SIGN-CHANGING AND MULTIPLE SOLUTIONS FOR THE p-LAPLACIAN

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We obtain a positive solution, a negative solution, and a sign-changing solution for a class of p-Laplacian problems with jumping nonlinearities using variational and super-subsolution methods.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$, be a bounded domain with a smooth boundary $\partial\Omega$. In this paper, we consider the quasilinear elliptic boundary value problem

$$u \in W_0^{1,p}(\Omega) : -\Delta_p u = f(x,u) \quad \text{in } W^{-1,p}(\Omega), \tag{1.1}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian, $1 < p < \infty$. By $W^{1,p}(\Omega)$ we denote the usual Sobolev space with dual space $(W^{1,p}(\Omega))^*$, and $W_0^{1,p}(\Omega)$ denotes its subspace whose elements have generalized homogeneous boundary values and whose dual space is given by $W^{-1,p}(\Omega)$. We assume the following growth and asymptotic behaviour of the nonlinear right-hand side f of (1.1):

(H1) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying

$$\begin{gathered} |f(x,t)| \leq C\left(|t|^{p-1} + 1\right), \\ f(x,t) = a\left(t^+\right)^{p-1} - b\left(t^-\right)^{p-1} + g(x,t), \end{gathered} \tag{1.2}$$

where

$$\lim_{t \to 0} \frac{g(x,t)}{|t|^{p-1}} = 0 \quad \text{uniformly in } x. \tag{1.3}$$

Here C denotes some generic positive constants.

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The set Σ_p of those points $(a,b) \in \mathbb{R}^2$ for which the asymptotic problem

$$u \in W_0^{1,p}(\Omega) : -\Delta_p u = a(u^+)^{p-1} - b(u^-)^{p-1} \quad \text{in } W^{-1,p}(\Omega) \tag{1.4}$$

has a nontrivial solution, is called the Fučík spectrum of the p-Laplacian on Ω, where $u^+ = \max\{u,0\}$ and $u^- = \max\{-u,0\}$. The Fučík spectrum was introduced in the semilinear case $p = 2$ by Dancer [8] and Fučík [14] who recognized its significance for the solvability of problems with jumping nonlinearities. In the semilinear ODE case $p = 2$, $N = 1$, Fučík [14] showed that Σ_2 consists of a sequence of hyperbolic-like curves passing through the points (λ_l, λ_l), where $\{\lambda_l\}_{l\in\mathbb{N}}$ are the eigenvalues of $-d^2/dx^2$, with one or two curves going through each point. Drábek [12] has recently shown that Σ_p has this same general shape for all $p > 1$ in the ODE case.

In the PDE case $N \geq 2$, much of the work to date on Σ_p has been done for the semilinear case $p = 2$. It is now known that Σ_2 consists, at least locally, of curves emanating from the points (λ_l, λ_l) (see, e.g., [2, 7, 8, 10, 14, 25]). Schechter [28] has shown that Σ_2 contains two continuous and strictly decreasing curves through (λ_l, λ_l), which may coincide, such that the points in the square $(\lambda_{l-1}, \lambda_{l+1})^2$ that are either below the lower curve or above the upper curve are not in Σ_2, while the points between them may or may not belong to Σ_2 when they do not coincide.

In the quasilinear PDE case $p \neq 2$, $N \geq 2$, it is known that the first eigenvalue λ_1 of $-\Delta_p$ is positive, simple, and admits a positive eigenfunction φ_1 (see Lindqvist [20]), so Σ_p clearly contains the two lines $\lambda_1 \times \mathbb{R}$ and $\mathbb{R} \times \lambda_1$. In addition, $\sigma(-\Delta_p)$ has an unbounded sequence of variational eigenvalues $\{\lambda_l\}$ satisfying a standard min-max characterization, and Σ_p contains the corresponding sequence of points $\{(\lambda_l, \lambda_l)\}$. A first nontrivial curve $\mathscr{C}$ in Σ_p through (λ_2, λ_2) asymptotic to $\lambda_1 \times \mathbb{R}$ and $\mathbb{R} \times \lambda_1$ at infinity was recently constructed and variationally characterized by a mountain-pass procedure by Cuesta et al. [6] (see Figure 1.1). More recently, unbounded sequences of curves (analogous to the lower and upper curves of Schechter) have been constructed and variationally characterized by min-max procedures by Micheletti and Pistoia [26] for $p \geq 2$ and by the second author [27] for all $p > 1$.

The main goal of this paper is to identify the set of points (a,b) relative to the Fučík spectrum which ensure the existence of sign-changing solutions of (1.1). More precisely, assuming the existence of a positive supersolution $\overline{u}$ and a negative subsolution $\underline{u}$ of (1.1) and (a,b) located above the curve $\mathscr{C}$, we prove the existence of at least three nontrivial solutions within the order interval $[\underline{u}, \overline{u}]$; a positive solution, a negative solution, and a sign-changing solution.

There are many existence and multiplicity results for (1.1) in the literature (see, e.g., [5, 6, 9, 13, 23, 27]). However, to the best of our knowledge, the first results on sign-changing solutions were obtained only recently by Li and Zhang [17]. In their paper the authors assume that $p > N$ and f is independent of x and locally Lipschitz in t. All these assumptions can be relaxed by our approach

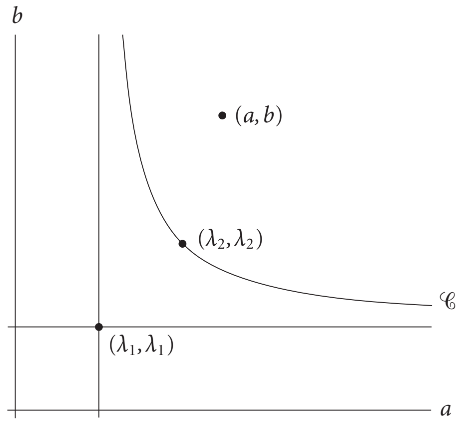

Figure 1.1

which is very different from that of Li and Zhang. Our main result, which will be proved in Section 3 (Theorem 3.1), improves upon their results.

2. Preliminaries

We denote the norm in $W^{1,p}(\Omega)$ and $L^p(\Omega)$ by $\|\cdot\|$ and $\|\cdot\|_p$, respectively, and recall the notion of sub- and supersolutions.

Definition 2.1. A function $\overline{u} \in W^{1,p}(\Omega)$ is a *supersolution* of (1.1) if the following holds:

(i) $\overline{u} \geq 0$ on $\partial\Omega$,
(ii) $\int_\Omega |\nabla \overline{u}|^{p-2}\nabla\overline{u}\cdot\nabla\varphi\, dx \geq \int_\Omega f(x,\overline{u})\varphi\, dx$ for all $\varphi \in W_0^{1,p}(\Omega)\cap L_+^p(\Omega)$.

Similarly, $\underline{u}$ is a *subsolution* of (1.1) if the reversed inequalities of Definition 2.1 hold with $\overline{u}$ replaced by $\underline{u}$. Here $L_+^p(\Omega)$ stands for the positive cone of $L^p(\Omega)$. Consider the boundary value problem

$$u \in W_0^{1,p}(\Omega) : -\Delta_p u = h \quad \text{in } W^{-1,p}(\Omega). \tag{2.1}$$

Besides the hypothesis (H1) we will assume the following hypotheses to hold throughout the rest of the paper.

(H2) There exist a positive supersolution $\overline{u}$ and a negative subsolution $\underline{u}$ of (1.1), and the point $(a,b) \in \mathbb{R}^2$ is above the curve $\mathscr{C}$ of the Fučík spectrum.
(H3) Any solution u of (2.1) with $h \in L^\infty(\Omega)$ belongs to $C^1(\overline{\Omega})$.

Remark 2.2. (i) Assuming the existence of super- and subsolutions as in hypothesis (H2) is a weaker assumption than the usual condition on the jumping nonlinearity at infinity.

(ii) By (H3) we impose $C^1(\overline{\Omega})$-regularity of the solution of (2.1). As for regularity results up to the boundary ($C^{1,\alpha}$-regularity) we refer, for example, to Giaquinta and Giusti [16], Giaquinta [15], Liu and Barrett [21, 22], Lieberman [18, 19], or Manfredi [24].

Lemma 2.3. *If $u \geq$ (resp., $\leq$) 0 is a solution of (1.1), then either $u >$ (resp., $<$) 0 or $u \equiv 0$. Moreover, if $u > 0$, then there is an $\varepsilon > 0$ such that $u \geq \varepsilon\varphi_1$, where φ_1 is the eigenfunction of the first eigenvalue of $-\Delta_p$.*

Proof. First, we note that by the results of Anane [1] and DiBenedetto [11] any solution u of (1.1) belongs to $L^\infty(\Omega) \cap C^1(\Omega)$, and thus the right-hand side of (1.1) yields a function $h \in L^\infty(\Omega)$, which by (H2) implies that $u \in C^1(\overline{\Omega})$. If $u \geq 0$ is a solution of (1.1) which is not identically zero, then by means of the Harnack inequality (Trudinger [29, Theorem 1.1]) u must be positive in Ω. For $\varrho > 0$, let $\Omega_\varrho = \{x \in \overline{\Omega} : \operatorname{dist}(x, \partial\Omega) \leq \varrho\}$. Then for ϱ sufficiently small, we have $f(x, u(x)) \geq 0$ for all $x \in \Omega_\varrho$ by (H1) and (H2). This allows us to apply the strong maximum principle due to Vázquez [30] to get the strict inequality $\partial u/\partial\nu\,(x) > 0$ for all $x \in \partial\Omega$, where ν is the interior normal at x. The eigenfunction φ_1 of the first eigenvalue of $-\Delta_p$ is positive, is of class $C^{1,\alpha}(\overline{\Omega})$ for $\alpha \in (0,1)$, and also satisfies $\partial\varphi_1/\partial\nu\,(x) > 0$ (see [1, 20]). Therefore, for ε sufficiently small, we obtain $u \geq \varepsilon\varphi_1$ in $\overline{\Omega}$. $\square$

Lemma 2.4. *Given a bounded sequence $\{u_n\} \subset W_0^{1,p}(\Omega)$ and a sequence of positive reals $\{\varepsilon_n\}$ with $\varepsilon_n \to 0$ as $n \to \infty$, for a subsequence,*

$$\frac{1}{\varepsilon_n^{p-1}} \int_\Omega |g(x, \varepsilon_n u_n(x))|\,dx \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{2.2}$$

Further, if G is the primitive of g, that is, $G(x,t) = \int_0^t g(x,s)ds$, then

$$\frac{1}{\varepsilon_n^{p}} \int_\Omega |G(x, \varepsilon_n u_n(x))|\,dx \longrightarrow 0 \quad \text{as } n \longrightarrow \infty \tag{2.3}$$

for a subsequence.

Proof. Passing to a subsequence (again denoted by (u_n)), we may assume that $u_n \to u$ a.e. and in $L^p(\Omega)$. By Egoroff's theorem, for any $\mu > 0$ there is a measurable subset Ω_μ of Ω such that $|\Omega \setminus \Omega_\mu| \leq \mu$ and $u_n \to u$ uniformly on Ω_μ. Thus $\varepsilon_n u_n \to 0$ a.e. in Ω_μ. We have

$$\begin{aligned} &\frac{1}{\varepsilon_n^{p-1}} \int_\Omega |g(x, \varepsilon_n u_n(x))|\,dx \\ &\qquad = \int_{\Omega_\mu} \frac{|g(x, \varepsilon_n u_n(x))|}{\varepsilon_n^{p-1}|u_n(x)|^{p-1}} |u_n(x)|^{p-1} dx \\ &\qquad\quad + \int_{\Omega\setminus\Omega_\mu} \frac{|g(x, \varepsilon_n u_n(x))|}{\varepsilon_n^{p-1}|u_n(x)|^{p-1}} |u_n(x)|^{p-1} dx. \end{aligned} \tag{2.4}$$

By (1.2) and (1.3),

$$\frac{|g(x,t)|}{|t|^{p-1}} \le C. \tag{2.5}$$

The first integral on the right-hand side of (2.4) tends to zero by the asymptotic behavior (1.3) of g, (2.5), and Lebesgue's dominated convergence theorem (observe that the integrand is majorized by $C(|u(x)|+\delta)^{p-1}$ for any $\delta>0$ due to the uniform convergence in Ω_μ). The second integral is bounded by

$$C|\Omega\setminus\Omega_\mu|^{1/p}\|u_n\|^{(p-1)/p} \le C\mu^{1/p} \longrightarrow 0 \quad \text{as } \mu \longrightarrow 0, \tag{2.6}$$

which proves (2.2). Observing that the elementary inequality

$$|G(x,\varepsilon_n u_n(x))| \le \varepsilon_n |u_n(x)|\,|g(x,\tau_n(x)\,\varepsilon_n u_n(x))| \tag{2.7}$$

holds, where $0\le \tau_n(x)\le 1$, which yields

$$\frac{1}{\varepsilon_n^p}\int_\Omega |G(x,\varepsilon_n u_n(x))|\,dx \le \int_\Omega \frac{|g(x,\tau_n(x)\varepsilon_n u_n(x))|}{\tau_n(x)^{p-1}\varepsilon_n^{p-1}|u_n(x)|^{p-1}}|u_n(x)|^p dx, \tag{2.8}$$

we see that (2.3) follows similarly. □

Lemma 2.5. *Problem (1.1) has a positive solution $u>0$ within the order interval $[0,\overline{u}]$ and a negative solution $u<0$ within the order interval $[\underline{u},0]$.*

Proof. In the proof we focus on the existence of a positive solution only since the existence of a negative solution can be shown in a similar way.

As is well known, solutions of (1.1) are the critical points of

$$\Phi(u) = \int_\Omega (|\nabla u|^p - pF(x,u))\,dx, \quad u\in W_0^{1,p}(\Omega), \tag{2.9}$$

where $F(x,t)=\int_0^t f(x,s)ds$. Let $\overline{f}$ be the following truncated nonlinearity:

$$\overline{f}(x,t) = \begin{cases} 0, & t\le 0,\\ f(x,t), & 0<t<\overline{u}(x),\\ f(x,\overline{u}(x)), & t\ge \overline{u}(x),\end{cases} \tag{2.10}$$

and $\overline{F}$ its associated primitive given by

$$\overline{F}(x,t) = \int_0^t \overline{f}(x,s)ds. \tag{2.11}$$

Consider the functional

$$\overline{\Phi}(u) = \int_\Omega (|\nabla u|^p - p\overline{F}(x,u))\,dx \tag{2.12}$$

whose critical points are the solutions of the auxiliary boundary value problem

$$u \in W_0^{1,p}(\Omega) : -\Delta_p u = \overline{f}(x,u) \quad \text{in } W^{-1,p}(\Omega). \tag{2.13}$$

Obviously, $\overline{\Phi} : W_0^{1,p}(\Omega) \to \mathbb{R}$ is bounded from below, weakly lower semicontinuous, and coercive. Thus, there is a global minimizer, that is, a critical point u of $\overline{\Phi}$

$$0 = \langle \overline{\Phi}'(u), \varphi \rangle = \int_\Omega \left(|\nabla u|^{p-2} \nabla u \cdot \nabla \varphi - \overline{f}(x,u)\varphi \right) dx. \tag{2.14}$$

We will show that this global minimizer is in fact a positive solution of (1.1) within $[0, \overline{u}]$. Taking in (2.14) the special test function $\varphi = u^- := \max(-u, 0)$ we get in view of the definition of $\overline{f}$ the equation

$$0 = \int_\Omega \left(|\nabla u|^{p-2} \nabla u \cdot \nabla u^- - \overline{f}(x,u)u^- \right) dx = \|u^-\|^p, \tag{2.15}$$

which shows that $u^- = 0$, and thus $u \geq 0$. Since $\overline{u}$ is a supersolution, $\varphi = (u - \overline{u})^+ \in W_0^{1,p}(\Omega) \cap L_+^p(\Omega)$, so by Definition 2.1 and (2.14) we obtain

$$\begin{aligned} 0 &\geq \int_\Omega \Big[\left(|\nabla u|^{p-2} \nabla u - |\nabla \overline{u}|^{p-2} \nabla \overline{u} \right) \cdot \nabla (u - \overline{u})^+ \\ &\qquad - \left(\overline{f}(x,u) - f(x,\overline{u}) \right)(u - \overline{u})^+ \Big] dx \\ &= \int_{\{u > \overline{u}\}} \left(|\nabla u|^{p-2} \nabla u - |\nabla \overline{u}|^{p-2} \nabla \overline{u} \right) \cdot (\nabla u - \nabla \overline{u}) dx \geq 0, \end{aligned} \tag{2.16}$$

which implies that $\nabla (u - \overline{u})^+ = 0$, and thus $u \leq \overline{u}$. This shows that the global minimizer u of the functional $\overline{\Phi}$ satisfies $u \in [0, \overline{u}]$, and thus u is a solution of (1.1) due to the definition of $\overline{f}$. Since $a > \lambda_1$, we get by hypothesis (H1) that

$$\overline{\Phi}(\varepsilon \varphi_1) < 0, \quad \varepsilon > 0 \text{ small.} \tag{2.17}$$

As u is a global minimizer of $\overline{\Phi}$, it follows $\overline{\Phi}(u) \leq \overline{\Phi}(\varepsilon \varphi) < 0$, and thus u must be a positive solution of (1.1). □

Definition 2.6. A solution u_+ is called the *least positive solution* of (1.1) if any other positive solution u of problem (1.1) satisfies $u \geq u_+$. Similarly, u_- is the *greatest negative solution* of (1.1) if any other negative solution u satisfies $u \leq u_-$.

Lemma 2.7. *Problem (1.1) has a least positive solution u_+ and a greatest negative solution u_-.*

Proof. We are going to prove the existence of the least positive solution only, since the proof of the existence of the greatest negative solution is analogous. In view of Lemma 2.5, there exists a positive solution $u \in [0, \overline{u}]$, and applying

Lemma 2.3 there is a $\varepsilon > 0$ small enough such that $\varepsilon\varphi_1 \le u$, where φ_1 is the eigenfunction that belongs to the first eigenvalue λ_1 of $-\Delta_p$. Since $a > \lambda_1$, one readily verifies that $\varepsilon\varphi_1$ is a subsolution of problem (1.1) for sufficiently small $\varepsilon > 0$. Thus there is an $\varepsilon_0 > 0$ such that $\varepsilon_0\varphi_1$ and $\overline{u}$ forms an ordered pair of sub- and supersolutions. Applying [3, Corollary 5.1.2] on the existence of extremal solutions for general quasilinear elliptic problems, we obtain the existence of a least and greatest solution of (1.1) with respect to the order interval $[\varepsilon_0\varphi_1, \overline{u}]$. We denote the least solution within this interval by u_0. Now let $(\varepsilon_n)_{n=0}^{\infty}$ be a decreasing sequence with $\varepsilon_n \to 0$ as $n \to \infty$, and denote by u_n the corresponding least solution of (1.1) with respect to the order interval $[\varepsilon_n\varphi_1, \overline{u}]$. Then obviously (u_n) is a decreasing sequence of least positive solutions of (1.1) which converges to its nonnegative pointwise limit u_* in $L^p(\Omega)$. We will show that u_* is in fact the least positive solution, that is, $u_* = u_+$. First we verify that u_* is a solution of (1.1). Since the u_n are solutions of (1.1) we get from the equation

$$||u_n||^p = \int_\Omega |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla u_n \, dx = \int_\Omega f(x, u_n) u_n \, dx, \tag{2.18}$$

which by the growth condition (H1) and the boundedness in $L^p(\Omega)$ of the sequence (u_n) implies its boundedness in $W_0^{1,p}(\Omega)$, that is, $\|u_n\| \le c$. Thus there exists a subsequence weakly convergent in $W_0^{1,p}(\Omega)$, and due to the strong convergence of (u_n) in $L^p(\Omega)$ even the entire sequence is weakly convergent in $W_0^{1,p}(\Omega)$ with weak limit u_*. From (1.1) with the test function $u_n - u_*$, we obtain

$$\begin{aligned}\langle -\Delta_p u_n, u_n - u_*\rangle &= \int_\Omega |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla (u_n - u_*) dx \\ &= \int_\Omega f(x, u_n)(u_n - u_*) dx,\end{aligned} \tag{2.19}$$

which implies

$$\limsup_n \langle -\Delta_p u_n, u_n - u_*\rangle \le 0. \tag{2.20}$$

The weak convergence of (u_n) and (2.20) along with the S_+-property of the operator $-\Delta_p$ (see, e.g., [3, Chapter D]) yield its strong convergence in $W_0^{1,p}(\Omega)$. This allows the passage to the limit in (1.1) with u replaced by u_n, and hence u_* is a solution of problem (1.1). To show that $u_* > 0$, our argument is by contradiction. Suppose $u_* = 0$, that is, $u_n \to 0$ in $W_0^{1,p}(\Omega)$. Since $u_n > 0$ we may consider $\tilde{u}_n := u_n/\|u_n\|$ which satisfies

$$\int_\Omega |\nabla \tilde{u}_n|^{p-2}\nabla \tilde{u}_n \cdot \nabla \varphi \, dx = \int_\Omega \left[a\tilde{u}_n^{p-1} + \frac{g(x, u_n)}{||u_n||^{p-1}}\right]\varphi \, dx. \tag{2.21}$$

By definition $\|\tilde{u}_n\| = 1$, so there is a subsequence $(\tilde{u}_n)$ that converges weakly in $W_0^{1,p}(\Omega)$ and strongly in $L^p(\Omega)$ to $\tilde{u}$ due to the compact embedding of $W_0^{1,p}(\Omega) \subset L^p(\Omega)$. Taking in (2.21) as special test function $\varphi = \tilde{u}_n - \tilde{u}$, we get for the right-hand side of (2.21)

$$\int_\Omega \left[a\tilde{u}_n^{p-1} + \frac{g(x,u_n)}{|u_n(x)|^{p-1}} |\tilde{u}_n(x)|^{p-1} \right] (\tilde{u}_n - \tilde{u})\, dx \longrightarrow 0, \tag{2.22}$$

as $n \to \infty$, because the terms in parentheses are $L^q(\Omega)$-bounded. Hence (2.21) implies

$$\limsup_n \langle -\Delta_p \tilde{u}_n, \tilde{u}_n - \tilde{u} \rangle \le 0, \tag{2.23}$$

which due to the S_+-property of $-\Delta_p$ implies the strong convergence of $\tilde{u}_n \to \tilde{u}$ in $W_0^{1,p}(\Omega)$. Moreover, the third integral term on the right-hand side of (2.21) converges to zero by Lemma 2.4, so we may pass to the limit to get

$$\int_\Omega |\nabla \tilde{u}|^{p-2} \nabla \tilde{u} \cdot \nabla \varphi\, dx = \int_\Omega a\tilde{u}^{p-1} \varphi\, dx \quad \forall \varphi \in C_0^\infty(\Omega), \tag{2.24}$$

that is, $\tilde{u}$ satisfies the boundary value problem

$$\tilde{u} \in W_0^{1,p}(\Omega) : -\Delta_p \tilde{u} = a\tilde{u}^{p-1} \quad \text{in } W^{-1,p}(\Omega). \tag{2.25}$$

Since $\|\tilde{u}_n\| = 1$ and $\tilde{u}_n > 0$, by Lemma 2.3 we have the same properties for $\tilde{u}$, which, however, contradicts the fact that a nontrivial solution of (2.25) changes sign. So far we have shown that the limit u_* of the least solutions $u_n \in [\varepsilon_n \varphi_1, \overline{u}]$ is a positive solution of (1.1). Finally, to prove that u_* is the least positive solution, let w be any positive solution of (1.1). Then by Lemma 2.3 there is a $\varepsilon_n > 0$ for n sufficiently large such that $\varepsilon_n \varphi_1 \le w$ which by definition of the sequence of least solutions (u_n) yields $u_* \le u_n \le w$, which proves that $u_* = u_+$ is in fact the least positive one. □

3. Main result

THEOREM 3.1. *Let hypotheses (H1), (H2), and (H3) be satisfied. Then the boundary value problem (1.1) has at least three nontrivial solutions: a positive solution, a negative solution, and a sign-changing solution.*

Proof. Let

$$\tilde{f}_+(x,t) = \begin{cases} 0, & t \le 0, \\ f(x,t), & 0 < t < u_+(x), \\ f(x,u_+(x)), & t \ge u_+(x), \end{cases} \quad \tilde{F}_+(x,t) = \int_0^t \tilde{f}_+(x,s)\, ds, \tag{3.1}$$

and consider

$$\widetilde{\Phi}_+(u) = \int_\Omega (|\nabla u|^p - p\widetilde{F}_+(x,u))\,dx. \tag{3.2}$$

Arguments similar to those in the proof of Lemma 2.5 show that critical points of $\widetilde{\Phi}_+$ are solutions of (1.1) in the order interval $[0,u_+]$, so 0 and u_+ are the only critical points of $\widetilde{\Phi}_+$ by Lemmas 2.3 and 2.7. Now, $\widetilde{\Phi}_+$ is bounded from below and coercive, and

$$\widetilde{\Phi}_+(\varepsilon\varphi_1) < 0, \quad \varepsilon > 0 \text{ small} \tag{3.3}$$

since $a > \lambda_1$, so $\widetilde{\Phi}_+$ has a global minimizer at a negative critical level. It follows that u_+ is the (strict) global minimizer of $\widetilde{\Phi}_+$ and $\widetilde{\Phi}_+(u_+) < 0$.

Now let

$$\begin{gathered}\widetilde{f}(x,t) = \begin{cases} f(x,u_-(x)), & t \le u_-(x), \\ f(x,t), & u_-(x) < t < u_+(x), \\ f(x,u_+(x)), & t \ge u_+(x), \end{cases} \quad \widetilde{F}(x,t) = \int_0^t \widetilde{f}(x,s)\,ds, \\ \widetilde{\Phi}(u) = \int_\Omega (|\nabla u|^p - p\widetilde{F}(x,u))\,dx.\end{gathered} \tag{3.4}$$

As before, critical points of $\widetilde{\Phi}$ are solutions of (1.1) in the order interval $[u_-,u_+]$, so it follows from Lemmas 2.3 and 2.7 that any nontrivial critical point other than $u_\pm$ is a sign-changing solution.

LEMMA 3.2. *The solutions $u_\pm$ are strict local minimizers of $\widetilde{\Phi}$, and $\widetilde{\Phi}(u_\pm) < 0$.*

Proof. We only consider u_+ as the argument for u_- is similar. Suppose that there is a sequence $u_j \to u_+$ in $W_0^{1,p}(\Omega)$, $u_j \ne u_+$ with $\widetilde{\Phi}(u_j) \le \widetilde{\Phi}(u_+)$. By (1.2) and (1.3) we have

$$|\widetilde{F}(x,t)| \le C|t|^p, \tag{3.5}$$

so

$$\begin{aligned}\widetilde{\Phi}(u_j) &= \int_\Omega \left(|\nabla u_j^+|^p - p\widetilde{F}(x,u_j^+)\right)dx + \int_\Omega \left(|\nabla u_j^-|^p - p\widetilde{F}(x,u_j^-)\right)dx \\ &\ge \widetilde{\Phi}_+(u_j^+) + \|u_j^-\|^p - C\|u_j^-\|_p^p.\end{aligned} \tag{3.6}$$

If $u_j^- = 0$, then $u_j^+ \ne u_+$ and

$$\widetilde{\Phi}_+(u_j^+) \le \widetilde{\Phi}(u_j) \le \widetilde{\Phi}(u_+) = \widetilde{\Phi}_+(u_+), \tag{3.7}$$

contradicting the fact that u_+ is the unique global minimizer of $\widetilde{\Phi}_+$, so $u_j^- \neq 0$. We will show that

$$\|u_j^-\|^p > C\|u_j^-\|_p^p, \quad j \text{ large}. \tag{3.8}$$

Assuming this for the moment, we have the contradiction $\widetilde{\Phi}_+(u_j^+) < \widetilde{\Phi}_+(u_+)$.

To see that (3.8) holds, we first note that the measure of the set $\Omega_j = \{x \in \Omega : u_j(x) < 0\}$ goes to zero. To see this, given $\varepsilon > 0$, take a compact subset Ω^ε of Ω such that $|\Omega \setminus \Omega^\varepsilon| < \varepsilon$ and let $\Omega_j^\varepsilon = \Omega^\varepsilon \cap \Omega_j$. Then

$$\|u_j - u_+\|_p^p \geq \int_{\Omega_j^\varepsilon} |u_j - u_+|^p dx \geq \int_{\Omega_j^\varepsilon} u_+^p \, dx \geq c^p |\Omega_j^\varepsilon|, \tag{3.9}$$

where $c = \min_{\Omega^\varepsilon} u_+ > 0$, so $|\Omega_j^\varepsilon| \to 0$. Since $\Omega_j \subset \Omega_j^\varepsilon \cup (\Omega \setminus \Omega^\varepsilon)$ and $\varepsilon > 0$ is arbitrary, the claim follows.

If (3.8) does not hold, setting $\tilde{u}_j = u_j^- / \|u_j^-\|_p$, $\|\tilde{u}_j\|$ is bounded for a subsequence, so $\tilde{u}_j \to \tilde{u}$ in $L^p(\Omega)$ and a.e. for a further subsequence, where $\|\tilde{u}\|_p = 1$ and $\tilde{u} \geq 0$. But then $\Omega_\mu = \{x \in \Omega : \tilde{u}(x) \geq \mu\}$ has positive measure for all sufficiently small $\mu > 0$ and

$$\|\tilde{u}_j - \tilde{u}\|_p^p \geq \int_{\Omega_\mu \setminus \Omega_j} |\tilde{u}_j - \tilde{u}|^p \, dx = \int_{\Omega_\mu \setminus \Omega_j} \tilde{u}^p \, dx \geq \mu^p (|\Omega_\mu| - |\Omega_j|), \tag{3.10}$$

a contradiction. □

Now a standard deformation argument gives a mountain-pass point u_1 at the critical value

$$c := \inf_{\gamma \in \Gamma} \max_{u \in \gamma([-1,1])} \widetilde{\Phi}(u) > \widetilde{\Phi}(u_\pm), \tag{3.11}$$

where $\Gamma = \{\gamma \in C([-1,1], W_0^{1,p}(\Omega)) : \gamma(\pm 1) = u_\pm\}$ is the class of paths joining $u_\pm$. To show that $u_1 \neq 0$, we will construct a path that lies in $\widetilde{\Phi}^0 := \{u \in W_0^{1,p}(\Omega) : \widetilde{\Phi}(u) < 0\}$.

First we show that, for all sufficiently small $\varepsilon > 0$, $\pm\varepsilon\varphi_1$ can be joined by a path γ_ε in $\widetilde{\Phi}^0$. We have

$$\widetilde{\Phi}(u) = I_{(a,b)}(u) - \int_\Omega \widetilde{G}(x,u) dx, \tag{3.12}$$

where

$$I_{(a,b)}(u) = \int_\Omega \left(|\nabla u|^p - a(u^+)^p - b(u^-)^p\right) dx \tag{3.13}$$

is the functional associated with (1.4) and

$$\widetilde{G}(x,t) = \widetilde{F}(x,t) - a(t^+)^p - b(t^-)^p = o(|t|^p) \quad \text{as } t \longrightarrow 0. \tag{3.14}$$

Since (a,b) is above $\mathscr{C}$, there is a path γ_0 in $\{u \in W_0^{1,p}(\Omega) : I_{(a,b)}(u) < 0,\ \|u\|_p = 1\}$ joining $\pm\varphi_1$ by the construction of $\mathscr{C}$ [6]. For $u \in \gamma_0([-1,1])$,

$$\widetilde{\Phi}(\varepsilon u) \le \varepsilon^p \left[\max I_{(a,b)}\big(\gamma_0([-1,1])\big) + \int_\Omega \frac{|\widetilde{G}(x,\varepsilon u)|}{\varepsilon^p} dx\right], \tag{3.15}$$

and the last integral goes to 0 uniformly on the compact set $\gamma_0([-1,1])$ as $\varepsilon \to 0$ by Lemma 2.4, so we can take $\gamma_\varepsilon = \varepsilon\gamma_0$.

We complete the proof by showing that $\pm\varepsilon\varphi_1$ and $u_\pm$ can be joined by paths in $\widetilde{\Phi}^0$. Again we only consider $\varepsilon\varphi_1$ and u_+. Setting $\alpha = \inf \widetilde{\Phi}_+ = \widetilde{\Phi}_+(u_+)$ and $\beta = \widetilde{\Phi}_+(\varepsilon\varphi_1) = \widetilde{\Phi}(\varepsilon\varphi_1) < 0$, by the second deformation lemma (see, e.g., Chang [4]), the sublevel set $\widetilde{\Phi}_+^\alpha := \{u \in W_0^{1,p}(\Omega) : \widetilde{\Phi}_+(u) \le \alpha\} = \{u_+\}$ is a strong deformation retract of $\widetilde{\Phi}_+^\beta$, that is, there is an $\eta \in C([0,1] \times \widetilde{\Phi}_+^\beta, \widetilde{\Phi}_+^\beta)$ such that

(i) $\eta(0,u) = u$ for all $u \in \widetilde{\Phi}_+^\beta$,
(ii) $\eta(t,u_+) = u_+$ for all $t \in [0,1]$,
(iii) $\eta(1,u) = u_+$ for all $u \in \widetilde{\Phi}_+^\beta$.

In particular, $\gamma = \eta(\cdot,\varepsilon\varphi_1)$ is a path in $\widetilde{\Phi}_+^\beta$ joining $\varepsilon\varphi_1$ and u_+. Now the path γ_+ defined by $\gamma_+(t) = \gamma(t)^+$ also joins $\varepsilon\varphi_1$ and u_+, and

$$\widetilde{\Phi}(\gamma_+(t)) = \widetilde{\Phi}_+(\gamma(t)) - \int_\Omega |\nabla\gamma(t)^-|^p \le \beta. \tag{3.16}$$

□

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