HOW THE CONSTANTS IN HILLE-NEHARI THEOREMS DEPEND ON TIME SCALES

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We present criteria of Hille-Nehari-type for the linear dynamic equation $(r(t)y^{\Delta})^{\Delta} + p(t)y^{\sigma} = 0$, that is, the criteria in terms of the limit behavior of $(\int_a^t 1/r(s)\Delta s)\int_t^{\infty} p(s)\Delta s$ as $t \to \infty$. As a particular important case, we get that there is a (sharp) critical constant in those criteria which belongs to the interval $[0,1/4]$, and its value depends on the graininess μ and the coefficient r. Also we offer some applications, for example, criteria for strong (non-) oscillation and Kneser-type criteria, comparison with existing results (our theorems turn out to be new even in the discrete case as well as in many other situations), and comments with examples.

1. Introduction

Consider the linear dynamic equation

$$(r(t)y^{\Delta})^{\Delta} + p(t)y^{\sigma} = 0, \tag{1.1}$$

where $r(t) > 0$ and $p(t)$ are rd-continuous functions defined on a time-scale interval $[a,\infty]$, $a \in \mathbb{T}$, and a time scale $\mathbb{T}$ is assumed to be unbounded from above. As a special case of (1.1), when $\mathbb{T} = \mathbb{R}$, we get the well-studied Sturm-Liouville differential equation

$$(r(t)y')' + p(t)y = 0, \tag{1.2}$$

with continuous coefficients $r(t) > 0$ and $p(t)$. There is very extensive literature concerning qualitative theory of (1.2), where large and important part is comprised by oscillation theory originated in [25] by Sturm in 1836. See, for example, Hartman [11], Reid [24], and Swanson [26] for some survey works. Many effective conditions that guarantee oscillation or nonoscillation of (1.2) have been established. The following Hille-Nehari criteria, see, for example, Nehari [18], Swanson [26], Willett [27], belong to the

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most famous ones: if $\liminf_{t\to\infty}(\int_a^t 1/r(s)ds)\int_t^\infty p(s)\,ds > 1/4$, then (1.2) is oscillatory; if $\limsup_{t\to\infty}(\int_a^t 1/r(s)ds)\int_t^\infty p(s)\;ds < 1/4$, then (1.2) is nonoscillatory. In these criteria we assume that $\int_a^\infty 1/r(s)ds = \infty$ and $\int_t^\infty p(s)ds \geq 0\ (\not\equiv 0)$ for large t, in particular, $\int_a^\infty p(s)ds$ converges. Various techniques have been used to prove Hille-Nehari theorems with sundry additional conditions, like those related to the sign of $p(t)$. The study of a discrete counterpart to (1.2), namely, the difference equation $\Delta(r(t)\Delta y(t)) + p(t)y(t+1) = 0$, which is nothing but (1.1) with $\mathbb{T} = \mathbb{Z}$, has also a long history. The discrete Hille-Nehari criteria, however with $r(t) \equiv 1$ or with some additional assumptions on $r(t)$, may be found, for example, in [7, 8, 14, 16, 17, 19]. Very early after the concept of time scales was introduced, equations of type (1.1) have started to be studied, see Erbe and Hilger [9]. Among others, some effort has been devoted to extensions of Hille-Nehari criteria and other related topics to time scales, like Kneser's criteria and oscillatory properties of Euler's equation, see Bohner and Saker [4], Bohner and Ünal [5], Erbe et al. [10], Hilscher [13], and Řehák [22, 23]. The results in quoted papers which are related to our subject are interesting and valuable (the claims come as consequences of various techniques and they may serve as a good inspiration) but the problem is that they contain restrictions that disable examination of many remaining important cases. Those additive conditions mainly concern two following facts: constants on the right-hand sides that may be improved or strict requirements to the choice of time scales.

What we offer in our present paper is the result that enables to handle with a wide class of new situations that could have not been examined before; it is new even in general discrete case. Moreover, we describe how the constants on the right-hand sides of Hille-Nehari-type criteria depend on time scales. As a special case, when the limit $M := \lim_{t\to\infty}\mu(t)/(r(t)(\int_a^t 1/r(s)\Delta s))$ exists, we get that the above mentioned (sharp) constant 1/4 is replaced by the (sharp) constant $\gamma(M) = \lim_{x\to M}(\sqrt{x+1}+1)^{-2}$, we use the word "sharp" since such a constant forms a "sharp borderline" between oscillation and nonoscillation area. This value, which belongs to the interval $[0, 1/4]$ and is the same for both sufficient condition for oscillation and nonoscillation, will be called the *critical constant*. Our new result leads to many interesting conclusions: for example, the critical constant is equal to 1/4 in all situations where $M = 0$; the critical constant in the discrete case, when $r(t) \not\equiv 1$, may be different from 1/4; if $\mu(t) = (q-1)t$ with $q > 1$ and $r(t) \equiv 1$, then $M = q-1$ and $\gamma(q-1) = (\sqrt{q}+1)^{-2} \in (0, 1/4)$; or even the critical constant may be equal to 0, this happens when $M = \infty$. Finally note that the proof of the main results is based on the so-called function sequence technique which exploits the Riccati technique, and the transformation of dependent variable.

The paper is organized as follows. In Section 2 we recall some important concepts and state preliminary results that are crucial to prove the main results. Generalized Hille-Nehari theorems are presented in Section 3. Both cases are examined, $\int_a^\infty 1/r(s)\Delta s = \infty$ and $\int_a^\infty 1/r(s)\Delta s < \infty$. Section 4 is the most extensive. To be more precise, there we discuss the concept of critical constant and oscillation constant. Further we apply the main result to obtain criteria for strong (non-) oscillation. Then we discuss conditionally oscillatory equations. We also examine Euler-type and generalized Euler-type equations with showing how they may be used to derive Kneser's and Hille-Nehari theorems. Section 4 also contains examples from h-calculus and q-calculus. Finally we make a comparison with existing results from the papers that have already been mentioned in the first part of this introductory section.

2. Important concepts and preliminary results

We assume that the reader is familiar with the notion of time scales. Thus note just that $\mathbb{T}$, σ, f^σ, μ, f^Δ, and $\int_a^b f^\Delta(s)\Delta s$ stand for time scale, forward jump operator, $f\circ\sigma$, graininess, delta derivative of f, and delta integral of f from a to b, respectively. See [12], which is the initiating paper of the time-scale theory written by Hilger, and the monographs [2, 3] by Bohner and Peterson containing a lot of information on time-scale calculus.

We will proceed with some essentials of oscillation theory of (1.1). First note that we are interested only in nontrivial solutions of (1.1). We say that a solution y of (1.1) has a *generalized zero* at t in case $y(t)=0$. If $\mu(t)>0$, then we say that y has a *generalized zero* in $(t,\sigma(t))$ in case $y(t)y^\sigma(t)<0$. A nontrivial *solution* y of (1.1) is called *oscillatory* if it has infinitely many generalized zeros; note that the uniqueness of IVP excludes the existence of a cluster point which is less than ∞. Otherwise it is said to be *nonoscillatory*. In view of the fact that the Sturm-type separation theorem extends to (1.1) (see, e.g., [20]), we have the following equivalence: one solution of (1.1) is oscillatory if and only if every solution of (1.1) is oscillatory. Hence we may speak about *oscillation* or *nonoscillation of* (1.1). Recall that the principal statements, like the Sturmian theory (Reid-type roundabout theorem, Sturm-type separation, and comparison theorems) for (1.1), can be established under the mere assumption $r(t)\neq 0$ and the basic concepts, especially generalized zero, have to be adjusted, see, for example, [1] or [20]. However, our approach requires the positivity of $r(t)$; (1.1) is viewed as a perturbation of the *nonoscillatory* equation $(r(t)y^\Delta)^\Delta=0$. Note that we do not require the positivity of $p(t)$ even though many approaches in special cases need this assumption.

Next we recall the Sturm-type comparison theorem for (1.1).

Theorem 2.1 [20]. *Let $\tilde r(t)$ and $\tilde p(t)$ be subject to the same conditions as $r(t)$ and $p(t)$, respectively. If $\tilde r(t)\le r(t)$, $\tilde p(t)\ge p(t)$ for large t, and (1.1) is oscillatory, then the equation $(\tilde r(t)x^\Delta)^\Delta+\tilde p(t)x^\sigma=0$ is oscillatory.*

In the above theorem, the comparison of the coefficients is pointwise. In the following Hille-Wintner-type theorem, we compare the coefficients "on average."

Theorem 2.2 [10, 23]. *Let $\int_a^\infty 1/r(s)\Delta s=\infty$. Assume that $0\le\int_t^\infty p(s)\Delta s\le\int_t^\infty \tilde p(s)\Delta s$ for large t (in particular, these integrals converge and are eventually nontrivial). If $(r(t)x^\Delta)^\Delta+\tilde p(t)x^\sigma=0$ is nonoscillatory, then (1.1) is nonoscillatory.*

The next lemma, called the function sequence technique, plays a crucial role in proving the main results. Its proof, as well as that of the previous theorem, is based on the equivalence between nonoscillation of (1.1) and solvability of the Riccati-type integral inequality $w(t)\ge\int_t^\infty p(s)\Delta s+\int_t^\infty w^2(s)/(r(s)+\mu(s)w(s))\Delta s$.

Lemma 2.3 [23]. *Assume that $\int_a^\infty 1/r(s)\Delta s=\infty$ and $\int_t^\infty p(s)\Delta s\ge 0$ $(\not\equiv 0)$ for large t. Define the function sequence $\{\varphi_k(t)\}$ by*

$$\varphi_0(t)=\int_t^\infty p(s)\Delta s,\qquad \varphi_k(t)=\varphi_0(t)+\int_t^\infty \frac{\varphi_{k-1}^2(s)}{r(s)+\mu(s)\varphi_{k-1}(s)}\Delta s,\quad k=1,2,\dots. \tag{2.1}$$

Then (1.1) is nonoscillatory if and only if there exists $t_0 \in [a,\infty)$ *such that* $\lim_{k\to\infty}\varphi_k(t) = \varphi(t)$ *for* $t \ge t_0$*, that is, the sequence* $\{\varphi_k(t)\}$ *is well defined and pointwise convergent.*

The following lemma will be useful in the case when $\int_a^\infty 1/r(s)\Delta s$ converges.

Lemma 2.4 [10]. *Assume that* h *is an rd-continuously delta differentiable function with* $h(t) \neq 0$*. Then* $y = hu$ *transforms (1.1) into the equation* $(\widetilde{r}(t)u^\Delta)^\Delta + \widetilde{p}(t)u^\sigma = 0$ *with* $\widetilde{r} = rhh^\sigma$ *and* $\widetilde{p} = h^\sigma[(rh^\Delta)^\Delta + ph^\sigma]$*. This transformation preserves oscillatory properties.*

We conclude this section with oscillatory criterion which may apply in the case when the value of $\liminf_{t\to\infty}(\int_a^t 1/r(s)\Delta s)\int_t^\infty p(s)\Delta s$ is less than the critical constant. We emphasize that the constant in the next theorem, in contrast to that in Theorem 3.1, does not depend on time scales.

Theorem 2.5 [23]. *Assume that* $\int_a^\infty 1/r(s)\Delta s = \infty$ *and* $\int_a^\infty p(s)\Delta s$ *converges with* $p(t) \ge 0$ *for large* t*. If* $\limsup_{t\to\infty}(\int_a^t 1/r(s)\Delta s)\int_t^\infty p(s)\Delta s > 1$*, then (1.1) is oscillatory. The following improvement of the criterion is possible: the integral* $\int_t^\infty p(s)\Delta s$ *can be replaced by* $\varphi_k(t)$ *and inequality has to hold for some* $k \in \mathbb{N}\cup\{0\}$*.*

3. Main results

In this section we prove the main results: Hille-Nehari-type criteria for (1.1). First we recall that $\int_a^\infty 1/r(s)\Delta s = \infty = \int_a^\infty p(s)\Delta s$ implies (1.1) to be oscillatory, see, for example, [20] for a time-scale extension of the well-known Leighton-Wintner-type criterion. Thus it is reasonable to assume that $\int_a^\infty p(s)\Delta s$ is convergent.

Theorem 3.1. *Let*

$$\int_a^\infty \frac{1}{r(s)}\Delta s = \infty. \tag{3.1}$$

Assume that

$$\int_t^\infty p(s)\Delta s \ge 0 \quad \text{and nontrivial for large } t. \tag{3.2}$$

Denote

$$\begin{aligned} M_* &:= \liminf_{t\to\infty}\frac{\mu(t)}{r(t)\int_a^t 1/r(s)\Delta s}, & M^* &:= \limsup_{t\to\infty}\frac{\mu(t)}{r(t)\int_a^t 1/r(s)\Delta s}, \\ \gamma(x) &:= \lim_{t\to x}\frac{1}{\left(\sqrt{t+1}+1\right)^2}, & \mathcal{A}(t) &:= \left(\int_a^t \frac{1}{r(s)}\Delta s\right)\int_t^\infty p(s)\Delta s. \end{aligned} \tag{3.3}$$

If

$$\liminf_{t\to\infty}\mathcal{A}(t) > \gamma(M_*), \tag{3.4}$$

then (1.1) is oscillatory. If

$$\limsup_{t\to\infty} \mathcal{A}(t) < \gamma(M^*), \tag{3.5}$$

then (1.1) is nonoscillatory.

Proof. Oscillatory part. We will apply Lemma 2.3 and use its notation. Denote $R(t) := \int_a^t 1/r(s)\Delta s$. Condition (3.4) can be rewritten as $\varphi_0(t) \geq \gamma_0/R(t)$ for large t, say $t \geq t_0 > a$, where $\gamma_0 > \gamma(M_*)$. Then, since $x \mapsto x^2/(y+zx)$ is increasing for $x > 0$, $y > 0$, $z > 0$, using the equalities $(1/R(t))^\Delta = -1/(r(t)R(t)R^\sigma(t))$ and

$$\frac{R\sigma(t)}{R(t)} = \frac{R(t) + \int_t^{\sigma(t)} 1/r(s)\Delta s}{R(t)} = 1 + \frac{\mu(t)}{r(t)R(t)} \tag{3.6}$$

we have

$$\begin{aligned}
\varphi_1(t) = \varphi_0(t) + \int_t^\infty \frac{\varphi_0^2(s)}{r(s)+\mu(s)\varphi_0(s)}\Delta s &\geq \frac{\gamma_0}{R(t)} + \int_t^\infty \frac{\gamma_0^2/R^2(s)}{r(s)+\gamma_0\mu(s)/R(s)}\Delta s \\
&= \frac{\gamma_0}{R(t)} + \gamma_0^2 \int_t^\infty \frac{1}{r(s)R(s)R\sigma(s)} \cdot \frac{R\sigma(s)}{R(s)} \cdot \frac{1}{1+\gamma_0\mu(s)/(r(s)R(s))}\Delta s \\
&= \frac{\gamma_0}{R(t)} + \gamma_0^2 \int_t^\infty \frac{1}{r(s)R(s)R\sigma(s)} \cdot \frac{r(s)R(s)+\mu(s)}{r(s)R(s)+\gamma_0\mu(s)}\Delta s \geq \frac{\gamma_1}{R(t)},
\end{aligned} \tag{3.7}$$

where

$$\gamma_1 = \gamma_0 + \gamma_0^2\Gamma_*(t_0,\gamma_0) \quad \text{with } \Gamma_*(t_0,\gamma_0) := \inf_{t\geq t_0} \frac{r(t)R(t)+\mu(t)}{r(t)R(t)+\gamma_0\mu(t)}. \tag{3.8}$$

Similarly, by induction, $\varphi_k(t) \geq \gamma_k/R(t)$, where

$$\gamma_k = \gamma_0 + \gamma_{k-1}^2\Gamma_*(t_0,\gamma_{k-1}), \quad k = 1,2,\ldots. \tag{3.9}$$

Observe that the function $x \mapsto x^2\Gamma_*(t_0,x)$ is increasing for $x > 0$. Hence, $\gamma_k < \gamma_{k+1}$, $k = 0,1,2,\ldots$. We claim that $\lim_{k\to\infty}\gamma_k = \infty$. If not, let $\lim_{k\to\infty}\gamma_k = L < \infty$. Then from (3.9) we have

$$L = \gamma_0 + L^2\Gamma_*(t_0,L). \tag{3.10}$$

First assume that $M := M_* = M^*$. Letting t_0 to ∞ in Γ_* we obtain $\Gamma_*(\infty,L) = (1+M)/(1+ML)$ when $M \in [0,\infty)$ and $\Gamma_*(\infty,L) = 1/L$ when $M = \infty$. Next we show that (3.10) after this limiting process has no real positive solution. Indeed, if $M = \infty$, then (3.10) yields $L = \gamma_0 + L$, but we have $\gamma_0 > 0$. If $M \in [0,\infty)$, then (3.10) yields $L^2 + (\gamma_0 M - 1)L + \gamma_0 = 0$, and a simple analysis shows that this equation is not solvable in the set of positive reals since $\gamma_0 > 1/(\sqrt{M+1}+1)^2$; in particular, the discriminant for this equation attains zero when $\gamma_0 = 1/(\sqrt{M+1}+1)^2$ and the function $x \mapsto L^2 + (xM-1)L + x$ is increasing. Hence we must have $\gamma_k \to \infty$ as $k \to \infty$, which implies $\varphi_k(t) \to \infty$ as $k \to \infty$ for $t \geq t_0$, where

t_0 is sufficiently large. Consequently, (1.1) is oscillatory by Lemma 2.3. Now we examine the case when $M_* < M^*$. We show that (3.10) taken as $t_0 \to \infty$ with $\gamma_0 > \gamma(M_*)$ has no real positive solution. Observe that $\lim_{t_0\to\infty} \Gamma_*(t_0,L) = \lim_{x\to\bar{M}}(1+x)/(1+xL)$, where $\bar{M} \in [M_*,M^*]$. Using the arguments as above, the equation $L = \bar{\gamma}_0 + L^2 \lim_{t_0\to\infty}\Gamma_*(t_0,L)$ has no real positive solution provided $\bar{\gamma}_0 > \gamma(\bar{M})$. Since $x \mapsto \gamma(x)$ is decreasing for $x > 0$, we have $\gamma_0 > \gamma(M_*) \geq \gamma(\bar{M})$, and so neither does the last equation with γ_0 instead of $\bar{\gamma}_0$ have a real solution. The rest of the proof is the same as in the case $M_* = M^*$. Note that M_* in (3.4) is the best value which can be attained when proceeding as in this proof since the function $x \mapsto (1+x)/(1+Lx)$ is nondecreasing when $L \in [0,1]$, and a closer examination shows that we are interested just in such L's.

Nonoscillatory part. First note that the case $M^* = \infty$ (i.e., $\gamma(M^*) = 0$) may obviously be excluded, in view of the assumptions of the theorem. Condition (3.5) can be rewritten as $\varphi_0(t) \leq \delta_0/R(t)$ for large t, say $t \geq t_0 > a$, where $0 < \delta_0 < \gamma(M^*)$. Similarly as in the previous part of this proof, we get

$$\varphi_k(t) \leq \frac{\delta_k}{R(t)}, \quad t \geq t_0 > a, \tag{3.11}$$

where

$$\delta_k = \delta_0 + \delta_{k-1}^2 \Gamma^*(t_0,\delta_{k-1}), \quad \Gamma^*(t_0,\delta_{k-1}) := \sup_{t\geq t_0} \frac{r(t)R(t)+\mu(t)}{r(t)R(t)+\delta_{k-1}\mu(t)}, \tag{3.12}$$

$k = 1,2,\ldots$. Clearly, $\{\delta_k\}$ is increasing. We claim that it converges. First assume that $M := M_* = M^*$. To show the convergence, consider the fixed point problem $x = g(x)$, where $g(x) = \lambda + x^2(1+M)/(1+Mx)$ with a positive constant λ, and the "perturbed" problem $x = \tilde{g}(x)$, where $\tilde{g}(x) = \lambda + x^2\Gamma^*(t_0,x)$. First consider $x = g(x)$, which can be rewritten as $x = x^2 + \lambda M x + \lambda =: g_1(x)$; note that we are particularly interested in the first quadrant. The fixed points of this problem will be found by means of the iteration scheme $x_k = g_1(x_{k-1})$, $k = 1,2,\ldots$. If $\lambda = 1/(\sqrt{M+1}+1)^2$, then the graph of g_1 is a parabola which has a unique minimum at $x = -M/[2(\sqrt{M+1}+1)^2]$ and touches the line $y = x$ at $(x,y) = (1/(\sqrt{M+1}+1), 1/(\sqrt{M+1}+1))$. Therefore, if we choose $x_0 = \lambda = 1/(\sqrt{M+1}+1)^2$, then we see that the approximating sequence $\{x_k\}$ for the problem $x = g_1(x)$, that is, satisfying the relation $x_k = g_1(x_{k-1})$ is strictly increasing and converges to $1/(\sqrt{M+1}+1)$. Clearly, if $0 < y_0 = \lambda < 1/(\sqrt{M+1}+1)^2$, then the approximating sequence $\{y_k\}$ for the same problem that is satisfying $y_k = g_1(y_{k-1})$ is increasing as well and permits $y_k < x_k < 1/(\sqrt{M+1}+1)$; therefore, $\{y_k\}$ converges. Thus we have solved the fixed point problem $x = g_1(x)$, and consequently, $x = g(x)$. Now we take into account that $\lim_{t_0\to\infty}\Gamma^*(t_0,x) = (1+M)/(1+Mx)$. Hence the function $\tilde{g}$ in the perturbed problem can be made as close to g as we need (locally, on the interval under consideration) provided t_0 is sufficiently large. This closeness of g to $\tilde{g}$ along with the inequality $\delta_0 < \gamma(M)$ lead to the fact that the sequence $\{\delta_k\}$ for the original problem (3.12) converges for t_0 large. Thus $\{\varphi_k(t)\}$ converges by (3.11), and so (1.1) is nonoscillatory by Lemma 2.3. The case when $M_* < M^*$ can be treated similarly, using ideas from the last part of the proof of oscillation. □

If there exists a limit of the expression in (3.3), then we may establish the critical constant (which is sharp) for the Hille-Nehari criteria.

Corollary 3.2. *Let $M := M_* = M^*$ in Theorem 3.1. Then $\gamma(M)$ is the critical constant (the constants on the right-hand sides of criteria (3.4) and (3.5) are equal). In particular,*

$$\gamma(M) = \begin{cases} \dfrac{1}{4} & \text{if } M = 0, \\ \dfrac{1}{(\sqrt{M+1}+1)^2} & \text{if } 0 < M < \infty, \\ 0 & \text{if } M = \infty. \end{cases} \tag{3.13}$$

Using the transformation of dependent variable and Theorem 3.1 we can easily treat the complementary case to (3.1), namely, $\int_a^\infty 1/r(s)\Delta s$ converges.

Theorem 3.3. *Let*

$$\int_a^\infty \frac{1}{r(s)}\Delta s < \infty. \tag{3.14}$$

Assume that

$$\int_t^\infty \left(\int_{\sigma(s)}^\infty \frac{1}{r(\tau)}\Delta_\tau\right)^2 p(s)\Delta s \ge 0 \quad \text{and nontrivial for large } t. \tag{3.15}$$

Denote

$$\widetilde{M}_* := \liminf_{t\to\infty} \frac{\mu(t)}{r(t)\int_{\sigma(t)}^\infty 1/r(s)\Delta s}, \qquad \widetilde{M}^* := \limsup_{t\to\infty} \frac{\mu(t)}{r(t)\int_{\sigma(t)}^\infty 1/r(s)\Delta s},$$
$$\widetilde{\mathcal{A}}(t) := \left(\int_t^\infty \frac{1}{r(s)}\Delta s\right)^{-1} \int_t^\infty \left(\int_{\sigma(s)}^\infty \frac{1}{r(\tau)}\Delta_\tau\right)^2 p(s)\Delta s. \tag{3.16}$$

If

$$\liminf_{t\to\infty} \widetilde{\mathcal{A}}(t) > \gamma(\widetilde{M}_*), \tag{3.17}$$

then (1.1) is oscillatory. If

$$\limsup_{t\to\infty} \widetilde{\mathcal{A}}(t) < \gamma(\widetilde{M}^*), \tag{3.18}$$

then (1.1) is nonoscillatory.

Proof. Denote $\widetilde{R}(t) := \int_t^\infty 1/r(s)\Delta s$. First note that by Lemma 2.4, the transformation $y = hu$ with $h(t) = \widetilde{R}(t)$ transforms (1.1) into the equation $(\widetilde{r}(t)u^\Delta)^\Delta + \widetilde{p}(t)u^\sigma = 0$, where $\widetilde{r}(t) = \widetilde{R}(t)\widetilde{R}^\sigma(t)r(t)$ and $\widetilde{p}(t) = (\widetilde{R}^\sigma(t))^2 p(t)$. Since $(1/\widetilde{R}(t))^\Delta = 1/\widetilde{r}(t)$, we get that $\int_a^\infty 1/\widetilde{r}(s)\Delta\widetilde{s} = \infty$. Further we obtain that the limit behavior (as $t \to \infty$) of $\mu(t)/(\widetilde{r}(t)\int_a^t 1/\widetilde{r}(s)\Delta s)$ is the same as that of $\mu(t)/(r(t)\widetilde{R}^\sigma(t))$, and the limit behavior (as $t \to \infty$) of $(\int_a^t 1/\widetilde{r}(s)\Delta s)\int_t^\infty \widetilde{p}(s)\Delta s$ is the same as that of $\widetilde{\mathcal{A}}(t)$. Applying now Theorem 3.1 and using the fact that oscillatory properties of transformed equation are preserved, we get the statement. □

Similarly as for Theorem 3.1, there is a corollary of Theorem 3.3 where the condition $\widetilde{M}_* = \widetilde{M}^*$ leads to the existence of a sharp critical constant.

4. Consequences, comparisons, and examples

(i) *Critical and oscillation constants.* As already said in introduction, in the continuous case it is well known that if $\liminf_{t\to\infty} \mathcal{A}_{\mathbb{R}}(t) > 1/4$, where

$$\mathcal{A}_{\mathbb{R}}(t) := \left(\int_a^t \frac{1}{r(s)}\,ds\right)\int_t^\infty p(s)\,ds, \tag{4.1}$$

then (1.2) is oscillatory, and the constant 1/4 is the best possible constant: it cannot be lowered since $\limsup_{t\to\infty} \mathcal{A}_{\mathbb{R}}(t) < 1/4$ implies nonoscillation of (1.2). Note that the latter condition is sufficient for nonoscillation provided $\int_t^\infty p(s)\,ds \geq 0$ for large t. If there is no such sign condition on $p(t)$, then we need to assume that $\liminf_{t\to\infty} \mathcal{A}_{\mathbb{R}}(t) > -3/4$, see, for example, [6]. On the other hand, oscillation is still possible even when $\liminf_{t\to\infty} \mathcal{A}_{\mathbb{R}}(t) < 1/4$, see Theorem 2.5 and [6]. The constant on the right-hand sides of the above Hille-Nehari criteria (but also of other ones that are of a similar type, like Kneser's one, see (iv)) is called a *critical constant*; in particular, it is the same for both oscillation and nonoscillation, and equals 1/4. Sometimes this constant is said to be an *oscillation constant*. However, we prefer to use the former terminology (and its extension to the time-scale case) since the second one has sometimes another meaning, see the next item devoted to conditionally oscillatory equations. As we will see, there is a connection between critical and oscillation constants: Hille-Nehari criteria involving the critical constant can be used to derive the oscillation constant. Note that sometimes (this particularly concerns various extensions, for example, higher-order, nonlinear, or discrete cases) the constant on the right-hand side of oscillatory [nonoscillatory] criteria (like that of Hille-Nehari-type) is called *oscillation [nonoscillation] constant*. In general, one may not be completely successful in extending, and the oscillation constant in the latter sense may be strictly greater than the nonoscillation one. Thus using the later terminology in Theorem 3.1, $\gamma(M_*)$ is oscillation constant and $\gamma(M^*)$ is nonoscillation constant. The above defined term "critical constant" reflects the fact that this constant cannot be improved and forms a sharp border between oscillation and nonoscillation. Note that the strict inequalitities in Hille-Nehari criteria cannot be replaced by nonstrict ones since no conclusion can be drawn if either $\liminf_{t\to\infty} \mathcal{A}(t)$ or $\limsup_{t\to\infty} \mathcal{A}(t)$ equals the critical constant; both oscillation and nonoscillation may happen, as it has already been shown in the continuous case, see, for example, [26]. Our result shows that if $\liminf_{t\to\infty} \mathcal{A}(t) > 1/4$, then (1.1) is oscillatory (no matter what time scale is, since $\gamma(x) \leq 1/4$ for $x \in [0,\infty) \cup \{\infty\}$). However, in addition, our theorem says that 1/4 is not the best possible constant which is universal for all time scales (in particular, it may not be critical at all). In fact, the constant depends on a time scale and also on the coefficient r; the cases happen where it is strictly less than 1/4. If (3.3) is satisfied, then the critical constant is $\gamma(M) \in [0,1/4]$. Later we will present examples where $\gamma(M) < 1/4$. We conclude this item with noting that oscillation of (1.1) is still possible even when

$\liminf_{t\to\infty} \mathcal{A}(t) < \gamma(M)$. This follows from Theorem 2.5, and we emphasize that there is no additional condition on a time scale in that theorem.

(ii) *Strong and conditional oscillation.* Consider the equation

$$(r(t)y^{\Delta})^{\Delta} + \lambda p(t) y^{\sigma} = 0, \tag{4.2}$$

where $r(t) > 0$, $p(t) > 0$, and λ is a real parameter. In the continuous case, the concept of strong and conditional oscillation was introduced by Nehari [18]. We say that (4.2) is *conditionally oscillatory* if there exists a constant $0 < \lambda_0 < \infty$ such that (4.2) is oscillatory for $\lambda > \lambda_0$ and nonoscillatory for $\lambda < \lambda_0$. The value λ_0 is called the *oscillation constant* of (4.2). Since this constant depends on the coefficients of the equation, we often speak about the oscillation constant of the function p with respect to r. If (4.2) is oscillatory (resp., nonoscillatory) for every $\lambda > 0$, then this equation is said to be *strongly oscillatory* (resp., *strongly nonoscillatory*). Next we apply the results from the previous section to derive necessary and sufficient condition for strong (non-) oscillation.

Theorem 4.1. *Let (3.1) hold and $\int_a^\infty p(s)\Delta s$ converge with $p(t) \geq 0$ for large t. Assume that $M^* < \infty$. Then (4.2) is strongly oscillatory if and only if $\limsup_{t\to\infty} \mathcal{A}(t) = \infty$, and it is strongly nonoscillatory if and only if $\lim_{t\to\infty} \mathcal{A}(t) = 0$.*

Proof. Denote that $R(t) := \int_a^t 1/r(s)\Delta s$. If $\limsup_{t\to\infty} \mathcal{A}(t) = \infty$ does hold, then we have $\limsup_{t\to\infty} R(t) \int_t^\infty \lambda p(s)\Delta s > 1$ for every $\lambda > 0$, and so (4.2) is oscillatory for every $\lambda > 0$ by Theorem 2.5. Conversely, if (4.2) is strongly oscillatory, then

$$\limsup_{t\to\infty} R(t) \int_t^\infty \lambda p(s)\Delta s \geq \gamma(M^*) > 0 \tag{4.3}$$

for every $\lambda > 0$ by Theorem 3.1. This implies $\limsup_{t\to\infty} \mathcal{A}(t) = \infty$; otherwise, (4.3) would be violated for sufficiently small λ. The proof of the part concerning strong nonoscillation is based on similar arguments. The details are left to the reader. □

One could ask whether the condition $M^* < \infty$ in the last theorem may be dropped. In general, the answer is no. Realize that strong oscillation (strong nonoscillation) of (4.2) is nothing but $\lambda_0 = 0$ $[\lambda_0 = \infty]$, where λ_0 is the oscillation constant. Now assume that $M^* = \infty = M_*$ and $\lim_{t\to\infty} \mathcal{A}(t) = L \in (0,\infty)$ exists. Then $\lim_{t\to\infty} R(t) \int_t^\infty \lambda p(s)\Delta s = \lambda L > 0$ for every $\lambda > 0$. This implies strong oscillation of (4.2), however the condition $\limsup_{t\to\infty} R(t) \int_t^\infty \lambda p(s)\Delta s = \infty$ does not hold. A particular example of such strongly oscillatory equation will be given later. Similar criteria as those in Theorem 4.1 can obviously be established also in the case when $\int_a^\infty 1/r(s)\Delta s < \infty$. Then they involve the expression $\widetilde{\mathcal{A}}(t)$. For the proof we use Theorem 3.3 and the counterpart—in the sense of $\int_a^\infty 1/r(s)\Delta s < \infty$—to Theorem 2.5 which can be derived by means of Lemma 2.4.

(iii) *Euler-type dynamic equation.* Consider the equation

$$y^{\Delta\Delta} + \frac{\lambda}{t\sigma(t)} y^{\sigma} = 0, \tag{4.4}$$

where λ is a positive parameter. Note that we are interested only in positive λ's since for $\lambda = 0$, (4.4) is readily explicitly solvable, it is nonoscillatory, and thus for $\lambda < 0$ is nonoscillatory as well by the Sturm-type comparison theorem (Theorem 2.1). Equation (4.4) will be called an *Euler dynamic equation* since for $\mathbb{T} = \mathbb{R}$ it reduces to the well known Euler differential equation $y'' + \lambda t^{-2} y = 0$. Applying Theorem 3.1 we get that (4.4) is oscillatory provided $\lambda > \gamma(M_*)$ and nonoscillatory provided $\lambda < \gamma(M^*)$. Assume that $M := M_* = M^*$. Then $M = \lim_{t\to\infty} \mu(t)/t$, $\gamma(M)$ is the critical constant, and $\lambda_0 = \gamma(M)$ is the oscillation constant. Now if, for example, $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = \mathbb{Z}$, then $M = 0$ and $\gamma(M) = 1/4$. This matches what we know from the classical differential and difference equations case, see, for example, [21, Section 8], [23, Example 2], and [28] for the discrete case. Note that $\gamma(M) = 1/4$ for all time scales whose graininess $\mu(t)$ is asymptotically less than t; for example, $\mathbb{T} = \{n^2 : n \in \mathbb{N}_0\}$ (then $\mu(t) = 1 + 2\sqrt{t}$). If we assume that $\mathbb{T} = q^{\mathbb{N}_0} := \{q^k : k \in \mathbb{N}_0\}$ with $q > 1$, then (4.4) reduces to the Euler q-difference equation, $\mu(t) = (q-1)t$, and $M = q - 1 > 0$. Hence the critical constant is $\gamma(M) = 1/(\sqrt{q} + 1)^2 < 1/4$. This matches the result by Bohner and Ünal [5] who solved (4.4) explicitly on $\mathbb{T} = q^{\mathbb{N}_0}$. Finally assume that $\mathbb{T} = 2^{\alpha^{\mathbb{N}_0}} := \{2^{\alpha^k} : k \in \mathbb{N}_0\}$ with $\alpha > 1$. Then $\mu(t) = t^\alpha - t$ and so $M = \infty$. Hence, the critical constant is $\gamma(M) = 0$. This implies that (4.4) on $2^{\alpha^{\mathbb{N}_0}}$ is oscillatory for all $\lambda > 0$. Therefore, (4.4) is strongly oscillatory when $\mathbb{T} = 2^{\alpha^{\mathbb{N}_0}}$ while it is conditionally oscillatory in all previous cases.

(iv) *Generalized Euler-type dynamic equation and Kneser-type criteria.* Consider the so-called *generalized Euler dynamic equation*

$$\left(r(t)y^\Delta\right)^\Delta + \frac{\lambda}{r(t)R(t)R^\sigma(t)} y^\sigma = 0, \tag{4.5}$$

where λ is a positive parameter and $R(t) := \int_a^t 1/r(s)\Delta s$ with $r(t) > 0$ and $R(\infty) = \infty$. First note that if $r(t) \equiv 1$, then (4.5) reduces to (4.4). In the continuous case, there is no essential difference between (4.4) and (4.5) owing to the transformation of independent variable $t \mapsto R(t)$, and so it suffices to examine (4.4) only. However, in general case such a transformation is not available, and so considering the case $r(t) \neq 0$ brings new observations. According to Corollary 3.2, the critical constant is $\gamma(M)$ provided $M := M_* = M^*$; for the associated oscillation constant we have $\lambda_0 = \gamma(M)$. Equations of type (4.5) may be very useful for comparison purposes: The Sturm-type comparison theorem (Theorem 2.2), where (1.1) and (4.5) are compared, leads to the following criteria.

(i) If $\liminf_{t\to\infty} r(t)R(t)R^\sigma(t)p(t) > \lambda_0$, then (1.1) is oscillatory.

(ii) If $\limsup_{t\to\infty} r(t)R(t)R^\sigma(t)p(t) < \lambda_0$, then (1.1) is nonoscillatory.

Since we know that $\lambda_0 = \gamma(M)$, we have derived *Kneser-type criteria* for (1.1), see, for example, [26] for the continuous case. A slight modification gives the Kneser-type criteria in the case when $M_* < M^*$. We omit details. Now imagine for a moment that Theorem 3.1is not at disposal but the oscillation constant λ_0 in (4.5) is known. Applying the Hille-Wintner-type comparison theorem (Theorem 2.2), where (1.1) and (4.5) are compared, we obtain Hille-Nehari-type criteria. Thus we have another method of how to get Hille-Nehari-type criteria. However, a disadvantage of this approach is that in a general case we do not know how to describe solutions of Euler-type equations, even when $r(t) \equiv 1$, in such a way which would provide an exact information about critical constants. Similar observations can be done also in the case when $\widetilde{R}(t) := \int_t^\infty 1/r(s)\Delta s$ converges. Then we consider the equation

$$\left(r(t)y^\Delta\right)^\Delta + \frac{\lambda}{r(t)\left(\widetilde{R}^\sigma(t)\right)^2}y^\sigma = 0, \tag{4.6}$$

which has the oscillation constant $\lambda_0 = \gamma(\widetilde{M})$ provided $\widetilde{M} := \widetilde{M}_* = \widetilde{M}^*$ by Theorem 3.3. Kneser-type and Hille-Nehari-type criteria can be again derived by means of suitable comparison theorems, the Sturm one and the modification of the Hille-Nehari one for the case $\widetilde{R}(t) < \infty$ (see [10, Theorem 2.5]), respectively. Details are omitted.

(v) *Example from h-calculus.* Let $h > 0$. Recall that the calculi developed on the time scales $\mathbb{T} = h\mathbb{Z} := \{hk : k \in \mathbb{Z}\}$ and the above- and below-mentioned $\mathbb{T} = q^{\mathbb{N}_0}$ are two important types of quantum calculus, see [15]. These calculi are called h-calculus (or calculus of finite differences) and q-calculus, respectively. Associated dynamic equations are called h-difference equations (or, especially when $h = 1$, difference equations) and q-difference equations, respectively. Consider the equation

$$\left(\frac{y^\Delta}{e^t}\right)^\Delta + \frac{\lambda}{e^t}y^\sigma = 0, \tag{4.7}$$

where λ is a real constant and e is a base of the natural logarithm. We start with the continuous case, that is, assume $\mathbb{T} = \mathbb{R}$. Applying Corollary 3.2, it is easy to see that for the oscillation constant of (4.7) we have $\lambda_0 = \gamma(M) = 1/4$. Moreover, $y(t) = \sqrt{e^t}$ is a nonoscillatory solution of (4.7) where $\lambda = \lambda_0$. Now assume $\mathbb{T} = h\mathbb{Z}$. Then $\sigma(t) = t + h$ and $\mu(t) \equiv h$. Since $(e^t)^\Delta = e^t(e^h - 1)/h$ and $(e^{-t})^\Delta = e^{-t}(e^{-h} - 1)/h$, we have $\lim_{t\to\infty} \mathcal{A}(t) = \lambda h^2 e^h/(e^h - 1)$, $M = e^h - 1$, and $\gamma(M) = 1/(\sqrt{e^h} + 1)^2$. Applying Corollary 3.2, we find that the oscillation constant of (4.7) when $\mathbb{T} = h\mathbb{Z}$ is $\lambda_0 = (\sqrt{e^h} - 1)^2/(h^2 e^h)$. Moreover, $y(t) = \sqrt{e^t}$ is a nonoscillatory solution of (4.7) where $\lambda = \lambda_0$. Note how the results resemble the continuous counterparts as $h \to 0$. In particular, $\gamma(M) \to 1/4$ and $\lambda_0 \to 1/4$ as $h \to 0$. Thus we have shown an example of difference equation where $\gamma(M) < 1/4$ which is not possible in the continuous case. As far as we know, finding Hille-Nehari-type criteria with the (sharp) critical constant when $M > 0$ has been an open problem even in the well-studied discrete case ($\mathbb{T} = \mathbb{Z}$), and this problem is solved now. One of the reasons for that problem may have been the absence of the above-mentioned transformation of the independent variable when $\mathbb{T} \neq \mathbb{R}$. Note that $M > 0$ may happen only when $r(t) \not\equiv 1$ in the case $\mathbb{T} = h\mathbb{Z}$. For this case, even we may have $M = \infty$, for example, when $r(t) = 2^{-t^t}$.

(vi) *Examples from q-calculus.* Assume that $\mathbb{T} = q^{\mathbb{N}_0} := \{q^k : k \in \mathbb{N}_0\}$ with $q > 1$. Then $\sigma(t) = qt$ and $\mu(t) = (q - 1)t$. We will compute the value of the critical constant γ for two different coefficients $r(t)$ and examine one q-difference equation, where $p(t)$ is not eventually of one sign. Let $r(t) = t^\alpha$, $\alpha \in \mathbb{R}$. First suppose $\alpha < 1$. Then, with $t = q^n$, $n \in \mathbb{N}_0$, we have

$$\lim_{t\to\infty}\frac{\mu(t)/r(t)}{\int_1^t 1/r(s)\Delta s}=\lim_{n\to\infty}\frac{(q-1)q^n/(q^n)^\alpha}{\sum_{j=0}^{n-1}\mu(q^j)/r(q^j)}=\lim_{n\to\infty}\frac{(q-1)(q^{1-\alpha})^n}{\sum_{j=0}^{n-1}(q-1)(q^{1-\alpha})^j}$$
$$=\lim_{n\to\infty}\frac{q^{(1-\alpha)n}(q^{1-\alpha}-1)}{q^{(1-\alpha)n}}=q^{1-\alpha}-1, \tag{4.8}$$

where L'Hôpital's rule is used. The same result is obtained when $\alpha=1$ since $(\ln t)^\Delta=\ln q/((q-1)t)$. If $\alpha>1$, then we proceed with the arguments similar to those in the case $\alpha<1$, and we get $\lim_{t\to\infty}(\mu(t)/r(t))/\int_{\sigma(t)}^\infty 1/r(s)\Delta s=q^{\alpha-1}-1$. Hence, for the critical constant associated to $r(t)=t^\alpha$ we have $\gamma=(\sqrt{q^{|\alpha-1|}}+1)^{-2}$ for $\alpha\in\mathbb{R}$ with the note that $\int_1^\infty 1/r(s)\Delta s=\infty$ if $\alpha\le 1$ and $\int_1^\infty 1/r(s)\Delta s<\infty$ if $\alpha>1$. Observe how in the above results the "limits" as $q\to 1$ correspond to the continuous counterparts. As another example, assume that $r(t)=\beta^{\log_q(1/t)}$, $\beta>0$. Then, with $t=q^n$, $n\in\mathbb{N}_0$, we have $r(t)=\beta^{-n}$. Applying again similar arguments as above, we obtain: if $q\beta\ge 1$, then $\int_1^\infty 1/r(s)\Delta s=\infty$ and $\gamma=(\sqrt{q\beta}+1)^{-2}$; if $0<q\beta<1$, then $\int_1^\infty 1/r(s)\Delta s<\infty$ and $\gamma=(\sqrt{1/(q\beta)}+1)^{-2}$. Considering now one of the above two $r(t)$'s and taking the relevant $p(t)$ as in (4.5) or (4.6), we get nice examples of conditionally oscillatory q-difference equations with known oscillation constant that can be further used for comparison purposes. The details are left to the reader. As the last example of a q-difference equation, consider (1.1) where

$$p(t)=\frac{\lambda_1}{t\log_q t\log_q(qt)}+\frac{\lambda_2(-1)^{\log_q t}}{t\log_q t},\qquad r(t)=t, \tag{4.9}$$

λ_1,λ_2 being nonnegative constants. Observe that $p(t)$ is not eventually of one sign if $\lambda_2\neq 0$. We have $((q-1)\log_q t)^\Delta=1/r(t)$. Hence, $R(t):=\int_1^t 1/r(s)\Delta s=(q-1)\log_q t$ and $M=\lim_{t\to\infty}1/\log_q t=0$. Further, it holds $((q-1)/\log_q t)^\Delta=-1/(t\log_q t\log_q(qt))$. Therefore,

$$\int_t^\infty p(s)\Delta s=\frac{\lambda_1(q-1)}{\log_q t}+\lambda_2(q-1)(-1)^{\log_q t}\left(\frac{1}{\log_q t}-\frac{1}{\log_q(qt)}+\frac{1}{\log_q(q^2t)}-\cdots\right). \tag{4.10}$$

Now it is easy to see that $(q-1)^2(\lambda_1-\lambda_2)\le\mathcal{A}(t)\le(q-1)^2(\lambda_1+\lambda_2)$. According to Corollary 3.2, we get the following: if $\lambda_1-\lambda_2>(q-1)^{-2}/4$, then (1.1) is oscillatory; if $\lambda_1+\lambda_2<(q-1)^{-2}/4$ and $\lambda_1\ge\lambda_2$, then (1.1) is nonoscillatory. Since $M=0$, in view of [20, Theorem 7], in the latter case, $\lambda_1\ge\lambda_2$ can be improved as $\lambda_1-\lambda_2\ge -3(q-1)^{-2}/4$. If $\lambda_2=0$, then there is the oscillation constant $\lambda_0=(q-1)^{-2}/4$. Note that

$$r(t)R(t)R^\sigma(t)p(t)=(q-1)\lambda_1+(q-1)\lambda_2(-1)^{\log_q t}\log_q(qt). \tag{4.11}$$

Since lim sup [lim inf] of $r(t)R(t)R^\sigma(t)p(t)$ as $t\to\infty$ is equal to ∞ $[-\infty]$, we see that the Kneser-type criteria derived in (iv) are not suitable to be applied here while the Hille-Nehari ones give the result.

(vii) *Comparisons.* Here we compare the results of this paper with related ones which have been achieved in previous works (in particular, with the criteria of a similar type or with examinations of Euler-type equations). We have already seen that in the cases $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = \mathbb{Z}$ our results reduce to the classical ones, see [7, 8, 14, 16–19, 21, 26, 28], with the note that our results from Section 3 are new in the discrete case when $r(t) \not\equiv 1$. In [4], it was shown that (4.4) is oscillatory if $\lambda > 1/4$ on any time scale. From the above, we can see that this result follows from our ones, but the constant 1/4 may be improved. Using a Wirtinger-type inequality, in [13] it was shown that (4.4) is nonoscillatory provided $\lambda < 1/\Psi$, where

$$\Psi = \limsup_{T\to\infty}\left\{\left(\sup_{t\ge T}\frac{\sigma(t)}{t}\right)^{1/2} + \left[\left(\sup_{t\ge T}\frac{\mu(t)}{t}\right) + \left(\sup_{t\ge T}\frac{\sigma(t)}{t}\right)\right]^{1/2}\right\}^2 \tag{4.12}$$

(on any time scale). If, for example, $\mathbb{T} = q^{\mathbb{N}_0}$, then $\Psi = (\sqrt{q} + \sqrt{2q-1})^2$. Comparing now $1/\Psi$ with the (critical) constant $\gamma(q-1)$ obtained from Corollary 3.2, we get $\gamma(q-1) = (\sqrt{q}+1)^{-2} > (\sqrt{q}+\sqrt{2q-1})^{-2} = 1/\Psi$, and we see that the constant $1/\Psi$ can be improved. Using a Hardy-type inequality, in [22] it was shown (even for a more general, namely, half-linear case) that the Euler-type equation

$$y^{\Delta\Delta} + \frac{\bar{\lambda}}{(\sigma(t))^2}y^{\sigma} = 0 \tag{4.13}$$

is nonoscillatory provided $\bar{\lambda} \le 1/4$ (on any time scale). Let $\mathbb{T} = q^{\mathbb{N}_0}$. Rewriting (4.13) into the form (4.4) we get $y^{\Delta\Delta} + (\bar{\lambda}/q)/(t\sigma(t))y^{\sigma} = 0$. Now Corollary 3.2 says that (4.13) is nonoscillatory provided $\bar{\lambda} < q/(\sqrt{q}+1)^2$. Since $1/4 < q/(\sqrt{q}+1)^2$ we have again an improvement. Note that in [22] the constant in the Hardy inequality which then corresponds to 1/4 in the Euler equation is shown to be the best possible constant when $\mu(t)/t \to 0$ as $t \to \infty$. Our observations now reveal justifiability of the additional condition $\mu(t)/t \to 0$ which is nothing but $M = 0$. As we have already pointed out, in [5] devoted to linear q-difference equations, (4.4) (when $\mathbb{T} = q_0^{\mathbb{N}}$) was explicitly solved, and the oscillation constant $\lambda_0 = (\sqrt{q}+1)^{-2}$ was derived which coincides with what we get from Corollary 3.2. In addition, (4.4) with $\lambda = \lambda_0$ was shown to have a nonoscillatory solution. The obtained results are used to establish Kneser-type criteria. Finally note that in [23] it was shown (even for half-linear case) that 1/4 is the critical constant in Hille-Nehari-type criteria for (1.1) provided $\lim_{t\to\infty}\mu(t)/(r(t)\int_a^t 1/r(s)\Delta s) = 0$ which again coincides with our results. The case when the limit is greater than zero or it does not exist is not discussed there, and for half-linear case it remains as challenging problem, in view of how the linear case is shown to work in the presented paper.

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