NONOSCILLATORY HALF-LINEAR DIFFERENCE EQUATIONS AND RECESSIVE SOLUTIONS

MARIELLA CECCHI, ZUZANA DOŠLÁ, AND MAURO MARINI

Received 30 January 2004 and in revised form 26 May 2004

Recessive and dominant solutions for the nonoscillatory half-linear difference equation are investigated. By using a uniqueness result for the zero-convergent solutions satisfying a suitable final condition, we prove that recessive solutions are the "smallest solutions in a neighborhood of infinity," like in the linear case. Other asymptotic properties of recessive and dominant solutions are treated too.

1. Introduction

Consider the second-order half-linear difference equation

$$\Delta\big(a_n\Phi(\Delta x_n)\big) + b_n\Phi(x_{n+1}) = 0, \tag{1.1}$$

where Δ is the forward difference operator $\Delta x_n = x_{n+1} - x_n$, $\Phi(u) = |u|^{p-2}u$ with $p > 1$, and $a = \{a_n\}$, $b = \{b_n\}$ are positive real sequences for $n \ge 1$.

It is known that there is a surprising similarity between (1.1) and the linear difference equation

$$\Delta(a_n\Delta x_n) + b_n x_{n+1} = 0. \tag{1.2}$$

In particular, for (1.1), the Sturmian theory continues to hold (see, e.g., [15]), and also Kneser- or Hille-type oscillation and nonoscillation criteria can be formulated (see, e.g., [10]).

Another concept recently extended to the half-linear case is the concept of a recessive solution (see [11]). We recall (see, e.g., [1, 8, 14]) that in the linear case, if (1.2) is nonoscillatory, then there exists a nontrivial solution $u = \{u_n\}$, uniquely determined up to a constant factor, such that

$$\lim_n \frac{u_n}{x_n} = 0, \tag{1.3}$$

where $x = \{x_n\}$ denotes an arbitrary nontrivial solution of (1.2), linearly independent of u. Solution u is called a *recessive solution* and x a *dominant solution*. Both solutions play

Advances in Difference Equations 2005:2 (2005) 193–204
DOI: 10.1155/ADE.2005.193

an important role in different contexts (see, e.g., [1, 4, 6] and references therein). In the linear case (see, e.g., [1, Chapter 6.3], [3, Theorem 6.8], [8, 14]), recessive solutions u and dominant solutions x can be equivalently characterized by the properties

$$\sum^{\infty} \frac{1}{a_n u_n u_{n+1}} = \infty, \qquad \sum^{\infty} \frac{1}{a_n x_n x_{n+1}} < \infty, \tag{1.4}$$

$$\frac{\Delta u_n}{u_n} < \frac{\Delta x_n}{x_n} \quad \text{for large } n. \tag{1.5}$$

As mentioned above, the concept of a recessive solution has been extended in [11] to the nonoscillatory half-linear equation (1.1) by the following way. Consider the generalized Riccati equation

$$\Delta w_n - b_n + \left(1 - \frac{a_n}{\Phi(\Phi^*(a_n) + \Phi^*(w_n))}\right) w_n = 0, \tag{1.6}$$

where Φ^* denotes the inverse function of Φ. If (1.1) is nonoscillatory, in [11] it is proved that there exists a solution $w^\infty = \{w_n^\infty\}$ of (1.6), such that $a_n + w_n^\infty > 0$ for large n, with the property that for any other solution $w = \{w_n\}$ of (1.6), with $a_n + w_n > 0$ in some neighborhood of ∞,

$$w_n^\infty < w_n \quad \text{for large } n. \tag{1.7}$$

Such solution w^∞ is called *(eventually) a minimal solution* of (1.6) and the solution $u = \{u_n\}$ of (1.1), given by

$$\Delta u_n = \Phi^*\left(\frac{w_n^\infty}{a_n}\right) u_n, \tag{1.8}$$

is called a *recessive solution* of (1.1). Since (1.1) is nonoscillatory, for any solution $x = \{x_n\}$ of (1.1), the sequence $w^x = \{w_n^x\}$, where

$$w_n^x = \frac{a_n \Phi(\Delta x_n)}{\Phi(x_n)}, \tag{1.9}$$

is, for large n, a solution of the generalized Riccati equation (1.6) and so property (1.7) coincides with (1.5), stated in the linear case.

In [11], the open problems whether analogous results as the limit characterization (1.3) and the summation property (1.4) hold in the half-linear case have been also posed. In the case when b is eventually negative, a complete answer to both questions has been given by the authors in a recent paper [6].

Our aim here is to continue this study, by considering the case when b_n is positive and

$$\sum_{n=1}^{\infty} b_n \Phi\left(\sum_{j=n+1}^{\infty} \frac{1}{\Phi^*(a_j)}\right) < \infty. \tag{1.10}$$

We will give a positive answer to the question posed in [11] concerning the limit characterization of the recessive solution, by showing that properties (1.3) and (1.5) are equivalent also in the half-linear case. In addition, two summation criteria, which reduce to

(1.4) in the linear case, are proved. These results are useful also in the numerical computation of recessive solutions. Indeed, as pointed out in [4, Chapter 5], the recessive behavior can be easily destroyed by numerical errors.

A similar problem has been studied and completely solved in [7] for the half-linear differential equations

$$\left(a(t)\Phi(x')\right)' + b(t)\Phi(x) = 0, \tag{1.11}$$

where a, b are continuous positive functions for $t \geq 0$, without any additional condition. One of the tools used in [7] for proving limit and integral characterization of principal solutions is based on certain properties of a suitable quadratic functional studied in [9]. Since in the discrete case such properties are not known, a different approach is used here and the additional condition (1.10) is required.

A discussion concerning the role of (1.10) and open problems completes the paper.

2. Preliminaries

Throughout the paper, for brevity, by "solution of (1.1)" we mean a nontrivial solution of (1.1). A solution $x = \{x_n\}$ of (1.1) is said to be *nonoscillatory* if there exists $N_x \geq 1$ such that $x_n x_{n+1} > 0$ for $n \geq N_x$. Since, as claimed, the Sturm-type separation theorem holds for (1.1), a solution of (1.1) is nonoscillatory if and only if every solution of (1.1) is nonoscillatory. Hence, (1.1) is called *nonoscillatory* if its solutions are nonoscillatory.

The half-linear equation is characterized by the *homogeneity property*, which means that if $x = \{x_n\}$ is a solution of (1.1), then also λx is a solution for any constant λ. This property will be used in our later consideration.

Let $x = \{x_n\}$ be a solution of (1.1) and denote its quasi-difference with $x^{[1]} = \{x_n^{[1]}\}$, $x_n^{[1]} = a_n \Phi(\Delta x_n)$. Observe that from (1.10), it follows that

$$\sum_{n=1}^{\infty} \frac{1}{\Phi^*(a_n)} < \infty. \tag{2.1}$$

Under assumption (1.10), equation (1.1) is nonoscillatory, as the following result shows.

Lemma 2.1. *If condition (1.10) is satisfied, then (1.1) is nonoscillatory. More precisely, if (1.10) holds, then (1.1) has a (nonoscillatory) solution $u = \{u_n\}$ satisfying*

$$\lim_n u_n = 0, \qquad \lim_n u_n^{[1]} = c_u, \quad c_u \in \mathbb{R} \setminus \{0\}. \tag{2.2}$$

Lemma 2.1 can be obtained from existing results. For instance it follows, with minor changes, from [13, 16], in which the same conclusion has been proved for systems, or equations, with delay. In particular in [16, Theorem 4.2], some additional assumptions on superlinearity are required. For the sake of completeness, a sketch of the proof is provided here.

Proof (sketch). Choose n_0 large so that

$$\sum_{k=n_0}^{\infty} b_k \Phi\left(\sum_{j=k+1}^{\infty} \frac{1}{\Phi^*(a_j)}\right) < \frac{1}{2}. \tag{2.3}$$

Consider the Banach space $B_{n_0}^{\infty}$ of all converging sequences defined for every integer $n \geq n_0$, endowed with the topology of the supremum norm, and consider the set $\Omega \subset B_{n_0}^{\infty}$ given by

$$\Omega = \left\{ u = \{u_n\} \in B_{n_0}^{\infty} : \frac{1}{2}\sum_{j=n}^{\infty} \frac{1}{\Phi^*(a_j)} \leq u_n \leq \sum_{j=n}^{\infty} \frac{1}{\Phi^*(a_j)},\ n \geq n_0 \right\}. \tag{2.4}$$

Consider the operator $\mathcal{T} : \Omega \to B_{n_0}^{\infty}$ defined by $\mathcal{T}(u) = y = \{y_n\}$, where

$$y_n = \sum_{k=n}^{\infty} \frac{1}{\Phi^*(a_k)} \Phi^*\left(-1 + \sum_{i=k}^{\infty} b_i \Phi(u_{i+1})\right). \tag{2.5}$$

It is easy to show that $\mathcal{T}(\Omega) \subset \Omega$. Using the discrete version of a well-known compactness result by Avramescu (see, e.g., [2, Remark 3.3.1]), one can check that $\mathcal{T}(\Omega)$ is relatively compact in $B_{n_0}^{\infty}$. Because $\mathcal{T}$ is also continuous in Ω, by the Schauder fixed-point theorem, there exists a fixed point $u = \{u_n\}$ of the operator $\mathcal{T}$ in Ω. Finally we have $\lim_n u_n = 0$ and $\lim_n u_n^{[1]} \neq 0$, that is, the assertion. □

The next lemma states the possible types of all nonoscillatory solutions of (1.1).

LEMMA 2.2. *Assume (1.10) and let $x = \{x_n\}$ be a solution of (1.1). Then*
- (i) *x and its quasi-difference $x^{[1]}$ are eventually strongly monotone;*
- (ii) *x is bounded;*
- (iii) *if $\lim_n x_n = 0$, then $\lim_n x_n^{[1]} = \mu_x$, where $-\infty \leq \mu_x < 0$ or $0 < \mu_x \leq \infty$ according to whether $x_n > 0$ or $x_n < 0$ for large n, respectively.*

Proof. Without loss of generality, assume that $x_n > 0$ for $n \geq n_0 \geq 1$.

Claim (i). From (1.1), the quasi-difference $x^{[1]}$ is eventually decreasing and so $\{\Delta x_n\}$ has eventually a fixed sign (different from zero), that is, x is eventually strongly monotone. Since for large n we have $\Delta x_n^{[1]} < 0$, $x^{[1]}$ is strongly monotone too.

Claim (ii). Since $x^{[1]}$ is eventually decreasing, we have for $n \geq n_0$,

$$\Delta x_n \leq \Phi^*(x_{n_0}^{[1]}) \frac{1}{\Phi^*(a_n)}; \tag{2.6}$$

by summation from n_0 to n, we obtain

$$x_{n+1} \leq x_{n_0} + \Phi^*(x_{n_0}^{[1]}) \sum_{k=n_0}^{n} \frac{1}{\Phi^*(a_k)}. \tag{2.7}$$

If x is unbounded, in view of (2.1), inequality (2.7) gives a contradiction as $n \to \infty$.

Claim (iii). Since x is eventually strongly monotone, positive and $\lim_n x_n = 0$, x is eventually decreasing and so $\Delta x_n < 0$ for large n. If $\lim_n x_n^{[1]} = 0$, then, by summation of (1.1) from n to ∞, we obtain $x_n^{[1]} > 0$ for large n, which is a contradiction. □

We close this section with the following version of the discrete Gronwall inequality.

LEMMA 2.3. *Let z, w be two nonnegative sequences for $n \geq N \geq 1$ such that $\sum_{j=N}^{\infty} w_j z_{j+1} < \infty$ and $\sum_{j=N}^{\infty} w_j < \infty$. If for $n \geq N$,*

$$z_n \leq \sum_{j=n}^{\infty} w_j z_{j+1}, \tag{2.8}$$

then $z_n = 0$ for every $n \geq N$.

Proof. Define the sequence $v = \{v_n\}$ as follows:

$$v_n = \sum_{j=n}^{\infty} w_j z_{j+1}. \tag{2.9}$$

In view of (2.8), we have $z_n \leq v_n$ for $n \geq N$. Then $\Delta v_n = -w_n z_{n+1} \geq -w_n v_{n+1}$ or

$$\Delta v_n + w_n v_{n+1} \geq 0. \tag{2.10}$$

Since $w_n \geq 0$ and $\sum_{n=N}^{\infty} w_n < \infty$, we have $0 < \prod_{j=N}^{\infty}(1 + w_j)^{-1} < \infty$. Putting

$$h_n = \prod_{j=n}^{\infty} \frac{1}{1 + w_j}, \tag{2.11}$$

we have $h_n > 0$ and $\Delta h_n = h_n w_n$. Multiplying (2.10) by h_n, we obtain $(n \geq N)$

$$h_n \Delta v_n + h_n w_n v_{n+1} = h_n \Delta v_n + \Delta h_n v_{n+1} = \Delta(h_n v_n) \geq 0. \tag{2.12}$$

Since $\lim_n v_n = 0$ and $\{h_n\}$ is bounded, from (2.12) we have $h_n v_n \leq 0$ and so $v_n = 0$. □

3. Recessive and dominant solutions

As already claimed, in [11] the notion of a recessive solution has been extended by using the Riccati equation approach, and for (1.1) reads as follows.

Definition 3.1. A solution $u = \{u_n\}$ of (1.1) is said to be a *recessive solution* of (1.1) if for every nontrivial solution $x = \{x_n\}$ of (1.1) such that $x \neq \lambda u$, $\lambda \in \mathbb{R} \setminus \{0\}$,

$$\frac{\Delta u_n}{u_n} < \frac{\Delta x_n}{x_n} \quad \text{for large } n. \tag{3.1}$$

The following theorem holds.

Theorem 3.2 (see [11]). *If (1.1) is nonoscillatory, recessive solutions of (1.1) exist and they are determined up to a constant factor.*

Analogously to the linear case, every solution of (1.1), which is not a recessive solution, is called a *dominant solution.*

The following result characterizes the recessive solution of (1.1).

Proposition 3.3. *Assume (1.10). If $u = \{u_n\}$ is a recessive solution of (1.1), then (2.2) holds and $u_n \Delta u_n < 0$ for large n.*

Proof. Without loss of generality, assume u eventually positive. If condition (2.2) does not hold, from Lemma 2.2 we obtain

$$\lim_n u_n = \ell_u > 0 \quad \text{or} \quad \lim_n u_n^{[1]} = -\infty. \tag{3.2}$$

In view of Lemma 2.1, there exists a solution $z = \{z_n\}$ of (1.1) satisfying (2.2). Then $z \neq \lambda u$ for every $\lambda \in \mathbb{R} \setminus \{0\}$ and so from (3.1),

$$\frac{\Delta u_n}{u_n} < \frac{\Delta z_n}{z_n} \quad \text{for large } n. \tag{3.3}$$

Without loss of generality, assume z eventually positive. Then (3.3) implies that $\Delta(u_n/z_n) < 0$ and so $\lim_n (u_n/z_n) = c$, $0 \leq c < \infty$, which gives a contradiction with (3.2). The second statement follows from Lemma 2.2(i). $\square$

The following uniqueness result will play an important role in our later consideration.

Theorem 3.4. *Assume (1.10). For any fixed $c \in \mathbb{R} \setminus \{0\}$, there exists a unique solution $u = \{u_n\}$ of (1.1) such that*

$$\lim_n u_n = 0, \qquad \lim_n u_n^{[1]} = c. \tag{3.4}$$

Proof. The existence follows from Lemma 2.1 and the homogeneity property. It remains to prove the uniqueness. The argument is suggested by [12, Theorem 4.3]. Without loss of generality, let $x = \{x_n\}$, $z = \{z_n\}$ be two eventually positive solutions of (1.1) satisfying $x_n > 0$, $z_n > 0$ for $N \geq 1$ and

$$\lim_n x_n = \lim_n z_n = 0, \qquad \lim_n x_n^{[1]} = \lim_n z_n^{[1]} = c < 0. \tag{3.5}$$

Since sequences $x^{[1]}$ and $z^{[1]}$ are eventually decreasing, we can assume also that for $n \geq N$,

$$0 < -\frac{c}{2} < -x_n^{[1]} < -c, \qquad 0 < -\frac{c}{2} < -z_n^{[1]} < -c. \tag{3.6}$$

For brevity, denote

$$A_n = \sum_{k=n}^{\infty} \frac{1}{\Phi^*(a_k)}. \tag{3.7}$$

Summing the equalities $x_n^{[1]} = a_n\Phi(\Delta x_n)$, $z_n^{[1]} = a_n\Phi(\Delta z_n)$, we obtain

$$x_n = \sum_{k=n}^{\infty} \frac{1}{\Phi^*(a_k)}\Phi^*\left(-x_k^{[1]}\right), \qquad z_n = \sum_{k=n}^{\infty} \frac{1}{\Phi^*(a_k)}\Phi^*\left(-z_k^{[1]}\right), \tag{3.8}$$

or, in view of (3.6),

$$-\Phi^*\left(\frac{c}{2}\right)A_n < x_n < -\Phi^*(c)A_n, \qquad -\Phi^*\left(\frac{c}{2}\right)A_n < z_n < -\Phi^*(c)A_n. \tag{3.9}$$

Recalling that $\Phi(r) = r^{p-1}$ for $r > 0$, by the mean-value theorem we obtain

$$|\Phi(x_n) - \Phi(z_n)| \le (p-1)(w_n)^{p-2}|x_n - z_n|, \tag{3.10}$$

where $w_n = \max\{x_n, z_n\}$ or $w_n = \min\{x_n, z_n\}$ or $w_n = 1$ according to $p > 2$, $1 < p < 2$, or $p = 2$, respectively. Then, in view of (3.9), for any $p > 1$, there exists a positive constant M such that

$$(p-1)(w_n)^{p-2} \le M(A_n)^{p-2}. \tag{3.11}$$

Taking into account (3.8), we have

$$\begin{aligned} |\Phi(x_n) - \Phi(z_n)| &\le M(A_n)^{p-2}|x_n - z_n| \\ &\le M(A_n)^{p-2}\sum_{k=n}^{\infty}\frac{1}{\Phi^*(a_k)}\left|\Phi^*\left(-x_k^{[1]}\right) - \Phi^*\left(-z_k^{[1]}\right)\right|. \end{aligned} \tag{3.12}$$

Similarly, again by applying the mean-value theorem and taking into account that $\lim_n \Phi^*(x_n^{[1]}) = \lim_n \Phi^*(z_n^{[1]}) = \Phi^*(c) < 0$, there exists a positive constant H such that

$$\left|\Phi^*\left(x_n^{[1]}\right) - \Phi^*\left(z_n^{[1]}\right)\right| \le H\left|x_n^{[1]} - z_n^{[1]}\right|. \tag{3.13}$$

Summing (1.1) from n to ∞, $n \ge N$, we obtain

$$x_n^{[1]} = c + \sum_{k=n}^{\infty} b_k\Phi(x_{k+1}), \qquad z_n^{[1]} = c + \sum_{k=n}^{\infty} b_k\Phi(z_{k+1}). \tag{3.14}$$

Thus from (3.12) and (3.13), we have

$$\begin{aligned} \left|\Phi^*\left(x_n^{[1]}\right) - \Phi^*\left(z_n^{[1]}\right)\right| &\le H\sum_{k=n}^{\infty} b_k|\Phi(x_{k+1}) - \Phi(z_{k+1})| \\ &\le HM\sum_{k=n}^{\infty} b_k(A_{k+1})^{p-2}\sum_{j=k+1}^{\infty}\frac{1}{\Phi^*(a_j)}\left|\Phi^*\left(z_j^{[1]}\right) - \Phi^*\left(x_j^{[1]}\right)\right|. \end{aligned} \tag{3.15}$$

Putting

$$u_n = \sup\left\{\left|\Phi^*\left(x_k^{[1]}\right) - \Phi^*\left(z_k^{[1]}\right)\right| : k \ge n\right\}, \tag{3.16}$$

we obtain

$$\begin{aligned} u_n &\leq HM \sum_{k=n}^{\infty} b_k (A_{k+1})^{p-2} \sum_{j=k+1}^{\infty} \frac{1}{\Phi^*(a_j)} u_j \\ &\leq HM \sum_{k=n}^{\infty} b_k u_{k+1} (A_{k+1})^{p-2} \sum_{j=k+1}^{\infty} \frac{1}{\Phi^*(a_j)} \\ &= HM \sum_{k=n}^{\infty} b_k \Phi(A_{k+1}) u_{k+1}. \end{aligned} \tag{3.17}$$

Taking into account (1.10), we can apply Lemma 2.3 and we obtain $u_n \equiv 0$ for $n \geq N$. This implies that $x_n^{[1]} = z_n^{[1]}$ for every $n \geq N$ and the assertion easily follows. □

In view of the homogeneity property, from Theorem 3.4 we obtain the following result.

Corollary 3.5. *Assume (1.10). If $u = \{u_n\}$ and $w = \{w_n\}$ are two solutions of (1.1) such that*

$$\lim_n u_n = \lim_n w_n = 0, \qquad \lim_n u_n^{[1]} = c_u, \qquad \lim_n w_n^{[1]} = d_w, \tag{3.18}$$

where $c_u, d_w \in \mathbb{R} \setminus \{0\}$, then there exists $\lambda \in \mathbb{R} \setminus \{0\}$ such that $u = \lambda w$.

Proof. Let $z = \{z_n\}$ be the solution of (1.1) given by $z_n = (c_u/d_w) w_n$. Then $\lim_n z_n = 0$, $\lim_n z_n^{[1]} = c_u$, and, in view of Theorem 3.4, we have $z = u$. □

Proposition 3.3 and Corollary 3.5 yield the following characterization of the recessive solution.

Corollary 3.6. *Assume (1.10). Any solution $u = \{u_n\}$ of (1.1) is a recessive solution if and only if (2.2) holds.*

Proof. If u is a recessive solution, then Proposition 3.3 gives the assertion. Now assume that u satisfies (2.2). In view of Theorem 3.2, there exists a recessive solution of (1.1), say $w = \{w_n\}$. From Proposition 3.3, we have $\lim_n w_n = 0$, $\lim_n w_n^{[1]} = c_w$, $c_w \in \mathbb{R} \setminus \{0\}$. Then, in view of Corollary 3.5, there exists $\mu \in \mathbb{R} \setminus \{0\}$ such that $u = \mu w$, so u is a recessive solution. □

Remark 3.7. Corollary 3.6 gives also an asymptotic estimate for the recessive solution. Indeed from (2.2), we have for the recessive solution u of (1.1)

$$\lim_n \frac{u_n}{A_n} = \Phi^*(c_u), \quad c_u \in \mathbb{R} \setminus \{0\}, \tag{3.19}$$

where A_n is defined in (3.7).

4. Applications

Using Proposition 3.3 and Corollaries 3.5 and 3.6, it is easy to show that the most characteristic property of the recessive solution to be the "*smallest solution in a neighborhood* *of infinity*," stated in the linear case, continues to hold also for (1.1). Indeed the following result, which gives a positive answer to the claimed open problem posed in [11], holds.

Theorem 4.1. *Assume (1.10) and let $u = \{u_n\}$ be a solution of (1.1). Then u is a recessive solution if and only if*

$$\lim_n \frac{u_n}{x_n} = 0 \tag{4.1}$$

for every solution $x = \{x_n\}$ of (1.1) such that $x \neq \lambda u$, $\lambda \in \mathbb{R} \setminus \{0\}$.

Proof. If u is a recessive solution of (1.1), from Proposition 3.3 we have $\lim_n u_n = 0$, $\lim_n u_n^{[1]} = c_u$, $c_u \in \mathbb{R} \setminus \{0\}$. Let $x = \{x_n\}$ be another solution of (1.1) such that $x \neq \lambda u$, $\lambda \in \mathbb{R} \setminus \{0\}$. Since the recessive solution is unique up to a constant factor, x is not the recessive solution. By Corollary 3.6 and Lemma 2.2, we have $\lim_n x_n = c_x$, $0 < |c_x| < \infty$, or $\lim_n x_n = 0$, $\lim_n |x_n^{[1]}| = \infty$ and so (4.1) holds.

Conversely assume (4.1) for every solution x of (1.1) such that $x \neq \lambda u$, $\lambda \in \mathbb{R} \setminus \{0\}$. By contradiction, suppose that u is not a recessive solution and let $z = \{z_n\}$ be a recessive solution of (1.1). Then $z \neq \lambda u$ for $\lambda \in \mathbb{R} \setminus \{0\}$ and so

$$\lim_n \frac{u_n}{z_n} = 0. \tag{4.2}$$

Since u is not a recessive solution, again from Lemma 2.2 and Corollary 3.6, we obtain $\lim_n u_n = c_u$, $(0 < |c_u| < \infty)$ or $\lim_n u_n = 0$, $\lim_n |u_n^{[1]}| = \infty$, which gives a contradiction with (4.2). $\square$

Recessive solutions satisfy the following summation properties.

Theorem 4.2. *Assume (1.10). If $u = \{u_n\}$ is a recessive solution of (1.1), then there exists $N \geq 1$ such that*

$$\sum_{n=N}^{\infty} \frac{1}{\Phi^*(a_n) u_n u_{n+1}} = \infty, \tag{4.3}$$

$$\sum_{n=N}^{\infty} \frac{\Delta u_n}{u_n^{[1]} u_n u_{n+1}} = \infty. \tag{4.4}$$

Proof. By Lemma 2.2, let u_n be eventually positive. From Proposition 3.3, we have $\lim_n u_n^{[1]} = c$, $c < 0$, and so, by the discrete L'Hopital rule (see [1, Theorem 1.8.7]), $\lim_n u_n / A_n = -c$, where A_n is defined by (3.7). Then there exists $N \geq 1$ such that $u_n < -2cA_n$ for $n \geq N$, which implies that $(N < m)$

$$\begin{aligned}\sum_{n=N}^{m} \frac{1}{\Phi^*(a_n) u_n u_{n+1}} &> \frac{1}{4c^2} \sum_{n=N}^{m} \frac{1}{\Phi^*(a_n) A_n A_{n+1}} = \frac{1}{4c^2} \sum_{n=N}^{m} \frac{-\Delta A_n}{A_n A_{n+1}} \\ &= \frac{1}{4c^2} \sum_{n=N}^{m} \Delta\left(\frac{1}{A_n}\right) = \frac{1}{4c^2}\left[\frac{1}{A_m} - \frac{1}{A_N}\right]\end{aligned} \tag{4.5}$$

and, as $m \to \infty$, we obtain (4.3).

We show that also (4.4) holds. In view of Proposition 3.3, without loss of generality, we can assume that $u_n > 0$, $u_n^{[1]} < 0$ for $n \geq N$. Since $\lim_n u_n^{[1]} = c$, $c < 0$, the series

$$\sum_{n=N}^{\infty} \frac{\Delta u_n}{u_n^{[1]} u_n u_{n+1}}, \qquad \sum_{n=N}^{\infty} \frac{-\Delta u_n}{u_n u_{n+1}} \tag{4.6}$$

have the same character. Because

$$\sum_{n=N}^{\infty} \frac{-\Delta u_n}{u_n u_{n+1}} = \sum_{n=N}^{\infty} \Delta\left(\frac{1}{u_n}\right), \tag{4.7}$$

and $\lim_n u_n = 0$, the assertion follows. □

Clearly, in the linear case, conditions (4.3) and (4.4) reduce to (1.4). When both series $\sum_{j=1}^{\infty}[\Phi^*(a_j)]^{-1}$, $\sum_{j=1}^{\infty} b_j$ converge, then the following stronger result holds.

Theorem 4.3. *Assume (2.1) and $\sum_{j=1}^{\infty} b_j < \infty$. Any solution $u = \{u_n\}$ of (1.1) is a recessive solution if and only if (4.3) holds or, equivalently, a solution $x = \{x_n\}$ of (1.1) is a dominant solution if and only if there exists $N \geq 1$ such that*

$$\sum_{n=N}^{\infty} \frac{1}{\Phi^*(a_n) x_n x_{n+1}} < \infty. \tag{4.8}$$

Proof. By Theorem 4.2, it is sufficient to prove that if (4.3) holds, then u is a recessive solution. Since, in view of Lemma 2.2(ii), every solution x of (1.1) is bounded, by summation of (1.1) from n to ∞ we obtain the boundedness of $x^{[1]}$. Hence from Corollary 3.5, we have $\lim_n x_n = c_x$, $0 < |c_x| < \infty$ and the assertion follows. □

The following example illustrates our results. It also shows that property (4.4) does not mean that u is necessarily a recessive solution.

Example 4.4. Consider the half-linear difference equation

$$\Delta(a_n \Phi(\Delta x_n)) + b_n \Phi(x_{n+1}) = 0, \tag{4.9}$$

where $\Phi(u) = u^2 \operatorname{sgn} u$ and

$$a_n = n(n+1)(n+2)^2, \qquad b_n = \frac{8(n+1)(n+2)}{n\left[(n+1)(n+2) - 1\right]^2}. \tag{4.10}$$

We have

$$\frac{1}{\Phi^*(a_n)} = \frac{1}{\Phi^*(n(n+1)(n+2)^2)} < \frac{1}{n(n+1)}, \tag{4.11}$$

so $\sum_{n=1}^{\infty} 1/\Phi^*(a_n) < \infty$, $\sum_{n=1}^{\infty} b_n < \infty$, and condition (1.10) is satisfied. It is easy to verify that the sequence $x = \{x_n\}$,

$$x_n = 1 - \frac{1}{n(n+1)}, \tag{4.12}$$

is a solution of (4.9). By Corollary 3.6 or Theorem 4.3, x is a dominant solution. However, x satisfies condition (4.4) because the series

$$\sum_{n=1}^{\infty} \frac{\Delta x_n}{x_n^{[1]} x_n x_{n+1}}, \qquad \sum_{n=1}^{\infty} \frac{1}{n+2} \tag{4.13}$$

have the same character. Moreover, since the limit

$$\lim_n nA_n = \lim_n n \sum_{k=n}^{\infty} \sqrt{\frac{1}{a_k}} \tag{4.14}$$

is finite and different from zero, in view of Remark 3.7, also the limit $\lim_n nu_n$ is finite and different from zero for any recessive solution u of (4.9).

5. Concluding remarks

Theorems 4.2 and 4.3, and Example 4.4 illustrate some difficulties concerning the characterization of the recessive solution via summation criteria. For instance, when (1.10) holds and $\sum_{n=1}^{\infty} b_n = \infty$, does property (4.3), or (4.4), imply that $u = \{u_n\}$ is a recessive solution?

When (1.1) is nonoscillatory and (1.10) is not satisfied, that is,

$$\sum_{n=1}^{\infty} b_n \Phi\left(\sum_{j=n+1}^{\infty} \frac{1}{\Phi^*(a_j)} \right) = \infty, \tag{5.1}$$

the asymptotic characterization of the recessive solution is different. In fact, in such a case, equation (1.1) does not have solutions u satisfying (2.2), as it can be proved using a similar argument, with minor change, like in [13, Theorems 1 and 9]. Moreover if (1.1) is nonoscillatory, (2.1) and (5.1) hold, then it may happen that every solution x of (1.1) satisfies

$$\lim_n x_n = 0, \qquad \lim_n |x_n^{[1]}| = \infty, \tag{5.2}$$

as follows from [13, Theorems 9 and 10] or [16, Theorems 3.4 and 3.5]. Hence it seems to be difficult to prove the limit characterization and the summation properties of recessive solutions using only the knowledge of the asymptotic behavior of solutions and their quasi-differences. This problem, when (1.10) fails, jointly with a discussion about related summation criteria, is considered in the forthcoming paper [5].

Acknowledgment

The work of the second author was supported by the Czech Grant Agency, Grant 201/04/0580.

References

[1] R. P. Agarwal, *Difference Equations and Inequalities*, 2nd ed., Monographs and Textbooks in Pure and Applied Mathematics, vol. 228, Marcel Dekker, New York, 2000.
[2] R. P. Agarwal and D. O'Regan, *Infinite Interval Problems for Differential, Difference and Integral Equations*, Kluwer Academic Publishers, Dordrecht, 2001.
[3] C. D. Ahlbrandt, *Dominant and recessive solutions of symmetric three term recurrences*, J. Differential Equations **107** (1994), no. 2, 238–258.
[4] C. D. Ahlbrandt and A. C. Peterson, *Discrete Hamiltonian Systems. Difference Equations, Continued Fractions, and Riccati Equations*, Kluwer Texts in the Mathematical Sciences, vol. 16, Kluwer Academic Publishers Group, Dordrecht, 1996.
[5] M. Cecchi, Z. Došlá, O. Došlý, and M. Marini, *Summation characterization of the recessive solution for half-linear equations*, in preparation.
[6] M. Cecchi, Z. Došlá, and M. Marini, *On recessive and dominant solutions for half-linear difference equations*, J. Differ. Equations Appl. **10** (2004), no. 9, 797–808.
[7] ———, *Half-linear equations and characteristic properties of the principal solution*, J. Differential Equations **208** (2005), no. 2, 494–507.
[8] S. S. Cheng, H. J. Li, and W. T. Patula, *Bounded and zero convergent solutions of second-order difference equations*, J. Math. Anal. Appl. **141** (1989), no. 2, 463–483.
[9] O. Došlý and Á. Elbert, *Integral characterization of the principal solution of half-linear second order differential equations*, Studia Sci. Math. Hungar. **36** (2000), no. 3-4, 455–469.
[10] O. Došlý and P. Řehák, *Nonoscillatory criteria for half-linear second order difference equations*, Comput. Math. Appl. **42** (2001), 453–464.
[11] ———, *Recessive solution of half-linear second order difference equations*, J. Differ. Equations Appl. **9** (2003), no. 1, 49–61.
[12] H. Hoshino, R. Imabayashi, T. Kusano, and T. Tanigawa, *On second-order half-linear oscillations*, Adv. Math. Sci. Appl. **8** (1998), no. 1, 199–216.
[13] W.-T. Li, *Classification schemes for nonoscillatory solutions of two-dimensional nonlinear difference systems*, Comput. Math. Appl. **42** (2001), no. 3–5, 341–355.
[14] W. T. Patula, *Growth and oscillation properties of second order linear difference equations*, SIAM J. Math. Anal. **10** (1979), no. 1, 55–61.
[15] P. Řehák, *Oscillatory properties of second order half-linear difference equations*, Czechoslovak Math. J. **51(126)** (2001), no. 2, 303–321.
[16] G. Zhang, S. S. Cheng, and Y. Gao, *Classification schemes for positive solutions of a second-order nonlinear difference equation*, J. Comput. Appl. Math. **101** (1999), no. 1-2, 39–51.

Mariella Cecchi: Department of Electronic and Telecommunications, University of Florence, Via S. Marta 3, 50139 Florence, Italy
E-mail address: cecchi@det.unifi.it

Zuzana Došlá: Department of Mathematics, Faculty of Science, Masaryk University in Brno, Janáčkovo nám. 2a, 66295 Brno, Czech Republic
E-mail address: dosla@math.muni.cz

Mauro Marini: Department of Electronic and Telecommunications, University of Florence, Via S. Marta 3, 50139 Florence, Italy
E-mail address: marini@ing.unifi.it