CONTINUITY AND GENERAL PERTURBATION OF THE DRAZIN INVERSE FOR CLOSED LINEAR OPERATORS

N. CASTRO GONZÁLEZ, J. J. KOLIHA, AND V. RAKOČEVIĆ

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We study perturbations and continuity of the Drazin inverse of a closed linear operator A and obtain explicit error estimates in terms of the gap between closed operators and the gap between ranges and nullspaces of operators. The results are used to derive a theorem on the continuity of the Drazin inverse for closed operators and to describe the asymptotic behavior of operator semigroups.

1. Introduction

In this paper, we investigate a perturbation of the Drazin inverse A^{D} of a closed linear operator A; the main tool for obtaining the estimates is the gap between subspaces and operators.

By $\mathscr{C}(X)$ we denote the set of all closed linear operators acting on a linear subspace of X to X, where X is a complex Banach space. We write $\mathscr{D}(A)$, $\mathscr{N}(A)$, $\mathscr{R}(A)$, $\rho(A)$, $\sigma(A)$, and $R(\lambda, A)$ for the domain, nullspace, range, resolvent set, spectrum, and the resolvent of an operator $A \in \mathscr{C}(X)$. All relevant concepts from the theory of closed linear operators can be found in [3, 14]. The set of all operators $T \in \mathscr{C}(X)$ with $\mathscr{D}(T) = X$ will be denoted by $\mathscr{B}(X)$; we recall that operators in $\mathscr{B}(X)$ are bounded, and the operator norm of $T \in \mathscr{B}(X)$ will be denoted by $\|T\|$. An operator $A \in \mathscr{C}(X)$ is called *quasi-polar* if 0 is a resolvent point or an isolated spectral point of A. The *spectral projection* of a quasi-polar operator $A \in \mathscr{C}(X)$ is the unique idempotent operator $A^\pi \in \mathscr{B}(X)$ such that $AA^\pi x = A^\pi Ax$ for all $x \in \mathscr{D}(A)$, AA^π is quasi-nilpotent, and $A + A^\pi$ is invertible in $\mathscr{C}(X)$.

With an eye to further development in this paper, we choose the following definition of the (generalized) Drazin inverse among the equivalent formulations given in [8].

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Definition 1.1. An operator $A \in \mathscr{C}(X)$ is *Drazin invertible* if A is quasi-polar. The *Drazin inverse* of A is defined by

$$A^{\mathrm{D}} = (A + A^{\pi})^{-1}(I - A^{\pi}), \tag{1.1}$$

where A^{π} is the spectral projection of A corresponding to 0 (see [14, Theorem V.9.2]); note that $A^{\mathrm{D}}, A^{\pi} \in \mathscr{B}(X)$, and that

$$A^{\pi} = I - AA^{\mathrm{D}}. \tag{1.2}$$

The *Drazin index* $i(A)$ of a Drazin invertible operator $A \in \mathscr{C}(X)$ is equal to 0 if A is invertible, to the nilpotency index of the operator AA^{π} if A is singular and AA^{π} nilpotent, and to ∞ if AA^{π} is quasi-nilpotent but not nilpotent.

This definition further generalizes the concept of the Drazin inverse introduced in [4], which in turn is a generalization of the original definition of pseudoinverse given by Drazin [1].

We need another characterization of the Drazin inverse for future use. A proof of the formula is given for completeness.

Lemma 1.2. *If $A \in \mathscr{C}(X)$ is Drazin invertible, then*

$$A^{\mathrm{D}} = f(A), \tag{1.3}$$

where $f(\lambda) = 0$ in Ω_0 and $f(\lambda) = \lambda^{-1}$ in Ω, where Ω_0 and Ω are disjoint open neighbourhoods of 0 *and $\sigma(A) \setminus \{0\}$, respectively.*

Proof. Since A is quasi-polar, we can choose open neighbourhoods Ω_0 and Ω of 0 and $\sigma(A) \setminus \{0\}$, respectively, such that the complement Ω^c of Ω is compact and $\Omega_0 \subset \Omega^c$. Let γ be a cycle in $\Omega \setminus \sigma(A)$ such that $\operatorname{ind}(\gamma, \lambda) = 0$ if $\lambda \in \sigma(A) \setminus \{0\}$ and $\operatorname{ind}(\gamma, \lambda) = 1$ if $\lambda \in \Omega^c$. Using the holomorphic calculus for A with f given in the statement of the lemma, we get

$$\begin{aligned} Af(A) &= -\frac{1}{2\pi i}\int_{\gamma} \lambda^{-1} AR(\lambda, A)\, d\lambda \\ &= \left(\frac{1}{2\pi i}\int_{\gamma} \frac{d\lambda}{\lambda}\right) I - \frac{1}{2\pi i}\int_{\gamma} R(\lambda, A)\, d\lambda = I - A^{\pi}. \end{aligned} \tag{1.4}$$

Since $A^{\pi} = g(A)$, where $g(\lambda) = 1$ in Ω_0 and $g(\lambda) = 0$ in Ω, we have $A^{\pi} f(A) = g(A)f(A) = (gf)(A) = 0$, and $(A + A^{\pi})f(A) = I - A^{\pi}$. Hence

$$f(A) = (A + A^{\pi})^{-1}(I - A^{\pi}) = A^{\mathrm{D}}. \tag{1.5}$$

□

From the last paragraph of the preceding proof we see that the Drazin inverse can also be given by the formula $A^{\mathrm{D}} = (A + \xi A^{\pi})^{-1}(I - A^{\pi})$ for any $\xi \neq 0$; in particular, we have

$$A^{\mathrm{D}} = (A - A^{\pi})^{-1}(I - A^{\pi}). \tag{1.6}$$

2. Some results on the gap between subspaces

First, we quantify results of [7] by obtaining explicit error bounds for the norm distance $\|P - Q\|$ between two idempotent operators $P, Q \in \mathcal{B}(X)$ in terms of the gap between their null spaces and ranges.

Let $\mathcal{S}(X)$ be the set of all closed subspaces of the Banach space X. The *gap between subspaces* $M, N \in \mathcal{S}(X)$ is defined in [2] and [3, Section IV.2.1]:

$$\operatorname{gap}(M,N) = \max\{\Delta(M,N), \Delta(N,M)\}, \tag{2.1}$$

where $\Delta(M,N) = \sup\{\operatorname{dist}(x,N) : x \in M, \ \|x\| = 1\}$. We summarize the properties of the gap function relevant to our investigation in the following lemma; these results can be found in [2, 3].

Lemma 2.1. *The following are true for any* $M, N, K \in \mathcal{S}(X)$*:*

(i) $0 \leq \operatorname{gap}(M,N) \leq 1$*;*
(ii) $\operatorname{gap}(M,N) = 0$ *implies that* $M = N$*;*
(iii) $\operatorname{gap}(M,N) = \operatorname{gap}(N,M)$*;*
(iv) $\operatorname{gap}(M,N) \leq 2(\operatorname{gap}(M,K) + \operatorname{gap}(K,N))$*;*
(v) $\operatorname{gap}(M^{\perp}, N^{\perp}) = \operatorname{gap}(M,N)$*, where* $M^{\perp}$*,* $N^{\perp}$ *are the annihilators of* M*,* N *in* X^**;*
(vi) *if* $\operatorname{gap}(M,N) < 1$*, then* M *and* N *have the same dimension.*

The sets $U(M,\varepsilon) = \{N \in \mathcal{S}(X) : \operatorname{gap}(M,N) < \varepsilon\}$, where $M \in \mathcal{S}(X)$ and $\varepsilon > 0$, form a base for the so-called *gap topology* on $\mathcal{S}(X)$; this is a complete metrizable topology on $\mathcal{S}(X)$ [3, Section IV.2.1]. For a detailed discussion of the gap topology see [10]. We write $M_n \xrightarrow{\mathcal{G}} M$ for $\operatorname{gap}(M_n, M) \to 0$ in $\mathcal{S}(X)$.

Proposition 2.2. *Let* $P, Q \in \mathcal{B}(X)$ *be idempotent operators and let*

$$\nu(P,Q) = \operatorname{gap}(\mathcal{N}(P), \mathcal{N}(Q)), \qquad \rho(P,Q) = \operatorname{gap}(\mathcal{R}(P), \mathcal{R}(Q)). \tag{2.2}$$

Then

$$\begin{aligned} \max\{\nu(P,Q), \rho(P,Q)\} &\leq \|P - Q\| \\ &\leq \frac{\|P\|\,\|I - P\|\,(\nu(P,Q) + \rho(P,Q))}{1 - \|P\|\nu(P,Q) - \|I - P\|\rho(P,Q)}, \end{aligned} \tag{2.3}$$

where the second inequality holds if

$$\|P\|\nu(P,Q) + \|I - P\|\rho(P,Q) < 1. \tag{2.4}$$

Proof. The first inequality in (2.3) is well known, see [2, equation (1.4)].

To prove the second inequality in (2.3), we apply [7, Lemma 3.1] to obtain the following string of inequalities:

$$\begin{aligned}
\|P-Q\| &\le \|(I-P)Q\| + \|P(I-Q)\| \\
&\le \|I-P\|\,\|Q\|\rho(P,Q) + \|P\|\,\|I-Q\|\nu(P,Q) \\
&\le \|I-P\|\big(\|P\|+\|P-Q\|\big)\rho(P,Q) \\
&\quad + \|P\|\big(\|I-P\|+\|P-Q\|\big)\nu(P,Q) \\
&= \|I-P\|\,\|P\|\big(\nu(P,Q)+\rho(P,Q)\big) \\
&\quad + \big(\|I-P\|\rho(P,Q)+\|P\|\nu(P,Q)\big)\|P-Q\|.
\end{aligned} \tag{2.5}$$

Then

$$\begin{aligned}
&\big(1-\|I-P\|\rho(P,Q)-\|P\|\nu(P,Q)\big)\|P-Q\| \\
&\qquad\le \|I-P\|\,\|P\|\big(\nu(P,Q)+\rho(P,Q)\big),
\end{aligned} \tag{2.6}$$

and the second inequality in (2.3) is proved provided that (2.4) holds. □

From Proposition 2.2, we recover [7, Lemma 3.3].

Corollary 2.3. *Let $P_n, P \in \mathcal{B}(X)$ be idempotent operators. Then*

$$\|P_n - P\| \longrightarrow 0 \Longleftrightarrow \mathcal{N}(P_n) \xrightarrow{\mathcal{G}} \mathcal{N}(P), \qquad \mathcal{R}(P_n) \xrightarrow{\mathcal{G}} \mathcal{R}(P). \tag{2.7}$$

Let H be a Hilbert space. Then the gap between closed subspaces M, N is equal to

$$\operatorname{gap}(M,N) = \|P_M - P_N\|, \tag{2.8}$$

with P_M, P_N as the orthogonal projections onto M, N, respectively, [3, Theorem I.6.34]. Rakočević recently proved [13] that, for an idempotent operator P, $\operatorname{gap}(\mathcal{N}(P)^\perp, \mathcal{R}(P)) = (1-\|P\|^{-2})^{1/2}$, that is,

$$\|P\| = \big(1-\operatorname{gap}^2\big(\mathcal{N}(P)^\perp,\mathcal{R}(P)\big)\big)^{-1/2}. \tag{2.9}$$

This yields a simple proof of the well-known equality

$$\|I-P\| = \|P\|. \tag{2.10}$$

3. Perturbations of the Drazin inverse

Let A and B be closed linear operators on X. In the case that both operators are Drazin invertible, we want to derive an error bound for the norm $\|B^{\mathrm{D}} - A^{\mathrm{D}}\|$. From [7, 12] we recall that for Drazin invertible bounded linear operators satisfying $A_n \to A$ (in the operator norm), we have

$$A_n^{\mathrm{D}} \longrightarrow A^{\mathrm{D}} \Longleftrightarrow A_n^{\pi} \longrightarrow A^{\pi}. \tag{3.1}$$

This suggests that we should consider perturbations B of A with B close to A in some sense and also with $\|B^\pi - A^\pi\|$ small.

If $A \in \mathscr{C}(X)$, we write $\mathrm{G}(A) = \{(u, Au) \in X \times X : u \in \mathscr{D}(A)\}$ for the graph of the operator A. For any $A \in \mathscr{C}(X)$, the set $\mathrm{G}(A)$ is a closed subspace of the product space $X \times Y$ equipped with the norm $\|(x, y)\| = (\|x\|^2 + \|y\|^2)^{1/2}$. We can then define the *gap between the operators* $A, B \in \mathscr{C}(X)$ [3, Section IV.4.4] by

$$\operatorname{gap}(A, B) = \operatorname{gap}\left(\mathrm{G}(A), \mathrm{G}(B)\right). \tag{3.2}$$

The sets $\mathscr{U}(A, \varepsilon) = \{B \in \mathscr{C}(X) : \operatorname{gap}(A, B) < \varepsilon\}$, where $A \in \mathscr{C}(X)$ and $\varepsilon > 0$, form a base for the so-called *operator gap topology* in $\mathscr{C}(X)$; this is (in general an incomplete) metrizable topology on $\mathscr{C}(X)$. For a detailed discussion of this topology see [11]. We write $A_n \xrightarrow{\mathscr{G}} A$ for $\operatorname{gap}(A_n, A) \to 0$ in $\mathscr{C}(X)$.

In the sequel, we need the following result on holomorphic calculus for closed operators that follows from [3, Theorem IV.3.15].

Lemma 3.1. *Let $A, A_n \in \mathscr{C}(X)$ $(n = 1, 2, \ldots)$. Let G be an open set containing $\sigma(A)$ and $\sigma(A_n)$ $(n = 1, 2, \ldots)$ and Ω an open neighborhood of $\bar{G}$ with Ω^c compact. If f is a function holomorphic on $\Omega \cup \{\infty\}$, then*

$$A_n \xrightarrow{\mathscr{G}} A \Longrightarrow \|f(A_n) - f(A)\| \longrightarrow 0. \tag{3.3}$$

For the rest of the section we assume that $A, B \in \mathscr{C}(X)$ are Drazin invertible. Then

$$\begin{aligned} B^{\mathrm{D}} - A^{\mathrm{D}} &= (B + B^\pi)^{-1}(I - B^\pi) - (A + A^\pi)^{-1}(I - B^\pi) \\ &\quad + (A + A^\pi)^{-1}(I - B^\pi) - (A + A^\pi)^{-1}(I - A^\pi) \\ &= \left[(B + B^\pi)^{-1} - (A + A^\pi)^{-1}\right](I - B^\pi) + (A + A^\pi)^{-1}(A^\pi - B^\pi). \end{aligned} \tag{3.4}$$

In order to obtain an error bound for $\|B^{\mathrm{D}} - A^{\mathrm{D}}\|$ we estimate $\|(B + B^\pi)^{-1} - (A + A^\pi)^{-1}\|$, $\|I - B^\pi\|$ in terms of the quantities $\operatorname{gap}(A, B)$ and $\|B^\pi - A^\pi\|$.

First we derive an upper estimate for $\|(B + B^\pi)^{-1} - (A + A^\pi)^{-1}\|$ writing

$$\alpha = 1 + \left\|(A + A^\pi)^{-1}\right\|^2, \qquad \beta = 1 + \|A^\pi\|^2 \tag{3.5}$$

and assuming that

$$\operatorname{gap}(A + A^\pi, B + B^\pi) < \alpha^{-1/2}. \tag{3.6}$$

By [3, Theorem IV.2.20],

$$\operatorname{gap}\left((A+A^\pi)^{-1},(B+B^\pi)^{-1}\right)=\operatorname{gap}(A+A^\pi,B+B^\pi). \tag{3.7}$$

When we apply [3, Theorem IV.4.13], we obtain the inequality

$$\left\|(B+B^\pi)^{-1}-(A+A^\pi)^{-1}\right\|\le\frac{\alpha\operatorname{gap}(A+A^\pi,B+B^\pi)}{1-\alpha^{1/2}\operatorname{gap}(A+A^\pi,B+B^\pi)}. \tag{3.8}$$

We estimate $\operatorname{gap}(A+A^\pi,B+B^\pi)$ with the help of [3, Theorem IV.2.17]:

$$\begin{aligned}\operatorname{gap}(A+A^\pi,B+B^\pi)&=\operatorname{gap}(A+A^\pi,(B+B^\pi-A^\pi)+A^\pi)\\&\le 2\left(1+\|A^\pi\|^2\right)\operatorname{gap}(A,B+B^\pi-A^\pi).\end{aligned} \tag{3.9}$$

To estimate $\operatorname{gap}(A,B+B^\pi-A^\pi)$ we use Lemma 2.1(iv) and [3, Theorem IV.2.14]:

$$\begin{aligned}\operatorname{gap}(A,B+B^\pi-A^\pi)&\le 2\left(\operatorname{gap}(A,B)+\operatorname{gap}(B,B+B^\pi-A^\pi)\right)\\&\le 2\left(\operatorname{gap}(A,B)+\|B^\pi-A^\pi\|\right).\end{aligned} \tag{3.10}$$

Then

$$\operatorname{gap}(A+A^\pi,B+B^\pi)\le 4\left(1+\|A^\pi\|^2\right)\left(\operatorname{gap}(A,B)+\|B^\pi-A^\pi\|\right). \tag{3.11}$$

If, in addition, we assume that

$$\operatorname{gap}(A,B)+\|B^\pi-A^\pi\|<(4\alpha^{1/2}\beta)^{-1}, \tag{3.12}$$

then inequality (3.6) is satisfied, and

$$\left\|(B+B^\pi)^{-1}-(A+A^\pi)^{-1}\right\|\le\frac{4\alpha\beta\left(\operatorname{gap}(A,B)+\|B^\pi-A^\pi\|\right)}{1-4\alpha^{1/2}\beta\left(\operatorname{gap}(A,B)+\|B^\pi-A^\pi\|\right)} \tag{3.13}$$

since the function $\varphi(t)=\alpha t/(1-\alpha^{1/2}t)$ is increasing for $0\le t<\alpha^{-1/2}$.

The estimate for $\|B^{\mathrm{D}}-A^{\mathrm{D}}\|$ based on (3.4) is obtained when we observe that

$$\|I-B^\pi\|\le\|I-A^\pi\|+\|B^\pi-A^\pi\|. \tag{3.14}$$

We can then summarize our results in the following theorem.

Theorem 3.2. *Let $A,B\in\mathscr{C}(X)$ be Drazin invertible. If*

$$\operatorname{gap}(A,B)+\|B^\pi-A^\pi\|<(4\alpha^{1/2}\beta)^{-1}, \tag{3.15}$$

where α and β are defined by (3.5), then

$$\begin{aligned} \|B^{\mathrm{D}} - A^{\mathrm{D}}\| \leq{}& \frac{4\alpha\beta(\operatorname{gap}(A,B) + \|B^{\pi} - A^{\pi}\|)}{1 - 4\alpha^{1/2}\beta(\operatorname{gap}(A,B) + \|B^{\pi} - A^{\pi}\|)} \\ &\times (\|I - A^{\pi}\| + \|B^{\pi} - A^{\pi}\|) + \left\|(A + A^{\pi})^{-1}\right\|\|B^{\pi} - A^{\pi}\|. \end{aligned} \tag{3.16}$$

4. The continuity of the Drazin inverse

Theorem 3.2 will enable us to obtain the following result on the continuity of the Drazin inverse of closed linear operators.

Theorem 4.1. *Let $A, A_n \in \mathscr{C}(X)$ be Drazin invertible operators ($n = 1, 2, \ldots$) such that $A_n \xrightarrow{\mathscr{G}} A$. Then the following conditions are equivalent:*

(i) $\|A_n^{\mathrm{D}} - A^{\mathrm{D}}\| \to 0$;
(ii) $\sup_n \|A_n^{\mathrm{D}}\| < \infty$;
(iii) *there exists $r > 0$ such that $0 < |\lambda| < r$ implies that $\lambda \in \rho(A_n)$ for all n;*
(iv) $\|A_n A_n^{\mathrm{D}} - AA^{\mathrm{D}}\| \to 0$;
(v) $\|A_n^{\pi} - A^{\pi}\| \to 0$.

Proof. The implication (i)⇒(ii) is clear.

(ii)⇒(iii). Suppose that $\|A_n^{\mathrm{D}}\| \leq M$ for all n and some $M > 0$, and let $r = M^{-1}$. In accordance with Lemma 1.2 we write $A_n^{\mathrm{D}} = f_n(A_n)$, where $f_n(\lambda) = 0$ in a neighborhood Δ_n of 0 and $f(\lambda) = \lambda^{-1}$ in a neighborhood Ω_n of $\sigma(A_n) \setminus \{0\}$, where $\Delta_n \cap \Omega_n = \emptyset$. If $\lambda \in \sigma(A_n) \setminus \{0\}$, then

$$|\lambda^{-1}| = |f_n(\lambda)| \leq \|f_n(A_n)\| = \|A_n^{\mathrm{D}}\| \leq M, \tag{4.1}$$

which gives $|\lambda| \geq r$. This implies (iii).

(iii)⇒(v). Condition (iii) and the quasi-polarity of A imply that we can choose disjoint open sets Δ and Ω such that $0 \in \Delta$, Ω contains the nonzero spectrum of A and of all the A_n, and Ω^c is compact. Let f be defined on $\Delta \cup \Omega$ by setting $f(\lambda) = 1$ on Δ and $f(\lambda) = 0$ on Ω. According to Lemma 3.1, $\|f(A_n) - f(A)\| \to 0$, that is,

$$\|A_n^{\pi} - A^{\pi}\| \longrightarrow 0. \tag{4.2}$$

(iv) and (v) are equivalent since $T^{\pi} = I - TT^{\mathrm{D}}$ for any Drazin invertible operator $T \in \mathscr{C}(X)$.

Finally, the implication (v)⇒(i) follows from Theorem 3.2. □

5. An error bound for $\|B^{\mathrm{D}} - A^{\mathrm{D}}\|$ using an upper estimate for $\|B^\pi - A^\pi\|$ and $\|(A+A^\pi)^{-1}\|$

If Δ is an upper estimate for $\|B^\pi - A^\pi\|$ and if $\operatorname{gap}(A,B)+\Delta < (4\sqrt{\alpha}\beta)^{-1}$, where α, β are defined by (3.5), then according to Theorem 3.2,

$$\|B^{\mathrm{D}} - A^{\mathrm{D}}\| \le \frac{4\alpha\beta(\operatorname{gap}(A,B)+\Delta)}{1-4\sqrt{\alpha}\beta(\operatorname{gap}(A,B)+\Delta)}(\|I-A^\pi\|+\Delta)+\left\|(A+A^\pi)^{-1}\right\|\Delta. \tag{5.1}$$

In this section, we find a value of Δ satisfying (5.1) that can be calculated from the gaps between spaces associated with the operators A, B.

The spectral projection A^π of a Drazin invertible operator $A \in \mathscr{C}(X)$ is the bounded linear operator with $\mathscr{R}(A^\pi) = \mathscr{H}(A)$ and $\mathscr{N}(A^\pi) = \mathscr{K}(A)$, where the spaces $\mathscr{H}(A)$ and $\mathscr{K}(A)$ are defined by

$$\begin{aligned} \mathscr{H}(A) &= \Big\{x \in \mathscr{D}_\infty(A) : \limsup_{n\to\infty}\|A^n x\|^{1/n} = 0\Big\}, \\ \mathscr{K}(A) &= \Big\{x \in X : \exists x_n \in \mathscr{D}_n(A) \text{ such that} \\ &\qquad Ax_1 = x,\ Ax_{n+1} = x_n \text{ for } n = 1,2,\ldots \\ &\qquad\qquad \text{and } \limsup_{n\to\infty}\|x_n\|^{1/n} < \infty\Big\} \end{aligned} \tag{5.2}$$

(see Mbekhta [9]). It is known [6] that A is Drazin invertible if and only if X is the direct sum $X = \mathscr{H}(A) \oplus \mathscr{K}(A)$ with at least one of the component spaces closed. If 0 is a pole of A of order p, then

$$\mathscr{K}(A) = \mathscr{N}(A^p), \qquad \mathscr{H}(A) = \mathscr{R}(A^p). \tag{5.3}$$

Theorem 5.1. *Let $A, B \in \mathscr{C}(X)$ be Drazin invertible operators and let*

$$\Delta_H = \operatorname{gap}(\mathscr{H}(A),\mathscr{H}(B)), \qquad \Delta_K = \operatorname{gap}(\mathscr{K}(A),\mathscr{K}(B)). \tag{5.4}$$

If $\|I-A^\pi\|\Delta_H + \|A^\pi\|\Delta_K < 1$, then (5.1) holds with Δ defined by

$$\Delta = \frac{\|I-A^\pi\|\|A^\pi\|(\Delta_H+\Delta_K)}{1-\|I-A^\pi\|\Delta_H-\|A^\pi\|\Delta_K} \tag{5.5}$$

provided $\operatorname{gap}(A,B)+\Delta < (4\sqrt{\alpha}\beta)^{-1}$.

Proof. The proof follows from Theorem 3.2 and Proposition 2.2. □

Next we find an upper estimate for the quantity $\|(A+A^\pi)^{-1}\|$. We observe that from $(A+A^\pi)A^\pi = A^\pi(I+AA^\pi)$, it follows that

$$(A+A^\pi)^{-1}A^\pi = (I+AA^\pi)^{-1}A^\pi. \tag{5.6}$$

Since the operator AA^π is bounded and quasi-nilpotent, we have

$$\begin{aligned}(A+A^\pi)^{-1}A^\pi &= (I+AA^\pi)^{-1}A^\pi \\ &= \left(\sum_{i=0}^{\infty}(-1)^i(AA^\pi)^i\right)A^\pi \\ &= \left(I+\sum_{i=1}^{\infty}(-1)^iA^iA^\pi\right)A^\pi.\end{aligned} \tag{5.7}$$

Consequently,

$$\begin{aligned}\left\|(A+A^\pi)^{-1}\right\| &\le \left\|(A+A^\pi)^{-1}(I-A^\pi)\right\| + \left\|(A+A^\pi)^{-1}A^\pi\right\| \\ &\le \|A^{\mathrm{D}}\| + \|A^\pi\|\left(1+\left\|\sum_{i=1}^{\infty}(-1)^iA^iA^\pi\right\|\right),\end{aligned} \tag{5.8}$$

which gives

$$\left\|(A+A^\pi)^{-1}\right\| \le \|A^{\mathrm{D}}\| + \|A^\pi\|\left(1+\left\|\sum_{i=1}^{\infty}(-1)^iA^iA^\pi\right\|\right) =: \Theta. \tag{5.9}$$

Theorem 5.2. *Let $A,B \in \mathscr{C}(X)$ be Drazin invertible operators and let Δ be an upper estimate for $\|B^\pi - A^\pi\|$ such that* $\operatorname{gap}(A,B)+\Delta < (4\sqrt{\alpha}\beta)^{-1}$, *where α, β are defined by (3.5). Then*

$$\|B^{\mathrm{D}} - A^{\mathrm{D}}\| \le \frac{4\alpha\beta(\operatorname{gap}(A,B)+\Delta)}{1-4\sqrt{\alpha}\beta(\operatorname{gap}(A,B)+\Delta)}(\|I-A^\pi\|+\Delta)+\Theta\Delta, \tag{5.10}$$

where Θ is given by (5.9).

6. The case of bounded operators

We assume that the operators in question are in $\mathscr{B}(X)$. In this case we are able to use the operator norm rather than the operator gap to deduce the following explicit error estimate. Specializing the theorems of this section to matrices, we recover recent results of Koliha [5].

Theorem 6.1. *Let $A,B \in \mathscr{B}(X)$ be Drazin invertible operators. If*

$$(\|B-A\|+\|B^\pi - A^\pi\|)\left\|(A+A^\pi)^{-1}\right\| < 1, \tag{6.1}$$

then

$$\begin{aligned}\|B^{\mathrm{D}} - A^{\mathrm{D}}\| \le{}& \frac{\left\|(A+A^\pi)^{-1}\right\|^2(\|B-A\|+\|B^\pi-A^\pi\|)}{1-\left\|(A+A^\pi)^{-1}\right\|(\|B-A\|+\|B^\pi-A^\pi\|)} \\ &\times(\|I-A^\pi\|+\|B^\pi-A^\pi\|)+\left\|(A+A^\pi)^{-1}\right\|\|B^\pi-A^\pi\|.\end{aligned} \tag{6.2}$$

If $\|I - A^\pi\|\Delta_H + \|A^\pi\|\Delta_K < 1$ *and* $\Theta(\|B - A\| + \Delta) < 1$, *then*

$$\|B^{\mathrm{D}} - A^{\mathrm{D}}\| \leq \frac{\Theta^2(\|B - A\| + \Delta)}{1 - \Theta(\|B - A\| + \Delta)}(\|I - A^\pi\| + \Delta) + \Theta\Delta, \tag{6.3}$$

where Δ *and* Θ *are defined by (5.5) and (5.9).*

Proof. According to (3.4),

$$B^{\mathrm{D}} - A^{\mathrm{D}} = \left[(B + B^\pi)^{-1} - (A + A^\pi)^{-1}\right](I - B^\pi) + (A + A^\pi)^{-1}(A^\pi - B^\pi). \tag{6.4}$$

The standard error estimate for the perturbation of the ordinary inverse of operators $T, S \in \mathcal{B}(X)$ gives

$$\|S^{-1} - T^{-1}\| \leq \frac{\|T^{-1}\|^2 \|S - T\|}{1 - \|T^{-1}\|\|S - T\|} \tag{6.5}$$

under the assumption that $\|T^{-1}\|\|S - T\| < 1$. Setting $T = A + A^\pi$, $S = B + B^\pi$, we obtain (6.2). □

Finally, we apply the results obtained in this section to operators on Hilbert spaces. If $A \in \mathcal{B}(H)$, then

$$\|A^\pi\| = \|I - A^\pi\| \leq \kappa_{\mathrm{D}}(A), \tag{6.6}$$

where $\kappa_{\mathrm{D}}(A) = \|A\|\|A^{\mathrm{D}}\|$ is the Drazin condition number. The estimate (5.9) yields that

$$\left\|(A + A^\pi)^{-1}\right\| \leq \|A^{\mathrm{D}}\| + \kappa_{\mathrm{D}}(A)\left(1 + \left\|\sum_{i=1}^{\infty}(-1)^i A^i A^\pi\right\|\right) =: \Theta_1. \tag{6.7}$$

Suppose that $\kappa_{\mathrm{D}}(A)(\Delta_H + \Delta_K) < 1$. By (2.3),

$$\|B^\pi - A^\pi\| \leq \frac{\kappa_{\mathrm{D}}(A)^2(\Delta_H + \Delta_K)}{1 - \kappa_{\mathrm{D}}(A)(\Delta_H + \Delta_K)} =: \Delta_1. \tag{6.8}$$

Then (6.2) combined with Theorem 3.2 yields the following result.

Theorem 6.2. *Let* Θ_1 *and* Δ_1 *be defined by (6.7) and (6.8), and let* $A, B \in \mathcal{B}(H)$ *be operators such that* $\Theta_1(\|B - A\| + \Delta_1) < 1$ *and* $\kappa_{\mathrm{D}}(A)(\Delta_H + \Delta_K) < 1$. *Then*

$$\|B^{\mathrm{D}} - A^{\mathrm{D}}\| \leq \frac{\Theta_1^2(\|B - A\| + \Delta_1)}{1 - \Theta_1(\|B - A\| + \Delta_1)}(\kappa_{\mathrm{D}}(A) + \Delta_1) + \Theta_1\Delta_1. \tag{6.9}$$

7. Applications to operator semigroups

We give a representation of the Drazin inverse of the infinitesimal generator of a C_0-semigroup with a special asymptotic behavior.

Theorem 7.1. *Let $T(t)$ be a bounded C_0-semigroup with the infinitesimal generator A and let $P \in \mathscr{B}(X)$ be a nonzero idempotent operator such that*

(a) $T(t)P = PT(t)$ *for all* $t \geq 0$;
(b) $\mathscr{R}(P) \subset \mathscr{D}(A)$;
(c) $\|T(t)(I-P)\| \to 0$ *as* $t \to \infty$;
(d) $\sigma(AP) = \{0\}$.

Then A is Drazin invertible with $A^\pi = P$, there are positive constants M, μ such that

$$\|T(t)(I-P)\| \leq Me^{-\mu t} \quad \forall t \geq 0, \tag{7.1}$$

$$A^{\mathrm{D}} = -\int_0^\infty T(t)(I-P)\,dt. \tag{7.2}$$

Proof. We observe that $APx = PAx$ for all $x \in \mathscr{D}(A)$. Write

$$S(t) = T(t)\exp(-tP), \quad t \geq 0. \tag{7.3}$$

A direct verification shows that $S(t)$ is a C_0-semigroup whose generator C is calculated from

$$\frac{d^+}{dt}\bigg|_0 S(t)x = \frac{d^+}{dt}\bigg|_0 T(t)\exp(0P)x + T(0)\frac{d^+}{dt}\bigg|_0 \exp(-tP)x = Ax - Px \tag{7.4}$$

for all $x \in \mathscr{D}(A)$; hence, $C = A - P$. Observing that $\exp(-tP) = I - P + e^{-t}P$, we obtain that

$$\|S(t)\| \leq \|T(t)(I-P)\| + \|T(t)\|\|P\|e^{-t} \longrightarrow 0 \quad \text{as } t \longrightarrow \infty. \tag{7.5}$$

Applying results from [15, Chapter 3], we conclude that there exist positive constants K, μ such that

$$\|S(t)\| \leq Ke^{-\mu t}, \quad t \geq 0. \tag{7.6}$$

In addition, the spectrum of the generator C of $S(t)$ is contained in the open left-half plane, which means that C is invertible.

To prove that A is Drazin invertible with the spectral projection P, we need to show that 0 is an isolated singularity of the resolvent $R(\lambda, A)$ with P as the residue of the resolvent at 0. For each $x \in \mathscr{D}(A)$,

$$(\lambda I - A)x = (\lambda I - AP)Px + (\lambda I - C)(I-P)x. \tag{7.7}$$

There exists $\rho > 0$ such that $\lambda I - C$ is invertible if $|\lambda| < \rho$. Then, for each λ satisfying $0 < |\lambda| < \rho$,

$$R(\lambda, A) = (\lambda I - AP)^{-1}P + (\lambda I - C)^{-1}(I-P). \tag{7.8}$$

Integrating the resolvent along a sufficiently small circular loop ω centred at 0, we get

$$\begin{aligned} A^{\pi} &= \frac{1}{2\pi i}\int_{\omega} R(\lambda, A)\,d\lambda \\ &= \frac{1}{2\pi i}\int_{\omega}\sum_{n=0}^{\infty}\lambda^{-n-1}A^{n}P\,d\lambda + \frac{1}{2\pi i}\int_{\omega} R(\lambda, C)(I-P)\,d\lambda \\ &= \left(\frac{1}{2\pi i}\int_{\omega}\frac{d\lambda}{\lambda}\right)P = P, \end{aligned} \tag{7.9}$$

which completes the argument.

Since C is an invertible generator of a C_0-semigroup satisfying $\|S(t)\| \le Ke^{-\mu t}$, we have

$$\left(\int_0^{\infty} S(t)\,dt\right)Cx = C\left(\int_0^{\infty} S(t)\,dt\right)x = \int_0^{\infty}\frac{d}{dt}S(t)x\,dt = -x \tag{7.10}$$

for all $x \in \mathfrak{D}(A)$; hence

$$C^{-1} = -\int_0^{\infty} S(t)\,dt. \tag{7.11}$$

Inequality (7.1) holds with $M = K\|I-P\|$ if we use the equation $T(t)\,(I-P) = S(T)(I-P)$. Hence $T(t)(I-P)$ is integrable over $[0,\infty)$ with

$$\int_0^{\infty} T(t)(I-P)\,dt = \int_0^{\infty} S(t)(I-P)\,dt = -(A-P)^{-1}(I-P) = -A^{\mathrm{D}}. \tag{7.12}$$

This completes the proof. □

A special case of Theorem 7.1 occurs when $T(t)$ converges in the operator norm as $t \to \infty$; in this case P is the limit of $T(t)$.

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N. Castro González: Departmento de Matemática Aplicada, Facultad de Informática, Universidad Politécnica de Madrid, Boadilla del Monte, 28660 Madrid, Spain

E-mail address: nieves@fi.upm.es

J. J. Koliha: Department of Mathematics and Statistics, The University of Melbourne, Victoria 3010, Australia

E-mail address: j.koliha@ms.unimelb.edu.au

V. Rakočević: Department of Mathematics, Faculty of Philosophy, University of Niš, 1800 Niš, Yugoslavia-Serbia

E-mail address: vrakoc@bankerinter.net