We divide the proof of this lemma into two steps.

Step 1. Suppose that $|x_i - x_j|/\varepsilon = R$ for some $i \ne j$. We claim that $x \in K^{c_{\varepsilon,1}}$.

In fact, by using Proposition 2.1 and Lemma A.2, we obtain

$$\begin{aligned}
K(x) &= I\left(\sum_{l=1}^k U_{\varepsilon,x_l,V_M} \right) + O\left(\varepsilon^N |\alpha_\varepsilon - 1|^2 + \|v\|_\varepsilon^2 \right) \\
&= \varepsilon^N \sum_{j=1}^k V_M^{p/(p-2)-N/2} A - \sum_{i=1}^{k-1} \int_{\mathbb{R}^N} \left(\sum_{j=i+1}^k U_{\varepsilon,x_j,V_M} \right)^{p-1} U_{\varepsilon,x_i,V_M} \\
&\quad + O\left(\sum_{l=1}^k \left(\varepsilon^N |V(x_l) - V_M| + \sum_{m=1}^{[h]} \varepsilon^{N+m} |D^m V(x_l)| \right) \right) \\
&\quad + O\left(\varepsilon^N \sum_{i \ne j} w^{1+\tau} \left(\frac{|x_i - x_j|}{\varepsilon} \right) \right) \\
&= \varepsilon^N \sum_{j=1}^k V_M^{p/(p-2)-N/2} A - \sum_{i=1}^{k-1} \int_{\mathbb{R}^N} \left(\sum_{j=i+1}^k U_{\varepsilon,x_j,V_M} \right)^{p-1} U_{\varepsilon,x_i,V_M} \\
&\quad + O\left(\varepsilon^{N+\alpha h} + \varepsilon^N \sum_{i \ne j} w^{1+\tau} \left(\frac{|x_i - x_j|}{\varepsilon} \right) \right).
\end{aligned} \quad (3.4)$$

From (3.4), we have

$$K(x) \le \varepsilon^N \left(k V_M^{p/(p-2)-N/2} A - c w(R) \right) + O\left(\varepsilon^{N+\alpha h}\right) < c_{\varepsilon,1}. \tag{3.5}$$

Thus $x \in K^{c_{\varepsilon,1}}$.

Step 2. Suppose that $x_i \in \partial M_{\varepsilon^\alpha}$ for some i. Without loss of generality, we assume that $i = 1$. We claim that either $K(x) < c_{\varepsilon,1}$, or $\partial K/\partial n > 0$, where n is the outward unit normal of $\partial M_{\varepsilon^\alpha}$ at x_1.

For any $x_j \in M_{\varepsilon^\alpha}$ and $m \ge 2$, we have

$$\begin{aligned} &\left|V(x_j) - V_M\right| = O\left(\varepsilon^{\alpha h}\right), \\ &\left|D^m V(x_j)\right| \varepsilon^m = O\left(d(x_j, M)^{h-m} \varepsilon^m\right) = O\left(\varepsilon^{\alpha(h-m)} \varepsilon^m\right) = O\left(\varepsilon^{\alpha h + 2(1-\alpha)}\right). \end{aligned} \tag{3.6}$$

So, by Proposition 2.1 and Lemma A.4, we have

$$\begin{aligned} \frac{\partial K}{\partial x_{1l}} &= \frac{\partial J}{\partial x_{1l}} + \left\langle \frac{\partial J}{\partial v}, \frac{\partial v}{\partial x_{1l}} \right\rangle_\varepsilon = \frac{\partial J}{\partial x_{1l}} + \sum_{j=1}^{k} \sum_{h=1}^{N} G_{hj} \left\langle \frac{\partial U_{\varepsilon,x_j,V_M}}{\partial x_{jh}}, \frac{\partial v}{\partial x_{1l}} \right\rangle_\varepsilon \\ &= \frac{\partial J}{\partial x_{1l}} - \sum_{h=1}^{N} G_{1h} \left\langle \frac{\partial^2 U_{\varepsilon,x_1,V_M}}{\partial x_{h1} \partial x_{1l}}, v \right\rangle_\varepsilon \\ &= \frac{\partial J}{\partial x_{1l}} + O\left(\varepsilon^{N-1} \sum_{i \ne j} e^{-(1+\tau)(|x_i - x_j|/\varepsilon)} + \varepsilon^{N+\alpha h} \right) \\ &= c_1 \varepsilon^N D_l V(x_1) - (p-1) \sum_{j=2}^{k} \int_{\mathbb{R}^N} U_{\varepsilon,x_1,V_M}^{p-2} U_{\varepsilon,x_j,V_M} \frac{\partial U_{\varepsilon,x_1,V_M}}{\partial x_{1l}} \\ &\quad + O\left(\varepsilon^{N-1} \sum_{i \ne j} e^{-(1+\tau)(|x_i - x_j|/\varepsilon)} + \varepsilon^{N-1+\alpha h + 2(1-\alpha)} \right). \end{aligned} \tag{3.7}$$

Let $\eta = \min_{j \ne i} |x_i - x_j|$. We distinguish two cases.

(i) Suppose that $U(\eta/\varepsilon) > L\varepsilon^{\alpha h}$, where $L > T$ is a large constant. In this case, we claim that $K(x) < c_{\varepsilon,1}$. In fact, it follows from (3.4) that

$$\begin{aligned} K(x) &\le \varepsilon^N k V_M^{p/(p-2)-N/2} A - c' \varepsilon^N w\left(\frac{\eta}{\varepsilon}\right) + O\left(\varepsilon^{N+\alpha h}\right) \\ &\le \varepsilon^N k V_M^{p/(p-2)-N/2} A - c' L \varepsilon^{N+\alpha h} + C \varepsilon^{N+\alpha h} < c_{\varepsilon,1}, \end{aligned} \tag{3.8}$$

if $L > T$ is large enough.

(ii) Suppose that $U(\eta/\varepsilon) \le L\varepsilon^{\alpha h}$. In this case, we prove that $\partial K(x)/\partial n > 0$, where n is the outward unit normal of $\partial M_{\varepsilon^\alpha}$ at x_1. Since for any $j \ne 1$,

$$\int_{\mathbb{R}^N} U_{\varepsilon,x_1,V_M}^{p-2} U_{\varepsilon,x_j,V_M} \frac{\partial U_{\varepsilon,x_1,V_M}}{\partial x_{1l}} = \big(c+o(1)\big)\varepsilon^{N-1} w\left(\frac{|x_j-x_1|}{\varepsilon}\right)\frac{x_{jl}-x_{1l}}{|x_j-x_1|}, \tag{3.9}$$

and for small $\beta > 1-\alpha$,

$$\left\langle \frac{x_j-x_1}{|x_j-x_1|}, n\right\rangle \le \varepsilon^\beta, \quad \forall x_j \in M_{\varepsilon^\alpha} \cap B_{\varepsilon^\alpha}(x_1), \tag{3.10}$$

we see that

$$\begin{aligned}\int_{\mathbb{R}^N} U_{\varepsilon,x_1,V_M}^{p-2} U_{\varepsilon,x_j,V_M} \frac{\partial U_{\varepsilon,x_1,V_M}}{\partial n} &\le \varepsilon^\beta\big(c+o(1)\big)\varepsilon^{N-1} w\left(\frac{|x_1-x_j|}{\varepsilon}\right)\\ &\le \varepsilon^\beta\big(c+o(1)\big)\varepsilon^{N-1} L\varepsilon^{\alpha h}, \quad \forall x_j \in M_{\varepsilon^\alpha}\cap B_{\varepsilon^\alpha}(x_1).\end{aligned} \tag{3.11}$$

On the other hand, if $|x_j - x_1| > \varepsilon^\alpha$, then

$$\int_{\mathbb{R}^N} U_{\varepsilon,x_1,V_M}^{p-2} U_{\varepsilon,x_j,V_M} \frac{\partial U_{\varepsilon,x_1,V_M}}{\partial n} = O\Big(\varepsilon^{N-1} e^{-1/\varepsilon^{1-\alpha}}\Big). \tag{3.12}$$

Combining (3.7), (3.11), and (3.12), we obtain

$$\begin{aligned}\frac{\partial K(x)}{\partial n} &\ge c_1\varepsilon^N\big\langle DV(x_1), n\big\rangle - \varepsilon^\beta L\big(c+o(1)\big)\varepsilon^{N-1+\alpha h} + O\big(\varepsilon^{N-1+\alpha h+2(1-\alpha)}\big)\\ &\ge c'\varepsilon^{N+\alpha(h-1)} - L\big(c+o(1)\big)\varepsilon^{N-1+\alpha h+\beta} > 0.\end{aligned} \tag{3.13}$$

Combining Steps 1 and 2, we complete the proof of this lemma. □

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. In view of Lemma 3.1 and Proposition 2.3, we see that

$$\#\big\{x : DK(x) = 0, x \in K^{c_{\varepsilon,2}} \setminus K^{c_{\varepsilon,1}}\big\} \ge \mathrm{Cat}_{\Omega_{\varepsilon^\alpha}}\big(K^{c_{\varepsilon,2}}, K^{c_{\varepsilon,1}}\big). \tag{3.14}$$

It is easy to check that

$$K^{c_{\varepsilon,2}} = \Omega_{\varepsilon^\alpha}. \tag{3.15}$$

On the other hand, we claim that

$$T_{c'\varepsilon\ln\varepsilon^{-1}}\setminus T_{\varepsilon R}\subset K^{c_{\varepsilon,1}}\subset T_{C\varepsilon\ln\varepsilon^{-1}}\setminus T_{\varepsilon R}, \tag{3.16}$$

where $T_\tau=\cup_{i\neq j}\{|x_i-x_j|\le\tau,\ x_i,x_j\in M_{\varepsilon^\alpha}\}$, $C>c'>0$ are some suitable constants.

In fact, it follows from Lemma A.2 that

$$\begin{aligned} c_{\varepsilon,1}&=\varepsilon^N kV_M^{p/(p-2)-N/2}A-T\varepsilon^{N+\alpha h}>K(x)\\ &=\varepsilon^N kV_M^{p/(p-2)-N/2}A-\sum_{i\neq j}\int_{\mathbb{R}^N}U^{p-1}_{\varepsilon,x_i,V_M}U_{\varepsilon,x_j,V_M}+O\big(\varepsilon^{N+\alpha h}\big), \end{aligned} \tag{3.17}$$

which implies $|x_i-x_j|\le C\varepsilon\ln(1/\varepsilon)$ for some $i\neq j$ if $T>0$ is large enough. Thus,

$$K^{c_{\varepsilon,1}}\subset T_{C\varepsilon\ln\varepsilon^{-1}}\setminus T_{\varepsilon R}. \tag{3.18}$$

On the other hand, it is easy to check

$$T_{c'\varepsilon\ln\varepsilon^{-1}}\setminus T_{\varepsilon R}\subset K^{c_{\varepsilon,1}}, \tag{3.19}$$

if $c'>0$ is small enough. So the claim follows.

Since $T_{C\varepsilon\ln\varepsilon^{-1}}\setminus T_{\varepsilon R}$ can be deformed into $T_{c'\varepsilon\ln\varepsilon^{-1}}\setminus T_{\varepsilon R}$, we have

$$\mathrm{Cat}_{\Omega_{\varepsilon^\alpha}}\left(K^{c_{\varepsilon,2}},K^{c_{\varepsilon,1}}\right)=\mathrm{Cat}_{\Omega_{\varepsilon^\alpha}}\left(\Omega_{\varepsilon^\alpha},T_{c'\varepsilon\ln\varepsilon^{-1}}\setminus T_{\varepsilon R}\right). \tag{3.20}$$

As a result, we have

$$\#\left\{x:DK(x)=0,\ x\in K^{c_{\varepsilon,2}}\setminus K^{c_{\varepsilon,1}}\right\}\ge\mathrm{Cat}_{\Omega_{\varepsilon^\alpha}}\left(\Omega_{\varepsilon^\alpha},T_{c'\varepsilon\ln\varepsilon^{-1}}\setminus T_{\varepsilon R}\right), \tag{3.21}$$

On the other hand, it follows from the definition that

$$\mathrm{Cat}_{\Omega_{\varepsilon^\alpha}}\left(\Omega_{\varepsilon^\alpha},T_{c'\varepsilon\ln\varepsilon^{-1}}\setminus T_{\varepsilon R}\right)\ge\mathrm{Cat}_{M^k_{\varepsilon^\alpha}}\left(M^k_{\varepsilon^\alpha},T_{c'\varepsilon\ln\varepsilon^{-1}}\right). \tag{3.22}$$

So we have

$$\#\left\{x:DK(x)=0,\ x\in K^{c_{\varepsilon,2}}\setminus K^{c_{\varepsilon,1}}\right\}\ge\mathrm{Cat}_{M^k_{\varepsilon^\alpha}}\left(M^k_{\varepsilon^\alpha},T_{c'\varepsilon\ln\varepsilon^{-1}}\right). \tag{3.23}$$

By Proposition B.4, we know that $M^k_{\varepsilon^\alpha}$ cannot be deformed into $T_{c'\varepsilon\ln\varepsilon^{-1}}$. Hence the right-hand side of (3.23) is greater than or equal to 1, and thus we have proved the first part of Theorem 1.2.

By Proposition B.5, we know that if $k=2$,

$$\begin{aligned} &\#\left\{x:DK(x)=0,\ x\in K^{c_{\varepsilon,2}}\setminus K^{c_{\varepsilon,1}}\right\}\\ &\qquad\ge\mathrm{Cat}_{M^2_{\varepsilon^\alpha}}\left(M^2_{\varepsilon^\alpha},T_{c'\varepsilon\ln\varepsilon^{-1}}\right)\\ &\qquad\ge 2\,\mathrm{cuplength}\left(M_{\varepsilon^\alpha}\right)=2\,\mathrm{cuplength}(M). \end{aligned} \tag{3.24}$$

Thus, the number of the two-peak solutions for (1.1) is at least

$$\frac{1}{2}\#\left\{x : DK(x) = 0,\ x \in K^{c_{\varepsilon,2}} \setminus K^{c_{\varepsilon,1}}\right\} \geq \text{cuplength}(M). \tag{3.25}$$

So we complete the proof of Theorem 1.2. □

4. Multipeak solutions concentrating on the maximum set

In this section, we assume that M is a local compact maximum set of $V(y)$. Let

$$\Omega_\delta = \left\{x = (x_1, \ldots, x_k) : x_j \in M_\delta,\ j = 1, \ldots, k,\ \frac{|x_i - x_j|}{\varepsilon} \geq R, \forall i \neq j\right\},$$
$$A_{k,\delta} = \Omega_\delta/\sigma_k. \tag{4.1}$$

Define

$$K_1([x]) = J\left(\alpha_\varepsilon(x), x, v_\varepsilon(x)\right), \quad \forall\, [x] \in A_{k,\delta}. \tag{4.2}$$

By (2.8) of Proposition 2.1, $K_1([x])$ is well defined in $A_{k,\delta}$. It is not difficult to prove that $A_{k,\delta}$ is a covering space of Ω_δ. As a result, $[x] \in A_{k,\delta}$ is a critical point of K_1 if and only if $x \in \Omega_\delta$ is a critical point of K.

Proof of Theorem 1.3. First, fix $\delta > 0$ small such that

$$\gamma_1 =: V_M - \max_{\partial M_\delta} V(x) > 0. \tag{4.3}$$

Then take a small positive constant γ satisfying $\gamma < \min(\gamma_1, w(R))$. Let $\delta_1 > 0$ small enough such that

$$V(x) > V_M - \tau\gamma, \quad \forall x \in M_{\delta_1}, \tag{4.4}$$

where $\tau > 0$ is a small constant.

Define

$$c_\varepsilon = \varepsilon^N \left(k V_M^{p/(p-2)-N/2} - \gamma\right). \tag{4.5}$$

Then it follows from Lemma A.2 that

$$J\left(\alpha_\varepsilon(x), x, v_\varepsilon(x)\right) < c_{\varepsilon,k}, \tag{4.6}$$

if $d(x_i, M) = \delta$ for some i, or $|x_i - x_j| = \varepsilon R$ for some $i \neq j$. That is,

$$K_1([x]) < c_{\varepsilon,k}, \quad \forall [x] \in \partial\left(A_{k,\delta}\right). \tag{4.7}$$

Applying Proposition 2.2 to $-K_1$, we obtain

$$\#\left\{[x] : [x] \in A_{k,\delta}, DK_1([x]) = 0, K_1([x]) \geq c_{\varepsilon,k}\right\} \geq \text{Cat}_{A_{k,\delta}}\left(\left\{K_1([x]) \geq c_{\varepsilon,k}\right\}\right). \tag{4.8}$$

On the other hand, it is easy to check from (3.4) that

$$A'_{k,\delta_1} =: \left(\underbrace{M_{\delta_1}\times\cdots\times M_{\delta_1}}_{k}\setminus\bigcup_{i\neq j}\left\{\left(|x_i-x_j|/\varepsilon\right)\geq\frac{1}{\varepsilon^{1/2}}\right\}\right)/\sigma_k\subset\left\{K_1([x])\geq c_{\varepsilon,k}\right\}. \tag{4.9}$$

Combining (4.8) and (4.9), we obtain

$$\#\left\{[x]:[x]\in A_{k,\delta},\ DK_1\left([x]\right)=0,K_1\left([x]\right)\geq c_{\varepsilon,k}\right\}\geq \mathrm{Cat}_{A_{k,\delta}}\left(A'_{k,\delta_1}\right). \tag{4.10}$$

But $A_{k,\delta}$ and A'_{k,δ_1} are homotopically equivalent, so we see

$$\begin{aligned}\#\left\{[x]:[x]\in A_{k,\delta},DK_1\left([x]\right)=0,K_1\left([x]\right)\geq c_{\varepsilon,k}\right\}\\ \geq \mathrm{Cat}_{A_{k,\delta}}\left(A'_{k,\delta_1}\right)=\mathrm{Cat}_{A_{k,\delta}}\left(A_{k,\delta}\right).\end{aligned} \tag{4.11}$$

Thus we have completed the proof of Theorem 1.3. □

Proof of Theorem 1.5. Suppose that M is a connected compact local minimum set. Let $(\alpha_\varepsilon(x),v_\varepsilon(x))$ be the map obtained in Proposition 2.1 ($k=1$). Define

$$\begin{aligned}K_3(x)&=J\left(\alpha_\varepsilon(x),x,v_\varepsilon(x)\right),\quad \forall x\in M_\gamma,\\ c'_\varepsilon&=\varepsilon^N V_M^{p/(p-2)-N/2}A+\varepsilon^N\eta,\end{aligned} \tag{4.12}$$

where $\eta>0$ is a small constant satisfying $\eta<\min_{\partial M_\gamma}V(x)-V_M$. Then it is easy to check that

$$K_3(x)>c'_\varepsilon,\quad \forall x\in\partial M_\gamma,\ M\subset K_3^{c'_\varepsilon}. \tag{4.13}$$

As a result, we have

$$\#\left\{x:x\in M_\gamma,DK_3(x)=0,\ x\in K_3^{c'_\varepsilon}\right\}\geq \mathrm{Cat}_{M_\gamma}(M). \tag{4.14}$$

The case that M is a maximum set can be treated in a similar way. So we complete the proof of Theorem 1.5. □

5. Multipeak solution concentrating on different sets

Suppose that M_1 and M_2 are two disjoint minimum or maximum sets of $V(y)$. The aim of this section is to construct a solution u for (1.1) such that u has k_i peaks near M_i, $i=1,2$.

Proof of Theorem 1.4. For the simplicity of the notation, we only prove Theorem 1.4 for $l=2$.

For any $\gamma > 0$, define

$$\Omega_{1,\gamma} = \left\{ (x_1, \ldots, x_{k_1}) : x_i \in M_{1,\gamma}, \frac{|x_i - x_j|}{\varepsilon} \geq R, \forall i \neq j \right\},$$

$$\Omega_{2,\gamma} = \left\{ (x_{k_1+1}, \ldots, x_{k_1+k_2}) : x_i \in M_{2,\gamma}, \frac{|x_i - x_j|}{\varepsilon} \geq R, \forall i \neq j \right\}, \tag{5.1}$$

$$\Omega^*_\gamma = \Omega_{1,\gamma} \times \Omega_{2,\gamma}.$$

For any $x \in \Omega^*_\gamma$, let $(\alpha_\varepsilon(x), v_\varepsilon(x))$ be the map obtained in Proposition 2.1. Define

$$K(x) = J\big(\alpha_\varepsilon(x), x, v_\varepsilon(x)\big). \tag{5.2}$$

Since the interaction between the peaks near M_1 and the peaks near M_2 is exponentially small, we have

$$\begin{aligned} K(x) = {} & \varepsilon^N \sum_{i=1}^{2} k_i V_i^{p/(p-2)-N/2} A + \varepsilon^N B \sum_{j=1}^{k_1} \big(V(x_j) - V_{M_1}\big) V_{M_1}^{2/(p-2)-N/2} \\ & + \varepsilon^N B \sum_{j=k_1+1}^{k_1+k_2} \big(V(x_j) - V_{M_2}\big) V_{M_2}^{2/(p-2)-N/2} \\ & - \sum_{i=1}^{k_1-1} \int_{\mathbb{R}^N} \left(\sum_{j=i+1}^{k_1} U_{\varepsilon,x_j,M_1} \right)^{p-1} U_{\varepsilon,x_i,M_1} \\ & - \sum_{i=k_1+1}^{k_1+k_2-1} \int_{\mathbb{R}^N} \left(\sum_{j=i+1}^{k_1+k_2} U_{\varepsilon,x_j,M_2} \right)^{p-1} U_{\varepsilon,x_i,M_2} \\ & + O\left(\sum_{j=1}^{k_1+k_2} \sum_{m=1}^{[h]} \varepsilon^{N+m} \big| D^m V(x_j) \big| \right) \\ & + O\left(\varepsilon^N \sum_{1 \leq i < j \leq k_1+k_2} w^{1+\tau}\left(\frac{|x_i - x_j|}{\varepsilon} \right) \right) \\ & + O\left(\varepsilon^N \sum_{j=1}^{k_1} \big| V(x_j) - V_{M_1} \big|^2 + \varepsilon^N \sum_{j=k_1+1}^{k_1+k_2} \big| V(x_j) - V_{M_2} \big|^2 \right). \end{aligned} \tag{5.3}$$

Case 1. Suppose that both M_1 and M_2 are maximum sets. In this case, similar to the proof of Theorem 1.3, using (5.3), we can check that

$$\max_{x \in \partial\Omega^*_\delta} K(x) < \max_{x \in \Omega^*_\delta} K(x). \tag{5.4}$$

As a result, the maximum point x_ε of $K(x)$ in Ω_δ^* is an interior point of Ω_δ^*, and thus a critical point of $K(x)$.

Case 2. Suppose that both M_1 and M_2 are minimum sets with nontrivial topology. Let

$$\begin{aligned} c'_{\varepsilon,1} &= \varepsilon^N \sum_{i=1}^{2} V_{M_i}^{p/(p-2)-N/2} k_i A - T\varepsilon^{N+\alpha h}, \\ c'_{\varepsilon,2} &= \varepsilon^N \left(\sum_{i=1}^{2} V_{M_i}^{p/(p-2)-N/2} k_i A + \eta \right), \\ K^c &= \left\{ x : x \in \Omega_{\varepsilon^\alpha}^*, K(x) \le c \right\}, \end{aligned} \tag{5.5}$$

where $\alpha \in (0,1)$ is a fixed constant close to 1.

Using (5.3), we see that $K(x) < c'_{\varepsilon,1}$ if $|x_i - x_j| = \varepsilon R$ for some $x_i, x_j \in M_{1,\varepsilon^\alpha}$ or $x_i, x_j \in M_{2,\varepsilon^\alpha}$.

Suppose that $x_1 \in \partial M_{1,\varepsilon^\alpha}$. Define $\eta_1 = \min_{1\le i<j\le k_1+k_2} |x_i - x_j|$. Then, similar to Lemma 3.1, we have $K(x) < c'_{\varepsilon,1}$ if $w(\eta_1/\varepsilon) > L\varepsilon^{\alpha h}$, and $\partial K(x)/\partial n > 0$ if $w(\eta_1/\varepsilon) \le L\varepsilon^{\alpha h}$. So, in order to prove that $K(x)$ has critical point in $K^{c_{2,\varepsilon}} \setminus K^{c_{1,\varepsilon}}$, we just need to prove that $K^{c_{\varepsilon,2}}$ cannot be deformed into $K^{c_{\varepsilon,1}}$.

It is easy to check that $K^{c_{\varepsilon,2}} = \Omega_{\varepsilon^\alpha}^*$.

Since

$$\begin{aligned} &\Omega_{1,\varepsilon^\alpha} \times \left(T_{2,c'\varepsilon\ln\varepsilon^{-1}} \setminus T_{2,\varepsilon R}\right) \cup \left(T_{1,c'\varepsilon\ln\varepsilon^{-1}} \setminus T_{1,\varepsilon R}\right) \times \Omega_{2,\varepsilon^\alpha} \subset K^{c_{\varepsilon,1}} \\ &\qquad \subset \Omega_{1,\varepsilon^\alpha} \times \left(T_{2,C\varepsilon\ln\varepsilon^{-1}} \setminus T_{2,\varepsilon R}\right) \cup \left(T_{1,C\varepsilon\ln\varepsilon^{-1}} \setminus T_{1,\varepsilon R}\right) \times \Omega_{2,\varepsilon^\alpha}, \end{aligned} \tag{5.6}$$

where $T_{l,c} = \cup_{i\ne j}\{|x_i - x_j| \le c,\ x_i, x_j \in M_{l,\varepsilon^\alpha}\}$, $l = 1,2$, $C > 0$ is a large constant and $c' > 0$ is a small constant, we see that if $K^{c_{\varepsilon,2}}$ could be deformed into $K^{c_{\varepsilon,1}}$, then $M_{1,\varepsilon^\alpha}^{k_1} \times M_{2,\varepsilon^\alpha}^{k_2}$ could be deformed into $T_{1,c'\varepsilon\ln\varepsilon^{-1}} \times M_{\varepsilon^\alpha,2}^{k_2} \cup M_{\varepsilon^\alpha,1}^{k_1} \times T_{2,c'\varepsilon\ln\varepsilon^{-1}}$. This is a contradiction to Proposition B.2, since $H_*(M_{i,\varepsilon^\alpha}, T_{i,c'\varepsilon\ln\varepsilon^{-1}}) \ne 0$, $i = 1,2$. So we have completed the proof of Theorem 1.4(i). Using Proposition B.4, we can prove Theorem 1.4(ii) in a similar way.

Case 3. Suppose that M_1 is a minimum set and M_2 is a maximum set. We only consider the case that M_1 has nontrivial topology. Let

$$\begin{aligned} c'_{\varepsilon,1} &= \varepsilon^N \sum_{i=1}^{2} V_{M_i}^{p/(p-2)-N/2} k_i A - T\varepsilon^{N+\alpha h}, \\ c'_{\varepsilon,2} &= \varepsilon^N \left(\sum_{i=1}^{2} V_{M_i}^{p/(p-2)-N/2} k_i A + \eta \right). \end{aligned} \tag{5.7}$$

Denote

$$\Omega'_{\varepsilon^\alpha} = \Omega_{1,\varepsilon^\alpha} \times \Omega_{2,\delta}, \qquad K^c = \left\{ x : x \in \Omega'_{\varepsilon^\alpha}, K(x) \le c \right\}, \tag{5.8}$$

where $\alpha \in (0,1)$ is a fixed constant close to 1.

From (5.3), we have $K(x) < c'_{\varepsilon,1}$ for $x \in \Omega_{1,\varepsilon^\alpha} \times \partial\Omega_{2,\delta} \cup \partial T_{1,\varepsilon R} \times \Omega_{2,\delta}$. We claim that if $K(x) \geq c'_{1,\varepsilon}$, then

$$d(x_j, M_2) \leq C\varepsilon^\alpha, \tag{5.9}$$

$$\sum_{1\leq i<j\leq k_1} w\left(\frac{|x_i - x_j|}{\varepsilon}\right) + \sum_{k_1+1\leq i<j\leq k_1+k_2} w\left(\frac{|x_i - x_j|}{\varepsilon}\right) \leq C\varepsilon^{\alpha h}. \tag{5.10}$$

In fact, suppose that $K(x) \geq c'_{1,\varepsilon}$. Then we get from (5.3)

$$\begin{aligned}&\sum_{1\leq i<j\leq k_1} w\left(\frac{|x_i - x_j|}{\varepsilon}\right) + \sum_{k_1+1\leq i<j\leq k_1+k_2} w\left(\frac{|x_i - x_j|}{\varepsilon}\right)\\ &+ \sum_{j=k_1+1}^{k_1+k_2} (V_{M_2} - V(x_j)) + O\left(\sum_{j=k_1+1}^{k_1+k_2} \sum_{m=1}^{[h]} \varepsilon^m d(x_j, M_2)^{h-m}\right) \leq C\varepsilon^{\alpha h}.\end{aligned} \tag{5.11}$$

But by assumption, $V_{M_2} - V(x_j) \geq Cd(x_j, M_2)^h$. Thus, (5.11) implies (5.9). Using (5.11) again, we deduce (5.10).

Assume that $x_1 \in \partial M_{1,\varepsilon^\alpha}$ and $K(x) \geq c'_{1,\varepsilon}$. Similar to Lemma 3.1, using (5.9) and (5.10), we can prove $\partial K(x)/\partial n > 0$. So we conclude that if $K(x)$ has no critical point in $K^{c_{\varepsilon,2}} \setminus K^{c_{\varepsilon,1}}$, then $K^{c_{\varepsilon,2}}$ can be deformed into $K^{c_{\varepsilon,1}}$.

It is easy to check

$$\begin{aligned}&K^{c_{\varepsilon,2}} = \Omega'_{\varepsilon^\alpha},\\ &\Omega_{1,\varepsilon^\alpha} \times \partial\Omega_{2,\delta} \cup \left(T_{1,c'\varepsilon\ln\varepsilon^{-1}} \setminus T_{1,\varepsilon R}\right) \times \Omega_{2,\delta} \subset K^{c_{\varepsilon,1}}.\end{aligned} \tag{5.12}$$

On the other hand, if $K(x) \leq c'_{1,\varepsilon}$, then

$$\begin{aligned}&\sum_{1\leq i<j\leq k_1} w\left(\frac{|x_i - x_j|}{\varepsilon}\right) + \sum_{k_1+1\leq i<j\leq k_1+k_2} w\left(\frac{|x_i - x_j|}{\varepsilon}\right)\\ &+ \sum_{j=k_1+1}^{k_1+k_2} (V_{M_2} - V(x_j)) + O\left(\sum_{j=k_1+1}^{k_1+k_2} \varepsilon^m d(x_j, M_2)^{h-m}\right) \geq c\varepsilon^{\alpha h},\end{aligned} \tag{5.13}$$

which implies that

$$\begin{aligned}&\sum_{1\leq i<j\leq k_1} w\left(\frac{|x_i - x_j|}{\varepsilon}\right) + \sum_{k_1+1\leq i<j\leq k_1+k_2} w\left(\frac{|x_i - x_j|}{\varepsilon}\right)\\ &+ C\sum_{j=k_1+1}^{k_1+k_2} d(x_j, M_2)^h \geq c\varepsilon^{\alpha h}.\end{aligned} \tag{5.14}$$

Thus, we obtain

$$K^{c_{\varepsilon,1}} \subset A =: \left(T_{1,C\varepsilon\ln\varepsilon^{-1}} \setminus T_{1,\varepsilon R}\right) \times \Omega_{2,\delta} \cup \Omega_{1,\varepsilon^\alpha} \times B, \tag{5.15}$$

where

$$\begin{aligned} B = \left\{x' = \left(x_{k_1+1}, \ldots, x_{k_1+k_2}\right) \in \Omega_{2,\delta} : d\left(x_j, M_2\right) \geq c'\varepsilon^\alpha, \text{ for some } j\right\} \\ \cup \left(T_{2,C\varepsilon\ln\varepsilon^{-1}} \setminus T_{2,\varepsilon R}\right). \end{aligned} \tag{5.16}$$

But A can be deformed into $\Omega_{1,\varepsilon^\alpha} \times \partial\Omega_{2,\delta} \cup \partial T_{1,\varepsilon R} \times \Omega_{2,\delta}$, so we see that $\Omega'_{\varepsilon^\alpha}$ can be deformed into $\Omega_{1,\varepsilon^\alpha} \times \partial\Omega_{2,\delta} \cup \partial T_{1,\varepsilon R} \times \Omega_{2,\delta}$. But $H_*(\Omega_{1,\varepsilon^\alpha}, \partial T_{1,\varepsilon R}) \neq 0$, $H_*(\Omega_{2,\delta}, \partial\Omega_{2,\delta}) \neq 0$. Thus we get a contradiction. □

Remark 5.1. If both M_1 and M_2 are local maximum sets of $V(y)$, using the same technique as that in Section 4, we see that the number of the solutions with k_i peaks near $M_i, k_i \geq 2$, $i = 1, 2$, is at least

$$\mathrm{Cat}_{A_{1,k_1} \times A_{2,k_2}} \left(A_{1,k_1} \times A_{2,k_2}\right), \tag{5.17}$$

where

$$A_{l,k_1} = \left(\underbrace{M_{l,\gamma} \times \cdots \times M_{l,\gamma}}_{k_i} \setminus \bigcup_{i \neq j} \left\{ \left| x_i - x_j \right| < d \right\} \right) / \sigma_{k_l}, \quad l = 1, 2. \tag{5.18}$$

We have $\mathrm{Cat}_{A\times B}(A \times B) \geq \mathrm{cuplength}(A \times B) + 1$. On the other hand, it follows from the Künneth's formula that $\mathrm{cuplength}(A \times B) \geq \mathrm{cuplength}(A) + \mathrm{cuplength}(B)$. (We stress here that all the homologies in this paper are with $\mathbb{Z}_2$-coefficients.) Moreover, by the estimates in Appendix C, we have $\mathrm{cuplength}(A_{i,k_i}) \geq N - 1$. Thus we see that (1.1) has at least $2N - 1$ different solutions with k_i peaks near M_i, $i = 1, 2$. Similarly, if $M_1, \ldots, M_l$ are disjoint connected compact local maximum sets of $V(x)$, then the number of the solutions with k_j peaks near M_j, $k_j \geq 2$, is at least $l(N-1)+1$.

Remark 5.2. Suppose that M is a connected compact local minimum set of $V(x)$. Using a similar technique as that in Section 4, we can prove that if M has nontrivial reduced homology, then the number of the k-peak solutions with all the peaks near M is at least $\mathrm{Cat}_{A_k}(A_k, \partial T_d/\sigma_k)$, where $T_d = \cup_{i\neq j}\{|x_i - x_j| \leq d,\ x_i, x_j \in M_\gamma\}$ and $d > 0$ is a small constant.

Appendices

A. Basic estimates

LEMMA A.1. *There is*

$$I\bigl(U_{\varepsilon,z,\bar{V}}\bigr)=\varepsilon^N\bar{V}^{p/(p-2)-N/2}A+\varepsilon^N\bigl(V(z)-\bar{V}\bigr)\bar{V}^{2/(p-2)-N/2}B \\ +\varepsilon^N O\left(\sum_{m=1}^{[h]}\bigl|D^mV(z)\bigr|\varepsilon^m+\varepsilon^{[h]+1}\right), \tag{A.1}$$

where $A=(1/2-1/p)\int_{\mathbb{R}^N}w^p$ *and* $B=(1/2)\int_{\mathbb{R}^N}w^2$.

Proof. First, noting that $U_{\varepsilon,z,\bar{V}}=\bar{V}^{1/(p-2)}w(\bar{V}^{1/2}((y-z)/\varepsilon))$, we have

$$I\bigl(U_{\varepsilon,z,\bar{V}}\bigr)=\varepsilon^N\bar{V}^{p/(p-2)-N/2}A \\ +\frac{1}{2}\int_{\mathbb{R}^N}\bigl(V(z)-\bar{V}\bigr)U_{\varepsilon,z}^2+\frac{1}{2}\int_{\mathbb{R}^N}\bigl(V(y)-V(z)\bigr)U_{\varepsilon,z}^2. \tag{A.2}$$

But

$$|V(y)-V(z)|=O\left(\sum_{m=1}^{[h]}|y-z|^m\bigl|D^mV(z)\bigr|+\varepsilon^{[h]+1}\right), \tag{A.3}$$

and thus

$$\int_{\mathbb{R}^N}\bigl(V(y)-V(z)\bigr)U_{\varepsilon,z}^2=\varepsilon^NO\left(\sum_{m=1}^{[h]}\bigl|D^mV(z)\bigr|\varepsilon^m+\varepsilon^{[h]+1}\right). \tag{A.4}$$

So the result follows from (A.2) and (A.4). □

LEMMA A.2. *The following holds*

$$I\left(\sum_{j=1}^kU_{\varepsilon,z_j,V_j}\right)=A\sum_{j=1}^k\varepsilon^NV_j^{p/(p-2)-N/2} \\ +B\sum_{j=1}^k\varepsilon^N\bigl(V\bigl(z_j\bigr)-V_j\bigr)V_j^{2/(p-2)-N/2} \\ -\sum_{i=1}^{k-1}\int_{\mathbb{R}^N}U_{\varepsilon,z_i,V_i}\left(\sum_{j=i+1}^kU_{\varepsilon,z_j,V_j}\right)^{p-1}$$

$$+\varepsilon^N O\left(\sum_{j=1}^{k}\left(\left|V(z_j)-V_j\right|^2+\sum_{m=1}^{[h]}\left|D^m V(z_j)\right|\varepsilon^m\right)+\varepsilon^{[h]+1}\right)$$
$$+\varepsilon^N O\left(\sum_{i\neq j} w^{1+\tau}\left(\frac{|z_i-z_j|}{\varepsilon}\right)\right). \tag{A.5}$$

Proof. We have

$$I\left(\sum_{j=1}^{k} U_{\varepsilon,z_j,V_j}\right)=\sum_{j=1}^{k} I\left(U_{\varepsilon,z_j,V_j}\right)+\frac{1}{2}\sum_{i\neq j}\int_{\mathbb{R}^N}\left(V(y)-V_j\right)U_{\varepsilon,z_i,V_i}U_{\varepsilon,z_j,V_j}$$
$$+\sum_{i\neq j}\int_{\mathbb{R}^N} U^{p-1}_{\varepsilon,z_i,V_i}U_{\varepsilon,z_j,V_j}$$
$$+\frac{1}{p}\int_{\mathbb{R}^N}\left(\left(\sum_{j=1}^{k} U_{\varepsilon,z_j,V_j}\right)^p-\sum_{j=1}^{k} U^p_{\varepsilon,z_j,V_j}\right). \tag{A.6}$$

On the other hand, we also have

$$\sum_{i\neq j}\int_{\mathbb{R}^N}\left(V(y)-V\left(z_j\right)\right)U_{\varepsilon,z_i,V_i}U_{\varepsilon,z_j,V_j}$$
$$=O\left(\varepsilon^N\left(\left|V\left(z_j\right)-V_j\right|+\sum_{m=1}^{[h]}\left|D^m V\left(z_j\right)\right|\varepsilon^m+\varepsilon^{[h]+1}\right)\sum_{i\neq j} w\left(\frac{|z_i-z_j|}{\varepsilon}\right)\right), \tag{A.7}$$

$$\int_{\mathbb{R}^N}\left(\left(\sum_{j=1}^{k} U_{\varepsilon,z_j,V_j}\right)^p-\sum_{j=1}^{k} U^p_{\varepsilon,z_j,V_j}\right)$$
$$=p\sum_{i\neq j}\int_{\mathbb{R}^N} U^{p-1}_{\varepsilon,z_i,V_i}U_{\varepsilon,z_j,V_j}+p\sum_{i=1}^{k-1}\int_{\mathbb{R}^N} U_{\varepsilon,z_i,V_i}\left(\sum_{j=i+1}^{k} U_{\varepsilon,z_j,V_j}\right)^{p-1}$$
$$+\varepsilon^N O\left(\sum_{i\neq j} w^{1+\tau}\left(\frac{|z_i-z_j|}{\varepsilon}\right)\right). \tag{A.8}$$

Combining (A.6), (A.7), (A.8), and Lemma A.1, we get the desired estimate. □

LEMMA A.3. *For any* $v \in E_{\varepsilon,x,k}$,

$$\left|\sum_{j=1}^{k}\int_{\mathbb{R}^N}\big(V(y)-V_j\big)U_{\varepsilon,x_j,V_j}v\right|+\left|\int_{\mathbb{R}^N}\left(\sum_{j=1}^{k}U_{\varepsilon,x_j,V_j}^{p-1}-\left(\sum_{j=1}^{k}U_{\varepsilon,x_j,V_j}\right)^{p-1}\right)v\right|$$

$$=O\left(\varepsilon^{N/2}\left(\sum_{j=1}^{k}\left(\big|V\big(x_j\big)-V_j\big|+\sum_{m=1}^{[h]}\big|D^mV\big(x_j\big)\big|\varepsilon^m\right)+\varepsilon^{[h]+1}\right)\right)\|v\|_\varepsilon$$

$$+O\left(\varepsilon^{N/2}\sum_{i\neq j}w^{(1+\tau)/2}\left(\frac{|x_i-x_j|}{\varepsilon}\right)\right)\|v\|_\varepsilon,$$

$$\left|\sum_{j=1}^{k}\int_{\mathbb{R}^N}\big(V(y)-V\big(x_j\big)\big)U_{\varepsilon,x_j,V_j}U_{\varepsilon,x_i,V_i}\right|$$

$$+\left|\int_{\mathbb{R}^N}\left(\sum_{j=1}^{k}U_{\varepsilon,x_j,V_j}^{p-1}-\left(\sum_{j=1}^{k}U_{\varepsilon,x_j,V_j}\right)^{p-1}\right)U_{\varepsilon,x_i}\right|$$

$$=O\left(\varepsilon^{N}\left(\sum_{j=1}^{k}\left(\big|V\big(x_j\big)-V_j\big|+\sum_{m=1}^{[h]}\big|D^mV\big(x_j\big)\big|\varepsilon^m\right)+\varepsilon^{[h]+1}\right)\right)$$

$$+O\left(\varepsilon^{N}\sum_{i\neq j}w^{(1+\tau)/2}\left(\frac{|x_i-x_j|}{\varepsilon}\right)\right). \tag{A.9}$$

Proof. The proof of this lemma is similar to that of Lemmas A.1 and A.2, and thus we omit the details. □

LEMMA A.4. *Let* $(\alpha_\varepsilon(x), v_\varepsilon(x))$ *be the map obtained in Proposition 2.1. Then*

$$\frac{\partial J}{\partial x_{il}}=c_1\varepsilon^N D_lV\big(x_1\big)-(p-1)\sum_{j\neq i}\int_\Omega U_{\varepsilon,x_i,V_i}^{p-2}U_{\varepsilon,x_j,V_j}\frac{\partial U_{\varepsilon,x_1,V_1}}{\partial x_{1l}}$$

$$+O\left(\varepsilon^{N-1}\sum_{i\neq j}e^{-(1+\tau)(|x_i-x_j|/\varepsilon)}\right)$$

$$+O\left(\varepsilon^{N-1}\sum_{j=1}^{k}\sum_{m=2}^{[h]}\left|D^m V(x_j)\right|\varepsilon^m+\varepsilon^{N+h}\right)$$

$$+O\left(\varepsilon^{N-1}\sum_{j=1}^{k}\left(\left|V(x_j)-V_j\right|^2+\sum_{m=1}^{[h]}\left|D^m V(x_j)\right|^2\varepsilon^{2m}\right)\right), \tag{A.10}$$

where $c_1>0$ is a constant and

$$\left\langle\frac{\partial J}{\partial v},U_{\varepsilon,x_i,V_i}\right\rangle=O\left(\varepsilon^N\sum_{i\neq j}e^{-(1+\tau)(|x_i-x_j|/\varepsilon)}+\varepsilon^{N+h}\right)$$
$$+O\left(\varepsilon^N\sum_{j=1}^{k}\left(\left|V(x_j)-V_j\right|+\sum_{m=1}^{[h]}\left|D^m V(x_j)\right|\varepsilon^m\right)\right). \tag{A.11}$$

Proof. We have

$$\begin{aligned}\frac{\partial J}{\partial x_{il}}&=\sum_{j=1}^{k}\int_{\mathbb{R}^N}\left(V(y)-V_j\right)U_{\varepsilon,x_j,V_j}\frac{\partial U_{\varepsilon,x_i,V_i}}{\partial x_{il}}\\&\quad+\sum_{j=1}^{k}\int_{\mathbb{R}^N}U_{\varepsilon,x_j,V_j}^{p-1}\frac{\partial U_{\varepsilon,x_i,V_i}}{\partial x_{il}}-\int_{\mathbb{R}^N}\left(\sum_{j=1}^{k}U_{\varepsilon,x_j,V_j}\right)^{p-1}\frac{\partial U_{\varepsilon,x_i,V_i}}{\partial x_{il}}\\&\quad+O\left(\varepsilon^{N-1}\left(\sum_{i\neq j}e^{-(1+\tau)(|x_i-x_j|/\varepsilon)}+\varepsilon^{2([h]+1)}\right)\right)\\&\quad+O\left(\varepsilon^{N-1}\sum_{j=1}^{k}\left(\left|V(x_j)-V_j\right|^2+\sum_{m=1}^{[h]}\left|D^m V(x_j)\right|^2\varepsilon^{2m}\right)\right)\\&=\int_{\mathbb{R}^N}V(y)U_{\varepsilon,x_i,V_i}\frac{\partial U_{\varepsilon,x_i,V_i}}{\partial x_{il}}-(p-1)\sum_{j=2}^{k}\int_{\Omega}U_{\varepsilon,x_i,V_i}^{p-2}U_{\varepsilon,x_j,V_j}\frac{\partial U_{\varepsilon,x_i,V_i}}{\partial x_{il}}\\&\quad+O\left(\varepsilon^{N-1}\left(\sum_{i\neq j}e^{-(1+\tau)(|x_i-x_j|/\varepsilon)}+\varepsilon^{2([h]+1)}\right)\right)\\&\quad+O\left(\varepsilon^{N-1}\sum_{j=1}^{k}\left(\left|V(x_j)-V_j\right|^2+\sum_{m=1}^{[h]}\left|D^m V(x_j)\right|^2\varepsilon^{2m}\right)\right).\end{aligned} \tag{A.12}$$

But

$$\int_{\mathbb{R}^N} V(y)U_{\varepsilon,x_i,V_i}\frac{\partial U_{\varepsilon,x_i,V_i}}{\partial x_{il}} = \frac{1}{2}\int_{\mathbb{R}^N}\frac{\partial V(y)}{\partial y_l}U^2_{\varepsilon,x_i,V_i}$$
$$= c_1\varepsilon^N D_l V(x_i)$$
$$+ O\left(\varepsilon^{N-1}\sum_{j=1}^{k}\sum_{m=2}^{[h]}\left|D^m V(x_j)\right|\varepsilon^m + \varepsilon^{N+[h]+1}\right). \tag{A.13}$$

So we see that (A.10) follows from (A.12) and (A.13). Equation (A.11) can be proved similarly. □

B. Some topological results

In this section, we give some topological results needed in the proof of our main results. First, we recall the definition for the relative category (see [18]).

Definition B.1. Let Y and A be closed subsets of a topological space X. Then $\mathrm{Cat}_X(A, Y)$ is the least integer k such that $A = \cup_{j=0}^{k} A_j$, where, for $0 \le j \le k$, A_j is closed and there exists $h_j \in C([0,1]\times A_j, X)$ such that

(a) $h_j(0,x) = x$ for $x \in A_j$, $0 \le j \le k$,
(b) $h_0(1,x) \in Y$ for $x \in A_0$ and $h_0(t,x) = x$ for $x \in A_0 \cap Y$ and $t \in [0,1]$,
(c) $h_j(1,x) = x_j$ for $x \in A_j$ and some $x_j \in X$, $1 \le j \le k$.

From the definition, we see that $\mathrm{Cat}_X(A, Y) \ge 1$ if A cannot be deformed into a subset of Y within X.

From now on, we assume that all the sets appearing in the propositions of this section are subsets in $\mathbb{R}^m$ for some positive integer m.

Proposition B.2. *Suppose that there are positive integers p_1 and q_1 such that $H_{p_1}(A, A') \ne 0$, $H_{q_1}(B, B') \ne 0$. Then*

$$H_t\left(A\times B, A'\times B \cup A\times B'\right) \ne 0 \tag{B.1}$$

for some positive integer t. In particular, $A \times B$ cannot be deformed into $A' \times B \cup A \times B'$.

Proof. Choose the largest positive integers p and q satisfying $H_p(A, A') \ne 0$ and $H_q(B, B') \ne 0$. Then it follows from Künneth formula [17] that

$$H_{p+q}\left(A\times B, A'\times B\cup A\times B'\right) = H_p\left(A, A'\right)\otimes H_q\left(B, B'\right)\oplus \text{ other group.} \tag{B.2}$$

So, $H_*(A\times B, A'\times B\cup A\times B') \ne 0$. □

Proposition B.3. *Let M be a compact n-dimensional manifold with boundary. Then $H_n(M, \partial M) \neq 0$.*

Proof. This result is well known. See, for example, [20]. □

Proposition B.4. *Suppose that M has nontrivial reduced homology. Then $H_*(M^k, T) \neq 0$, where $T = \cup_{i \neq j}\{(x_1, \ldots, x_k) \in M^k : x_i = x_j\}$.*

Proof. For the proof of Proposition B.4, see [14]. □

Proposition B.5. *We have*

$$\operatorname{cuplength}\left(M_\gamma \times M_\gamma, T_2\right) \geq 2\operatorname{cuplength}\left(M_\gamma\right) - 1, \tag{B.3}$$

where $T_2 = \{x = (x_1, x_2) \in M_\gamma^2 : |x_1 - x_2| \leq d\}$ and $d > 0$ is a small constant. As a result,

$$\operatorname{Cat}_{M_\gamma \times M_\gamma}\left(M_\gamma \times M_\gamma, T_2\right) \geq 2\operatorname{cuplength}\left(M_\gamma\right). \tag{B.4}$$

Proof. For the proof of Proposition B.5, see [14]. □

C. Some estimates of the cuplength

Let $B(\mathbb{R}^N, k)$ be the configuration space of k distinct unordered points of $\mathbb{R}^N$ defined as follows:

$$B\left(\mathbb{R}^N, k\right) = F\left(\mathbb{R}^N, k\right)/\sigma_k, \tag{C.1}$$

where

$$F\left(\mathbb{R}^N, k\right) = \underbrace{\mathbb{R}^N \times \cdots \times \mathbb{R}^N}_{k} \setminus \bigcup_{i \neq j}\left\{\left(x_1, \ldots, x_k\right) \in \mathbb{R}^{kN} : x_i = x_j\right\}. \tag{C.2}$$

It is not difficult to check that both $F(\mathbb{R}^N, k)$ and $B(\mathbb{R}^N, k)$ are path connected if $N \geq 2$. The geometry of such configuration spaces has been extensively studied by topologists in recent years. For sophisticated techniques in this respect, the readers can refer to [9]. In this section, we give a lower bound of the category of $B(\mathbb{R}^N, k)$, obtained by elementary considerations of the cuplength, relying on the fact that the symmetric group σ_k contains the alternating group $\hat{\sigma}_k$ as a normal subgroup of index 2. The main result of this section is the following.

Proposition C.1. *If $k = 2$ or 3, $\operatorname{cuplength}(B(\mathbb{R}^N, k)) \geq N - 1$. If $k \geq 4$, $\operatorname{cuplength}(B(\mathbb{R}^N, k)) \geq 2^t - 1$, where t is the smallest positive integer satisfying $2^t > N - 1$.*

The proof of Proposition C.1 for $k = 2$ is quite easy. In fact, since $B(\mathbb{R}^N, 2)$ has the same homotopy type as the real projective space $\mathbb{R}P^{N-1}$, we see that

cuplength$(B(\mathbb{R}^N, 2)) =$ cuplength$(\mathbb{R}P^{N-1}) = N-1$. To prove Proposition C.1 for $k \geq 3$, we need to do more work.

First, we recall a general fact in algebraic topology. A two-to-one covering map $p : \tilde{B} \to B$ between connected spaces $\tilde{B}$ and B gives rise naturally to a 1-dimensional cohomology class $\theta_p \in H^1(B)$. One elegant way to describe θ_p is that for any connected B, $H^1(B)$ is identifiable with the group of homomorphisms from the fundamental group $\pi_1(B)$ to $\mathbb{Z}_2$. Another way of description is that θ_p is represented by the cocycle whose value on a loop ω in B is zero or one according to whether or not ω is the p-image of a loop $\tilde{\omega}$ in $\tilde{B}$.

For later purpose, we consider here two examples.

Let $p_1 : S^{N-1} \to \mathbb{R}P^{N-1}$ be the double covering from a sphere to a real projective space of dimension $N-1$. In this case, θ_{p_1} is the generator of the group $H^1(\mathbb{R}P^{N-1}) = \mathbb{Z}_2$. We know that the cuplength of θ_{p_1} is $N-1$, that is,

$$\underbrace{\theta_{p_1} \cup \cdots \cup \theta_{p_1}}_{l} = 0 \tag{C.3}$$

if and only if $l \geq N$. See, for example, [17].

For a further example consider $\tilde{B} = S^{N-1} \times_T S^{N-1}$, obtained from $S^{N-1} \times S^{N-1}$ by identifying each pair (x_1, x_2) with its opposite $(-x_1, -x_2)$. There is then a double covering map $q : \tilde{B} \to B$, where $B = \mathbb{R}P^{N-1} \times \mathbb{R}P^{N-1}$, q is defined via $q(x_1, x_2) = (p_1(x_1), p_2(x_2))$ and p_1, p_2 are the double coverings for the two individual factors, respectively. Now

$$H^1(B) = H^1\big(\mathbb{R}P^{N-1}\big) \oplus H^1\big(\mathbb{R}P^{N-1}\big) = \mathbb{Z}_2 \oplus \mathbb{Z}_2, \tag{C.4}$$

with generators denoted by $\theta_{p_1} \otimes 1$ and $1 \otimes \theta_{p_2}$, respectively. Because of symmetry, one clearly has

$$\theta_q = \theta_{p_1} \otimes 1 + 1 \otimes \theta_{p_2} \tag{C.5}$$

in this circumstance.

Consider the iterated cup product

$$\theta_q^l = \underbrace{\theta_q \cup \cdots \cup \theta_q}_{l}. \tag{C.6}$$

We have the following lemma.

LEMMA C.2. *The iterated cup product θ_q^l is zero if and only if $l \geq 2^t$, where 2^t is the smallest power of 2 strictly exceeding $N-1$.*

Proof. Remember that we are working mod 2 since $\mathbb{Z}_2$ is used as coefficient group for cohomology.

If $l = 2^t$, then

$$\theta_q^l = \big(\theta_{p_1} \otimes 1 + 1 \otimes \theta_{p_2}\big)^{2^t} = \theta_{p_1}^{2^t} \otimes 1 + 1 \otimes \theta_{p_2}^{2^t} = 0, \tag{C.7}$$

since $2^t \geq N$.

If $l < 2^t$, write

$$l = 2^{\beta_1} + 2^{\beta_2} + \cdots + 2^{\beta_s} \tag{C.8}$$

with $t > \beta_1 > \beta_2 > \cdots > \beta_s \geq 0$. Let $l_1 = 2^{\beta_1}$ and $l_2 = l - l_1$. Then $l = l_1 + l_2$ with $l_1 < N$, $l_2 < N$, and $\binom{l}{l_1}$ is odd. Note that $(\theta_{p_1} \otimes 1 + 1 \otimes \theta_{p_2})^l$ contains, upon binomial expansion, the nonzero term

$$\binom{l}{l_1} \left(\theta_{p_1} \otimes 1\right)^{l_1} \cup \left(1 \otimes \theta_{p_2}\right)^{l_2} = \binom{l}{l_1} \theta_{p_1}^{l_1} \otimes \theta_{p_2}^{l_2}, \tag{C.9}$$

which lies in $H^{l_1}(\mathbb{R}P^{N-1}) \otimes H^{l_2}(\mathbb{R}P^{N-1})$, a direct summand of $H^l(\mathbb{R}P^{N-1} \times \mathbb{R}P^{N-1})$. Thus $\theta_q^l \neq 0$. □

We are now ready to prove Proposition C.1.

Proof of Proposition C.1. Define $\tilde{B}(\mathbb{R}^N, k) = F(\mathbb{R}^N, k)/\hat{\sigma}_k$, where $\hat{\sigma}_k$ is the alternating group.

Suppose that $k \geq 4$. First, we consider the case $k = 2h$, where h is even. Construct a continuous map

$$\hat{f} : S^{N-1} \times S^{N-1} \longrightarrow \tilde{B}\left(\mathbb{R}^N, k\right), \tag{C.10}$$

by sending (a, b) in $S^{N-1} \times S^{N-1}$ to the orbit of

$$\left(a, -a, 2a, -2a, \ldots, (h-1)a, -(h-1)a, hb, -hb\right) \tag{C.11}$$

under the $\hat{\sigma}_k$-action on $F(\mathbb{R}^N, k)$. It can be easily checked that because h is even, $\hat{f}$ sends $(-a, -b)$ to the same $\hat{\sigma}_k$-orbit. Therefore, it induces a map $\tilde{f} : S^{N-1} \times_T S^{N-1} \to \tilde{B}(\mathbb{R}^N, k)$. Bringing in the double coverings p and q, we form the following commutative diagram:

$$\begin{array}{ccc} S^{N-1} \times_T S^{N-1} & \xrightarrow{\hat{f}} & \tilde{B}\left(\mathbb{R}^N, k\right) \\ \downarrow q & & \downarrow p \\ \mathbb{R}P^{N-1} \times \mathbb{R}P^{N-1} & \xrightarrow{F} & B\left(\mathbb{R}^N, k\right), \end{array} \tag{C.12}$$

where F is a uniquely induced map. Note once more that for any point u in $\mathbb{R}P^{N-1} \times \mathbb{R}P^{N-1}$ and its image point $v = F(u)$ in $B(\mathbb{R}^N, k)$, the restriction of $\tilde{f}$ maps $q^{-1}(u)$ to $p^{-1}(v)$ bijectively. By the natural way θ_p and θ_q arise from their respective double coverings, we see that for the induced homomorphism

$$F^* : H^1\left(B\left(\mathbb{R}^N, k\right)\right) \longrightarrow H^1\left(\mathbb{R}P^{N-1} \times \mathbb{R}P^{N-1}\right), \tag{C.13}$$

$F^*(\theta_p)$ naturally equals θ_q.

By Lemma C.2, $\theta_q^l \neq 0$ if l is less than 2^t, the smallest power of 2 exceeding $N-1$. Since F^* is a homomorphism of cohomology rings, the same must be true for θ_p^l. Thus we have exhibited in $B(\mathbb{R}^N, k)$ a nonzero cupproduct of length $2^t - 1$, and our claim follows.

If $k = 2h + j$, where h is even and $1 \leq j \leq 3$, we can use the same proof by modifying the map $\hat{f}$ into the one which sends (a, b) and $(-a, -b)$ to the $\hat{\sigma}_k$-orbit of

$$\big(a, -a, 2a, -2a, \ldots, (h-1)a, -(h-1)a, hb, -hb, c_1, \ldots, c_j\big) \tag{C.14}$$

instead, where $c_1, \ldots, c_j$ are j distinct points fixed in $\mathbb{R}^N$, each having distance greater than $h+1$ from the origin.

Suppose that $k = 3$. Define a map $\hat{f} : S^{N-1} \to \tilde{B}(\mathbb{R}^N, 3)$ by sending $a \in S^{N-1}$ to the orbit of $(a, -a, 0)$ in $F(\mathbb{R}^N, 3)$ under the $\hat{\sigma}_3$-action. We have the following commutative diagram:

$$\begin{array}{ccc} S^{N-1} & \xrightarrow{\hat{f}} & \tilde{B}\big(\mathbb{R}^N, k\big) \\ \big\downarrow p_1 & & \big\downarrow p \\ \mathbb{R}P^{N-1} & \xrightarrow{F} & B\big(\mathbb{R}^N, k\big), \end{array} \tag{C.15}$$

where F is a uniquely induced map. Since the nonzero element $\theta_{p_1} \in H^1(\mathbb{R}P^{N-1})$ has cuplength $N-1$ (see the first example above), we can prove in a similar way as above that $\theta_p^{N-1} \neq 0$. So we have proved Proposition C.1 for the case $k = 3$. As we mentioned before, the Proposition C.1 is quite easy to prove if $k = 2$. □

As a direct consequence of Proposition C.1, we have the following proposition.

PROPOSITION C.3. *If $k \geq 4$, then $\mathrm{Cat}_{A_k}(A_k) \geq 2^t$, where t is the smallest positive integer satisfying $2^t > N-1$. If $k = 2, 3$, then $\mathrm{Cat}_{A_k}(A_k) \geq N$. Here*

$$A_k = \Bigg(\underbrace{M_\gamma \times \cdots \times M_\gamma}_{k} \setminus \bigcup_{i \neq j} \big\{\big|x_i - x_j\big| < d\big\}\Bigg)/\sigma_k. \tag{C.16}$$

Proof. Let $x_0 \in M_\gamma$ and let $\delta > 0$ be so small that $B_\delta(x_0) \subset M_\gamma$. We also let

$$V_k = \Bigg(\underbrace{B_\delta\big(x_0\big) \times \cdots \times B_\delta\big(x_0\big)}_{k} \setminus \bigcup_{i \neq j} \big\{\big|x_i - x_j\big| \leq d\big\}\Bigg)/\sigma_k, \tag{C.17}$$

then $B(\mathbb{R}^N, k)$ and V_k are homotopically equivalent. So

$$\begin{aligned} \mathrm{Cat}_{A_k} A_k &\geq \mathrm{Cat}_{B(\mathbb{R}^N,k)}\big(V_k\big) = \mathrm{Cat}_{B(\mathbb{R}^N,k)}\big(B\big(\mathbb{R}^N, k\big)\big) \\ &\geq \mathrm{cuplength}\big(B\big(\mathbb{R}^N, k\big)\big) + 1. \end{aligned} \tag{C.18}$$

So Proposition C.3 follows from Proposition C.1. □

References

[1] A. Bahri, *Critical Points at Infinity in Some Variational Problems*, Pitman Research Notes in Mathematics Series, vol. 182, Longman Scientific & Technical, Harlow, 1989. MR 91h:58022. Zbl 676.58021.

[2] A. Bahri and J.-M. Coron, *On a nonlinear elliptic equation involving the critical Sobolev exponent: the effect of the topology of the domain*, Comm. Pure Appl. Math. **41** (1988), no. 3, 253–294. MR 89c:35053. Zbl 649.35033.

[3] V. Benci and G. Cerami, *The effect of the domain topology on the number of positive solutions of nonlinear elliptic problems*, Arch. Rational Mech. Anal. **114** (1991), no. 1, 79–93. MR 91j:35102. Zbl 727.35055.

[4] V. Benci, G. Cerami, and D. Passaseo, *On the number of the positive solutions of some nonlinear elliptic problems*, Nonlinear Analysis (A. Ambrosetti et al., eds.), Quaderni, Scuola Norm. Sup., Pisa, 1991, pp. 93–107. MR 94b:35109. Zbl 838.35040.

[5] D. Cao, E. N. Dancer, E. S. Noussair, and S. Yan, *On the existence and profile of multi-peaked solutions to singularly perturbed semilinear Dirichlet problems*, Discrete Contin. Dynam. Systems **2** (1996), no. 2, 221–236. MR 96m:35095. Zbl 947.35073.

[6] D. Cao and E. S. Noussair, *Multiplicity of positive and nodal solutions for nonlinear elliptic problems in* $\mathbb{R}^N$, Ann. Inst. H. Poincaré Anal. Non Linéaire **13** (1996), no. 5, 567–588. MR 97k:35058. Zbl 859.35032.

[7] D. Cao, E. S. Noussair, and S. Yan, *Existence and uniqueness results on single-peaked solutions of a semilinear problem*, Ann. Inst. H. Poincaré Anal. Non Linéaire **15** (1998), no. 1, 73–111. MR 99d:35053. Zbl 905.35033.

[8] ———, *Solutions with multiple peaks for nonlinear elliptic equations*, Proc. Roy. Soc. Edinburgh Sect. A **129** (1999), no. 2, 235–264. MR 2000a:35052. Zbl 928.35048.

[9] F. Cohen, *The homology of* C_{n+1}*-space,* $n \geq 0$, Lecture Notes in Math., vol. 533, pp. 207–351, Springer-Verlag, 1976.

[10] E. N. Dancer, *Some slightly subcritical or slightly supercritical problems*, preprint.

[11] ———, *Superlinear problems on domains with holes of asymptotic shape and exterior problems*, Math. Z. **229** (1998), no. 3, 475–491. MR 2000c:35058. Zbl 933.35068.

[12] E. N. Dancer and J. Wei, *On the effect of domain topology in a singular perturbation problem*, Topol. Methods Nonlinear Anal. **11** (1998), no. 2, 227–248. MR 2000a:35012. Zbl 926.35015.

[13] E. N. Dancer and S. Yan, *Multipeak solutions for a singularly perturbed Neumann problem*, Pacific J. Math. **189** (1999), no. 2, 241–262. MR 2000d:35010. Zbl 933.35070.

[14] ———, *A singularly perturbed elliptic problem in bounded domains with nontrivial topology*, Adv. Differential Equations **4** (1999), no. 3, 347–368. MR 2000d:35009. Zbl 947.35075.

[15] ———, *Singularly perturbed elliptic problems in exterior domains*, Differential Integral Equations **13** (2000), no. 4-6, 747–777. CMP 1 750 049. Zbl 992.34727.

[16] M. Del Pino and P. L. Felmer, *Multi-peak bound states for nonlinear Schrödinger equations*, Ann. Inst. H. Poincaré Anal. Non Linéaire **15** (1998), no. 2, 127–149. MR 99c:35228. Zbl 901.35023.

[17] A. Dold, *Lectures on Algebraic Topology*, Die Grundlehren der mathematischen Wissenschaften, vol. 200, Springer-Verlag, New York, 1972 (German). MR 54#3685. Zbl 234.55001.

[18] G. Fournier and M. Willem, *Relative category and the calculus of variations*, Variational Methods (Paris, 1988), Progr. Nonlinear Differential Equations Appl., vol. 4, Birkhäuser, Massachusetts, 1990, pp. 95–104. MR 93k:58046. Zbl 719.58010.

[19] B. Gidas, W. M. Ni, and L. Nirenberg, *Symmetry and related properties via the maximum principle*, Comm. Math. Phys. **68** (1979), no. 3, 209–243. MR 80h:35043. Zbl 425.35020.

[20] M. J. Greenberg and J. R. Harper, *Algebraic Topology: A First Course*, Mathematics Lecture Note Series, vol. 58, Benjamin/Cummings Publishing, Massachusetts, 1981. MR 83b:55001. Zbl 498.55001.

[21] C. Gui, *Existence of multi-bump solutions for nonlinear Schrödinger equations via variational method*, Comm. Partial Differential Equations **21** (1996), no. 5-6, 787–820. MR 98a:35122. Zbl 857.35116.

[22] Y. Y. Li, *On a singularly perturbed elliptic equation*, Adv. Differential Equations **2** (1997), no. 6, 955–980. MR 99b:35005.

[23] E. S. Noussair and S. Yan, *On positive multipeak solutions of a nonlinear elliptic problem*, J. London Math. Soc. (2) **62** (2000), no. 1, 213–227. CMP 1 772 182. Zbl 991.69987.

[24] O. Rey, *The role of the Green's function in a nonlinear elliptic equation involving the critical Sobolev exponent*, J. Funct. Anal. **89** (1990), no. 1, 1–52. MR 91b:35012. Zbl 786.35059.

[25] S. Yan, *On the number of interior multipeak solutions for singularly perturbed Neumann problems*, Topol. Methods Nonlinear Anal. **12** (1998), no. 1, 61–78. CMP 1 677 747. Zbl 929.35056.

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