ON SECOND-ORDER MULTIVALUED IMPULSIVE FUNCTIONAL DIFFERENTIAL INCLUSIONS IN BANACH SPACES

M. BENCHOHRA, J. HENDERSON, AND S. K. NTOUYAS

Received 13 May 2001

A fixed point theorem for condensing maps due to Martelli is used to investigate the existence of solutions to second-order impulsive initial value problem for functional differential inclusions in Banach spaces.

1. Introduction

Differential equations arise in many real world problems such as physics, population dynamics, ecology, biological systems, biotechnology, industrial robotics, pharmacokinetics, optimal control, and so forth. Much has been done under the assumption that the state variables and system parameters change continuously. However, one may easily visualize situations in nature where abrupt changes such as shock, harvesting, and disasters may occur. These phenomena are short-term perturbations whose duration is negligible in comparison with the duration of the whole evolution process. Consequently, it is natural to assume, in modelling these problems, that these perturbations act instantaneously, that is, in the form of impulses. For more details on this theory and on its applications we refer to the monographs of Baĭnov and Simeonov [2], Lakshmikantham, Baĭnov, and Simeonov [19], and Samoilenko and Perestyuk [24]. However, very few results are available for impulsive differential inclusions; see for instance, the papers of Benchohra and Boucherif [4, 5], Erbe and Krawcewicz [12], and Frigon and O'Regan [14].

Very recently an extension to functional differential equations of first order with impulsive effects has been done by Yujun [10] by using the coincidence degree theory, and by Benchohra and Ntouyas [7] with the aid of Schaefer's theorem. These results have been also generalized to the multivalued case by the authors in [6] by combining the a priori bounds and the Leray-Schauder

Abstract and Applied Analysis 6:6 (2001) 369–380
2000 Mathematics Subject Classification: 34A37, 34A60, 34G20
URL: http://aaa.hindawi.com/volume-6/S1085337501000604.html

nonlinear alternative for multivalued maps. For other results concerning functional differential equations, we refer the interested reader to the monographs of Erbe, Qingai, and Zhang [13], Hale [15], Henderson [16], and the survey paper of Ntouyas [23].

The fundamental tools used in the existence proofs of all the above-mentioned works are essentially fixed point arguments, nonlinear alternative, topological transversality [11], topological degree theory [22], or the monotone method combined with upper and lower solutions [18].

In this paper, we will be concerned with the existence of solutions of the second-order initial value problem for the impulsive functional differential inclusion

$$y'' \in F\big(t, y_t\big), \quad t \in J = [0, T],\ t \neq t_k,\ k = 1, \ldots, m, \tag{1.1}$$

$$\Delta y|_{t=t_k} = I_k\big(y\big(t_k^-\big)\big), \quad k = 1, \ldots, m, \tag{1.2}$$

$$\Delta y'|_{t=t_k} = \bar{I}_k\big(y\big(t_k^-\big)\big), \quad k = 1, \ldots, m, \tag{1.3}$$

$$y(t) = \phi(t), \quad t \in [-r, 0], \qquad y'(0) = y_0, \tag{1.4}$$

where $F : J \times C([-r,0], E) \to 2^E$ is a given multivalued map with compact and convex values, $(0 < r < \infty)$, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, $I_k, \bar{I}_k \in C(E, E)$ $(k = 1, 2, \ldots, m)$ are bounded, $y_0 \in E$, $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$, $\Delta y'|_{t=t_k} = y'(t_k^+) - y'(t_k^-)$ and $y(t_k^-)$, $y(t_k^+)$, $y'(t_k^-)$ and $y'(t_k^+)$ represent the left and right limits of $y(t)$ and $y'(t)$, respectively at $t = t_k$, and E a real Banach space with norm $|\cdot|$.

For any continuous function y defined on the interval $[-r, T] - \{t_1, \ldots, t_m\}$ and any $t \in J$, we denote by y_t the element of $C([-r,0], E)$ defined by

$$y_t(\theta) = y(t + \theta), \quad \theta \in [-r, 0]. \tag{1.5}$$

Here $y_t(\cdot)$ represents the history of the state from time $t - r$, up to the present time t.

In this paper, we will generalize the results of Benchohra and Ntouyas [8] considered for second-order impulsive functional differential equations to the multivalued case. Our approach is based on a fixed point theorem for condensing maps due to Martelli [21].

2. Preliminaries

In this section, we introduce notations, definitions, and results which are used throughout the paper.

Let $[a, b]$ denote a real compact interval of $\mathbb{R}$. Let $C([a,b], E)$ be the Banach space of continuous functions from $[a, b]$ into E with norm

$$\|y\|_\infty = \sup\big\{|y(t)| : t \in [a, b]\big\} \quad \forall y \in C\big([a, b], E\big). \tag{2.1}$$

Let $y : [a,b] \to E$ be a measurable function. By $\int_a^b y(s)ds$, we mean the Bochner integral of y, assuming it exists. A measurable function $y : [a,b] \to E$ is Bochner integrable if and only if $|y|$ is Lebesgue integrable. For properties of the Bochner integral, see Yosida [25].

$L^1([a,b],E)$ denotes the Banach space of functions Bochner integrable normed by

$$\|y\|_{L^1} = \int_a^b |y(t)|dt \quad \forall y \in L^1\big([a,b],E\big). \tag{2.2}$$

Let $(X,|\cdot|)$ be a Banach space. A multivalued map $G : X \to 2^X$ has convex (closed) values if $G(x)$ is convex (closed) for all $x \in X$. G is bounded on bounded sets if $G(B)$ is bounded in X for each bounded set B of X (i.e., $\sup_{x\in B}\{\sup\{|y| : y \in G(x)\}\} < \infty$).

G is called upper semicontinuous (u.s.c.) on X if for each $x_0 \in X$ the set $G(x_0)$ is a nonempty, closed subset of X, and if for each open set N of X containing $G(x_0)$, there exists an open neighbourhood M of x_0 such that $G(M) \subseteq N$. G is said to be completely continuous if $G(B)$ is relatively compact for every bounded subset $B \subseteq X$.

If the multivalued G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph (i.e., $x_n \to x_*$, $y_n \to y_*$, $y_n \in G(x_n)$ imply $y_* \in G(x_*)$). G has a fixed point if there is $x \in X$ such that $x \in G(x)$.

In the following $CC(E)$ denotes the set of all nonempty compact, convex subsets of E.

A multivalued map $G : [a,b] \to CC(X)$ is said to be measurable if for each $x \in E$ the function $Y : [a,b] \to \mathbb{R}$ defined by

$$Y(t) = d\big(x, G(t)\big) = \inf\big\{|x - z| : z \in G(t)\big\} \tag{2.3}$$

is measurable. For more details on multivalued maps see Aubin and Frankowska [1], Deimling [9], and Hu and Papageorgiou [17].

An upper semi-continuous map $G : X \to 2^X$ is said to be condensing [3] if for any subset $B \subseteq X$ with $\alpha(B) \neq 0$, we have $\alpha(G(B)) < \alpha(B)$, where α denotes the Kuratowski measure of noncompacteness [3].

We remark that a completely continuous multivalued map is the easiest example of a condensing map.

Definition 2.1. A multivalued map $F : J \times C([-r,0],E) \to 2^E$ is said to be an L^1-Carathéodory if

(i) $t \mapsto F(t,u)$ is measurable for each $u \in C([-r,0],E)$;

(ii) $u \mapsto F(t,u)$ is upper semicontinuous for a.a. $t \in J$;

(iii) for each $\rho > 0$, there exists $h_\rho \in L^1(J,\mathbb{R}_+)$ such that

$$\big\|F(t,u)\big\| = \sup\big\{|v| : v \in F(t,u)\big\} \leq h_\rho(t) \quad \forall \|u\| \leq \rho \text{ and for a.a. } t \in J. \tag{2.4}$$

In order to define the solution of (1.1), (1.2), (1.3), and (1.4) we will consider the following space $\Omega = \{y : [-r, T] \to E : y_k \in C(J_k, E),\ k = 0, \dots, m$ and there exist $y(t_k^-)$, and $y(t_k^+)$, $k = 1, \dots, m$ with $y(t_k^-) = y(t_k)$, $y(t) = \phi(t)$, for all $t \in [-r, 0]\}$ which is a Banach space with the norm

$$\|y\|_\Omega = \max\{\|y_k\|_{J_k},\ k = 0, \dots, m\}, \tag{2.5}$$

where y_k is the restriction of y to $J_k = [t_k, t_{k+1}]$, $k = 0, \dots, m$.

We will also consider the set $\Omega^1 = \{y : [-r, T] \to E : y_k \in W^{2,1}(J_k, E)$, $k = 0, \dots, m$ and there exist $y(t_k^-)$ and $y(t_k^+)$, $k = 1, \dots, m$ with $y(t_k^-) = y(t_k)$, $y(t) = \phi(t)$, for all $t \in [-r, 0]\}$, where $W^{2,1}(J_k, E)$ is the Sobolev space of functions $y : J_k \to E$ such that y and y' are absolutely continuous, and $y'' \in L^1(J_k, E)$. The set Ω^1 is a Banach space with the norm

$$\|y\|_{\Omega^1} = \max\{\|y_k\|_{W^{2,1}(J_k,E)},\ k = 0, \dots, m\}. \tag{2.6}$$

Let I be a compact real interval. For any $y \in C(I, E)$ we define the set

$$S_{F,y} = \{v \in L^1(I, E) : v(t) \in F(t, y) \text{ for a.e. } t \in I\}. \tag{2.7}$$

Definition 2.2. A function $y \in \Omega \cap \Omega^1$ is said to be a solution of (1.1), (1.2), (1.3), and (1.4) if y satisfies the differential inclusion $y''(t) \in F(t, y_t)$ a.e. on $J - \{t_1, \dots, t_m\}$ and the conditions $\Delta y|_{t=t_k} = I_k(y(t_k^-))$, $\Delta y'|_{t=t_k} = \bar{I}_k(y(t_k^-))$, $k = 1, \dots, m$.

The following lemmas are crucial in the proof of our main theorem.

LEMMA 2.3 [20]. *Let I be a compact real interval and X a Banach space. Let F be a multivalued map satisfying the Carathéodory conditions with the set of L^1-selections S_F is nonempty, and let Γ be a linear continuous mapping from $L^1(I, X)$ to $C(I, X)$. Then the operator*

$$\Gamma \circ S_F : C(I, X) \longrightarrow CC(C(I, X)), \qquad y \longmapsto (\Gamma \circ S_F)(y) := \Gamma(S_{F,y}), \tag{2.8}$$

is a closed graph operator in $C(I, X) \times C(I, X)$.

LEMMA 2.4 [21]. *Let $G : X \to CC(X)$ be an u.s.c. condensing map. If the set*

$$\mathcal{M} := \{y \in X : \lambda y \in G(y) \text{ for some } \lambda > 1\} \tag{2.9}$$

is bounded, then G has a fixed point.

We introduce the following hypotheses:

(H1) $F : J \times C([-r, 0], E) \to CC(E)$; $(t, u) \mapsto F(t, u)$ is an L^1-Carathéodory multivalued map and for each fixed $u \in C([-r, 0], E)$ the set

$$S_{F,u} = \{g \in L^1(J, E) : g(t) \in F(t, u) \text{ for a.e. } t \in J\} \tag{2.10}$$

is nonempty;

(H2) there exist constants c_k, d_k such that $|I_k(y)| \le c_k$, $|\bar{I}_k(y)| \le d_k$, $k = 1,\dots,m$ for each $y \in E$.

(H3) $\|F(t,u)\| := \sup\{|v| : v \in F(t,u)\} \le p(t)\psi(\|u\|)$ for a.a. $t \in J$ and all $u \in C(J_0,E)$, where $p \in L^1(J,\mathbb{R}_+)$ and $\psi : \mathbb{R}_+ \to (0,\infty)$ is continuous and increasing with

$$\int_0^T (T-s)p(s)ds < \int_c^\infty \frac{d\tau}{\psi(\tau)}; \tag{2.11}$$

where $c = \|\phi\| + T|y_0| + \sum_{k=1}^m [c_k + (T-t_k)d_k]$;

(H4) for each bounded $B \subseteq C([-r,T],E)$, and for each $t \in J$ the set

$$\begin{aligned}\Big\{\phi(0) + ty_0 + \int_0^t (t-s)g(s)ds \\ + \sum_{0<t_k<t} \big[I_k\big(y(t_k)\big) + \big(t-t_k\big)\bar{I}_k\big(y(t_k)\big)\big] : g \in S_{F,B}\Big\}\end{aligned} \tag{2.12}$$

is relatively compact in E, where $S_{F,B} = \cup\{S_{F,y}; y \in B\}$.

Remark 2.5. (i) If $\dim E < \infty$, then for each $u \in C([-r,0],E)$, $S_{F,u} \ne \emptyset$ (see Lasota and Opial [20]).

(ii) If $\dim E = \infty$ and $u \in C([-r,0],E)$ the set $S_{F,u}$ is nonempty if and only if the function $Y : J \to \mathbb{R}$ defined by

$$Y(t) := \inf\big\{|v| : v \in F(t,u)\big\} \tag{2.13}$$

belongs to $L^1(J,\mathbb{R})$ (see Hu and Papageorgiou [17]).

(iii) If $\dim E < \infty$, then (H4) is satisfied.

We have the following auxiliary result. In what follows we will use the notation $\sum_{0<t_k<t}[y(t_k^+) - y(t_k)]$ to mean 0, when $k = 0$ and $0 < t < t_1$, and to mean $\sum_{i=1}^k [y(t_k^+) - y(t_k)]$, when $k \ge 1$ and $t_k < t \le t_{k+1}$.

Lemma 2.6. *If $y \in \Omega \cap \Omega^1$, then*

$$\begin{aligned}y(t) = y(0) + ty'(0) + \int_0^t (t-s)y''(s)ds \\ + \sum_{0<t_k<t} \big\{\big[y\big(t_k^+\big) - y\big(t_k\big)\big] + \big(t-t_k\big)\big[y'\big(t_k^+\big) - y'\big(t_k\big)\big]\big\} \quad \textit{for } t \in J.\end{aligned} \tag{2.14}$$

Proof. Recall that $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$. We first show that

$$y(t) = y(0) + \int_0^t y'(s)ds + \sum_{0<t_k<t} \big\{\big[y\big(t_k^+\big) - y\big(t_k\big)\big]\big\} \quad \text{for } t \in J. \tag{2.15}$$

Suppose $t_k < t \le t_{k+1}$. Then

$$\begin{aligned} y(t_1) - y(0) &= \int_0^{t_1} y'(s)ds, \\ y(t_2) - y(t_1^+) &= \int_{t_1}^{t_2} y'(s)ds, \\ &\vdots \\ y(t_k) - y(t_{k-1}^+) &= \int_{t_{k-1}}^{t_k} y'(s)ds, \\ y(t) - y(t_k^+) &= \int_{t_k}^{t} y'(s)ds. \end{aligned} \tag{2.16}$$

Adding these equalities together, we get

$$y(t) - y(0) - \sum_{i=1}^{k} \left[y(t_i^+) - y(t_i)\right] = \int_0^t y'(s)ds. \tag{2.17}$$

Hence

$$y(t) = y(0) + \int_0^t y'(s)ds + \sum_{0<t_k<t} \left[y(t_k^+) - y(t_k)\right]. \tag{2.18}$$

Similarly, we have

$$y'(t) = y'(0) + \int_0^t y''(s)ds + \sum_{0<t_k<t} \left[y'(t_k^+) - y'(t_k)\right]. \tag{2.19}$$

Substituting (2.19) into (2.15), it is easy to get (2.14). □

3. Main result

THEOREM 3.1. *Suppose that hypotheses (H1), (H2), (H3), and (H4) are satisfied. Then the impulsive initial value problem (1.1), (1.2), (1.3), and (1.4) has at least one solution on* $[-r, T]$.

Proof. Transform the problem into a fixed point problem. Consider the multivalued map, $G : \Omega \to 2^{\Omega}$ defined by

$$G(y) = \left\{ h \in \Omega : h(t) = \begin{cases} \phi(t), \ t \in [-r, 0] \\ \phi(0) + t y_0 + \displaystyle\int_0^t (t-s)g(s)ds \\ + \displaystyle\sum_{0<t_k<t} \left[I_k(y(t_k)) + (t - t_k)\bar{I}_k(y(t_k))\right], \ t \in J \end{cases} \right\}, \tag{3.1}$$

where

$$g \in S_{F,y} = \{g \in L^1(J,E) : g(t) \in F(t, y_t) \text{ for a.e. } t \in J\}. \tag{3.2}$$

Remark 3.2. Clearly from Lemma 2.6 the fixed points of G are solutions to (1.1), (1.2), (1.3), and (1.4).

We will show that G satisfies the assumptions of Lemma 2.4. The proof will be given in several steps.

Step 1. $G(y)$ is convex for each $y \in \Omega$.

Indeed, if h_1, h_2 belong to $G(y)$, then there exist $g_1, g_2 \in S_{F,y}$ such that for each $t \in J$ we have

$$\begin{aligned} h_i(t) &= \phi(0) + t y_0 + \int_0^t (t-s) g_i(s)\,ds \\ &\quad + \sum_{0<t_k<t} \left[I_k(y(t_k)) + (t-t_k)\bar{I}_k(y(t_k)) \right], \quad i = 1,2. \end{aligned} \tag{3.3}$$

Let $0 \le d \le 1$. Then for each $t \in J$ we have

$$\begin{aligned} (dh_1 + (1-d)h_2)(t) &= \phi(0) + t y_0 + \int_0^t (t-s)\left[d g_1(s) + (1-d) g_2(s)\right] ds \\ &\quad + \sum_{0<t_k<t} \left[I_k(y(t_k)) + (t-t_k)\bar{I}_k(y(t_k)) \right]. \end{aligned} \tag{3.4}$$

Since $S_{F,y}$ is convex (because F has convex values) then

$$dh_1 + (1-d)h_2 \in G(y). \tag{3.5}$$

Step 2. G maps bounded sets into bounded sets in Ω.

Indeed, it is enough to show that there exists a positive constant ℓ such that for each $h \in G(y)$ with $y \in B_q = \{y \in \Omega : \|y\|_\infty \le q\}$ one has $\|h\|_\infty \le \ell$. If $h \in G(y)$, then there exists $g \in S_{F,y}$ such that for each $t \in J$ we have

$$\begin{aligned} h(t) &= \phi(0) + t y_0 + \int_0^t (t-s) g(s)\,ds \\ &\quad + \sum_{0<t_k<t} \left[I_k(y(t_k)) + (t-t_k)\bar{I}_k(y(t_k)) \right]. \end{aligned} \tag{3.6}$$

By (H2) and (H3) we have for each $t \in J$

$$\begin{aligned}
|h(t)| &\le \|\phi\| + t|y_0| + \int_0^t (t-s)|g(s)|ds \\
&\quad + \sum_{0<t_k<t} \big[|I_k(y(t_k))| + |(t-t_k)||\bar{I}_k(y(t_k))| \big] \\
&\le \|\phi\| + T|y_0| + \int_0^t (t-s)|g_q(s)|ds \\
&\quad + \sum_{k=1}^m \big[\sup\{|I_k(|y|)| : \|y\|_\infty \le q\} \\
&\qquad + (T-t_k)\sup\{|\bar{I}_k(|y|)| : \|y\|_\infty \le q\}\big].
\end{aligned} \tag{3.7}$$

Thus

$$\begin{aligned}
\|h\|_\infty &\le \|\phi\| + T|y_0| + \int_0^T (T-s)|g_q(s)|ds \\
&\quad + \sum_{k=1}^m \big[\sup\{|I_k(|y|)| : \|y\|_\infty \le q\} \\
&\qquad + (T-t_k)\sup\{|\bar{I}_k(|y|)| : \|y\|_\infty \le q\}\big] = \ell.
\end{aligned} \tag{3.8}$$

Step 3. G maps bounded sets into equicontinuous sets of Ω.

Let $r_1, r_2 \in J$, $r_1 < r_2$ and $B_q = \{y \in \Omega : \|y\|_\infty \le q\}$ a bounded set of Ω.

For each $y \in B_q$ and $h \in G(y)$, there exists $g \in S_{F,y}$ such that

$$h(t) = \phi(0) + ty_0 + \int_0^t (t-s)g(s)ds + \sum_{0<t_k<t} \big[I_k(y(t_k)) + (t-t_k)\bar{I}_k(y(t_k)) \big]. \tag{3.9}$$

Thus

$$\begin{aligned}
|h(r_2) - h(r_1)| &\le (r_2 - r_1)|y_0| + \int_{r_1}^{r_2} (s-r_1)g_q(s)ds + (r_2 - r_1)\int_0^{r_2} g_q(s)ds \\
&\quad + \sum_{0<t_k<r_2-r_1} \big[I_k(y(t_k)) + (r_2 - r_1)\bar{I}_k(y(t_k)) \big].
\end{aligned} \tag{3.10}$$

As $r_2 \to r_1$ the right-hand side of the above inequality tends to zero.

The equicontinuity for the cases $r_1 < r_2 \le 0$ and $r_1 \le 0 \le r_2$ are obvious.

Step 4. G has a closed graph.

Let $y_n \to y_*, h_n \in G(y_n)$, and $h_n \to h_*$. We will prove that $h_* \in G(y_*)$. $h_n \in G(y_n)$ means that there exists $g_n \in S_{F,y_n}$ such that for each $t \in J$

$$h_n(t) = \phi(0) + ty_0 + \int_0^t (t-s)g_n(s)ds + \sum_{0<t_k<t} \big[I_k(y_n(t_k)) + (t-t_k)\bar{I}_k(y_n(t_k)) \big]. \tag{3.11}$$

We must prove that there exists $g_* \in S_{F,y_*}$ such that for each $t \in J$

$$\begin{aligned} h_*(t) &= \phi(0) + ty_0 + \int_0^t (t-s) g_*(s)\,ds \\ &\quad + \sum_{0<t_k<t} \big[I_k\big(y_*(t_k)\big) + \big(t-t_k\big)\bar{I}_k\big(y_*(t_k)\big)\big]. \end{aligned} \tag{3.12}$$

Clearly, since I_k and $\bar{I}_k$, $k = 1, \dots, m$ are continuous we have

$$\begin{aligned} &\Bigg\| \Bigg(h_n - \phi(0) - ty_0 - \sum_{0<t_k<t} \big[I_k\big(y_n(t_k)\big) + \big(t-t_k\big)\bar{I}_k\big(y_n(t_k)\big)\big] \Bigg) \\ &\qquad - \Bigg(h_* - \phi(0) - ty_0 - \sum_{0<t_k<t} \big[I_k\big(y_*(t_k)\big) + \big(t-t_k\big)\bar{I}_k\big(y_*(t_k)\big)\big] \Bigg) \Bigg\|_\infty \\ &\quad \longrightarrow 0, \qquad \text{as } n \longrightarrow \infty. \end{aligned} \tag{3.13}$$

Consider the linear continuous operator

$$\begin{aligned} \Gamma : L^1(J,E) &\longrightarrow C(J,E), \\ g \longmapsto \Gamma(g)(t) &= \int_0^t (t-s) g(s)\,ds. \end{aligned} \tag{3.14}$$

From Lemma 2.3, it follows that $\Gamma \circ S_F$ is a closed graph operator.

Moreover, we have

$$\Bigg(h_n(t) - \phi(0) - ty_0 - \sum_{0<t_k<t} \big[I_k\big(y_n(t_k)\big) + \big(t-t_k\big)\bar{I}_k\big(y_n(t_k)\big)\big] \Bigg) \in \Gamma\big(S_{F,y_n}\big). \tag{3.15}$$

Since $y_n \to y_*$, it follows from Lemma 2.3 that

$$\Bigg(h_*(t) - \phi(0) - ty_0 - \sum_{0<t_k<t} \big[I_k\big(y_*(t_k)\big) + \big(t-t_k\big)\bar{I}_k\big(y_*(t_k)\big)\big] \Bigg) = \int_0^t g_*(s)\,ds \tag{3.16}$$

for some $g_* \in S_{F,y_*}$.

Step 5. Now it remains to show that the set

$$\mathcal{M} := \big\{ y \in C\big([-r,T],E\big) : \lambda y \in G(y), \text{ for some } \lambda > 1 \big\} \tag{3.17}$$

is bounded.

Let $y \in \mathcal{M}$. Then $\lambda y \in G(y)$ for some $\lambda > 1$. Thus there exists $g \in S_{F,y}$ such that

$$\begin{aligned} y(t) = \lambda^{-1}\phi(0) + \lambda^{-1} t y_0 + \lambda^{-1}\int_0^t (t-s)g(s)ds \\ + \lambda^{-1}\sum_{0<t_k<t}\left[I_k\big(y(t_k)\big) + \big(t-t_k\big)\bar{I}_k\big(y(t_k)\big)\right], \quad t \in J. \end{aligned} \tag{3.18}$$

This implies by (H2) and (H3) that for each $t \in J$, we have

$$\left|y(t)\right| \le \|\phi\| + T\left|y_0\right| + \int_0^t (T-s)p(s)\psi\big(\|y_s\|\big)ds + \sum_{k=1}^{m}\left[c_k + \big(T-t_k\big)d_k\right]. \tag{3.19}$$

We consider the function μ defined by

$$\mu(t) = \sup\{|y(s)| : -r \le s \le t\}, \quad 0 \le t \le T. \tag{3.20}$$

Let $t^* \in [-r, t]$ be such that $\mu(t) = |y(t^*)|$. If $t^* \in [0, T]$, by the previous inequality we have for $t \in [0, T]$

$$\mu(t) \le \|\phi\| + T\left|y_0\right| + \int_0^t (T-s)p(s)\psi\big(\mu(s)\big)ds + \sum_{k=1}^{k}\left[c_k + \big(T-t_k\big)d_k\right]. \tag{3.21}$$

If $t^* \in J_0$ then $\mu(t) = \|\phi\|$ and the previous inequality holds.

We take the right-hand side of the above inequality as $v(t)$, then we have

$$\begin{aligned} c = v(0) = \|\phi\| + T\left|y_0\right| + \sum_{k=1}^{m}\left[c_k + \big(T-t_k\big)d_k\right], \\ \mu(t) \le v(t), \quad t \in [0, T]. \end{aligned} \tag{3.22}$$

Using the nondecreasing character of ψ we get

$$v'(t) = (T-t)p(t)\psi\big(\mu(t)\big) \le (T-t)p(t)\psi\big(v(t)\big), \quad t \in [0, T]. \tag{3.23}$$

This implies for each $t \in J$ that

$$\int_{v(0)}^{v(t)} \frac{du}{\psi(u)} \le \int_0^T (T-s)p(s)ds < \int_{v(0)}^{\infty} \frac{du}{\psi(u)}. \tag{3.24}$$

This inequality implies that there exists a constant b such that $v(t) \le b,\ t \in J$, and hence $\mu(t) \le b,\ t \in J$. Since for every $t \in [0, T]$, $\|y_t\| \le \mu(t)$, we have

$$\|y\|_\infty := \sup\left\{\left|y(t)\right| : -r \le t \le T\right\} \le b, \tag{3.25}$$

where b depends only on T and on the functions p and ψ. This shows that $\mathcal{M}$ is bounded.

Set $X := C([-r,T],E)$. As a consequence of Lemma 2.4 we deduce that G has a fixed point y which is a solution of (1.1), (1.2), (1.3), and (1.4). □

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M. Benchohra: Department of Mathematics, University of Sidi Bel Abbes, BP 89, 22000 Sidi Bel Abbes, Algeria
E-mail address: benchohra@yahoo.com

J. Henderson: Department of Mathematics, Auburn University, Auburn, AL 36849-5310, USA
E-mail address: hendej2@mail.auburn.edu

S. K. Ntouyas: Department of Mathematics, University of Ioannina, 451 10 Ioannina, Greece
E-mail address: sntouyas@cc.uoi.gr