ON THIRD-ORDER LINEAR DIFFERENCE EQUATIONS INVOLVING QUASI-DIFFERENCES

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We study the third-order linear difference equation with quasi-differences and its adjoint equation. The main results of the paper describe relationships between the oscillatory and nonoscillatory solutions of both equations.

1. Introduction

Consider the third-order linear difference equation

$$\Delta\left(p_n\Delta\left(r_n\Delta x_n\right)\right)+q_nx_{n+1}=0 \tag{E}$$

and its adjoint equation

$$\Delta\left(r_{n+1}\Delta\left(p_n\Delta u_n\right)\right)-q_{n+1}u_{n+2}=0, \tag{E^A}$$

where Δ is the forward difference operator defined by $\Delta x_n = x_{n+1} - x_n$, (p_n), (r_n), and (q_n) are sequences of positive real numbers for $n \in \mathbb{N}$.

This paper has been motivated by the paper [9], where third-order difference equations

$$\begin{gathered}\Delta^3 v_n - p_{n+1}\Delta v_{n+1} + q_{n+1}v_{n+1} = 0,\\ \Delta\left(\Delta^2 u_n - p_{n+1}u_{n+1}\right) - q_{n+2}u_{n+2} = 0\end{gathered} \tag{1.1}$$

had been investigated. As it is noted here, these equations are not adjoint equations and are referred to as *quasi-adjoint* equations.

Equation (E) is a special case of linear nth-order difference equations with quasi-differences. Such equations have been widely studied in the literature, see, for example, [6, 11] and the references therein. The natural question which arises is to find the adjoint equation to (E) and to examine the connection between solutions of (E) and its adjoint one.

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In the continuous case, it holds (see, e.g., [5, Theorem 8.33]) that

$$\left(\frac{1}{p(t)}\left(\frac{1}{r(t)}x'(t)\right)'\right)' + q(t)x(t) = 0 \tag{1.2}$$

is oscillatory if and only if the adjoint equation

$$\left(\frac{1}{r(t)}\left(\frac{1}{p(t)}x'(t)\right)'\right)' - q(t)x(t) = 0 \tag{1.3}$$

has the same property. In addition, nonoscillatory solutions of these equations satisfy some interesting relationships, see, for example, [2, 5].

The aim of this paper is to investigate oscillatory and asymptotic properties of solutions of (E) and (E^A). We will prove that (E^A) is the adjoint equation to (E) and we will give discrete analogues of the above-quoted results for third-order differential equations. Moreover, the oscillation of (E) and (E^A) is characterized by means of second-order linear difference equations and the problem of the number of oscillatory solutions in a given basis for the solution space of (E) and (E^A) is investigated. Our results extend and complete results of [7–10] stated for the various forms of third-order difference equations.

A solution x of (E) is a real sequence (x_n) defined for all $n \in \mathbb{N}$ and satisfying (E) for all $n \in \mathbb{N}$. A solution of (E) is called *nontrivial* if for any $n_0 \geq 1$, there exists $n > n_0$ such that $x_n \neq 0$. Otherwise, the solution is called *trivial*. A nontrivial solution x of (E) is said to be *oscillatory* if for any $n_0 \geq 1$, there exists $n > n_0$ such that $x_{n+1}x_n \leq 0$. Otherwise, the nontrivial solution is said to be *nonoscillatory*. Equation (E) is *oscillatory* if it has an oscillatory solution. The same terminology is used for (E^A).

Denote quasi-differences $x^{[i]}$, $i = 0,1,2$, of a solution x of (E) as follows:

$$x_n^{[0]} = x_n, \qquad x_n^{[1]} = r_n \Delta x_n, \qquad x_n^{[2]} = p_n \Delta x_n^{[1]}, \qquad x_n^{[3]} = \Delta x_n^{[2]}. \tag{1.4}$$

Similarly, denote quasi-differences $u^{[i]}$, $i = 0,1,2$, of a solution u of (E^A) as follows:

$$u_n^{[0]} = u_n, \qquad u_n^{[1]} = p_n \Delta u_n, \qquad u_n^{[2]} = r_{n+1} \Delta u_n^{[1]}, \qquad u_n^{[3]} = \Delta u_n^{[2]}. \tag{1.5}$$

All nonoscillatory solutions x of (E) can be a priori classified to the following classes:

$$\begin{aligned}
N_0 &= \{x : \exists n_x \text{ s.t. } x_n x_n^{[1]} < 0,\ x_n x_n^{[2]} > 0\ \forall n \geq n_x\},\\
N_1 &= \{x : \exists n_x \text{ s.t. } x_n x_n^{[1]} > 0,\ x_n x_n^{[2]} < 0\ \forall n \geq n_x\},\\
N_2 &= \{x : \exists n_x \text{ s.t. } x_n x_n^{[1]} > 0,\ x_n x_n^{[2]} > 0\ \forall n \geq n_x\},\\
N_3 &= \{x : \exists n_x \text{ s.t. } x_n x_n^{[1]} < 0,\ x_n x_n^{[2]} < 0\ \forall n \geq n_x\},
\end{aligned} \tag{1.6}$$

and similarly solutions u of (E^A) can be classified to the same classes, whereby quasi-differences $u^{[i]}$, $i = 1,2$, are defined by (1.5), see [4, 3]. Solutions of (E) from the class N_0 are called *Kneser solutions* and solutions of (E^A) which belong to the class N_2 are called *strongly monotone solutions*.

2. Relationship between (E) and (E^A)

Solutions of (E) and (E^A) are related by the following properties.

Theorem 2.1. (a) *Let x, y be solutions of (E). Then the sequence $C = (C_n)$ $(n \geq 2)$ such that*

$$C_{n-1} = C(x_{n-1}, y_{n-1}) \equiv \begin{vmatrix} x_{n-1} & y_{n-1} \\ x^{[1]}_{n-1} & y^{[1]}_{n-1} \end{vmatrix} \tag{2.1}$$

is a solution of (E^A).

(b) *Let u, v be solutions of (E^A). Then the sequence $D = (D_n)$ $(n \geq 2)$ such that*

$$D_n = D(u_{n-1}, v_{n-1}) \equiv \begin{vmatrix} u_{n-1} & v_{n-1} \\ u^{[1]}_{n-1} & v^{[1]}_{n-1} \end{vmatrix} \tag{2.2}$$

is a solution of (E).

Proof. Claim (a). For any two solutions x, y of (E), we have

$$x_n \Delta y^{[2]}_{n-1} - y_n \Delta x^{[2]}_{n-1} = -x_n q_{n-1} y_n + y_n q_{n-1} x_n = 0. \tag{2.3}$$

Therefore,

$$\begin{aligned} \Delta C_{n-1} &= x_n \Delta y^{[1]}_{n-1} + y^{[1]}_{n-1} \Delta x_{n-1} - y_n \Delta x^{[1]}_{n-1} - x^{[1]}_{n-1} \Delta y_{n-1} \\ &= x_n \Delta y^{[1]}_{n-1} + r_{n-1} \Delta y_{n-1} \Delta x_{n-1} - y_n \Delta x^{[1]}_{n-1} - r_{n-1} \Delta x_{n-1} \Delta y_{n-1} \\ &= x_n \Delta y^{[1]}_{n-1} - y_n \Delta x^{[1]}_{n-1}. \end{aligned} \tag{2.4}$$

Using the fact $x^{[2]}_{n-1} = x^{[2]}_n - \Delta x^{[2]}_{n-1}$ and (2.3), we obtain

$$\begin{aligned} C^{[1]}_{n-1} &= p_{n-1} \Delta C_{n-1} = x_n y^{[2]}_{n-1} - y_n x^{[2]}_{n-1} \\ &= x_n \left(y^{[2]}_n - \Delta y^{[2]}_{n-1} \right) - y_n \left(x^{[2]}_n - \Delta x^{[2]}_{n-1} \right) = x_n y^{[2]}_n - y_n x^{[2]}_n. \end{aligned} \tag{2.5}$$

By a direct computation in view of (2.3), we get

$$\Delta C^{[1]}_{n-1} = x_{n+1} \Delta y^{[2]}_n + y^{[2]}_n \Delta x_n - y_{n+1} \Delta x^{[2]}_n - x^{[2]}_n \Delta y_n = y^{[2]}_n \Delta x_n - x^{[2]}_n \Delta y_n, \tag{2.6}$$

hence

$$C^{[2]}_{n-1} = r_n \Delta C^{[1]}_{n-1} = x^{[1]}_n y^{[2]}_n - y^{[1]}_n x^{[2]}_n. \tag{2.7}$$

Finally

$$\begin{aligned}\Delta C_{n-1}^{[2]} &= x_{n+1}^{[1]}\Delta y_n^{[2]} + y_n^{[2]}\Delta x_n^{[1]} - y_{n+1}^{[1]}\Delta x_n^{[2]} - x_n^{[2]}\Delta y_n^{[1]} \\ &= -x_{n+1}^{[1]}q_n y_{n+1} + p_n\Delta y_n^{[1]}\Delta x_n^{[1]} + y_{n+1}^{[1]}q_n x_{n+1} - p_n\Delta x_n^{[1]}\Delta y_n^{[1]} \\ &= q_n\left(x_{n+1}y_{n+1}^{[1]} - y_{n+1}x_{n+1}^{[1]}\right) = q_n C_{n+1},\end{aligned} \tag{2.8}$$

that is, C_{n-1} is a solution of (E^A).

Claim (b). By the similar argument as in (a), we get

$$\Delta D_n = u_n\Delta v_{n-1}^{[1]} - v_n\Delta u_{n-1}^{[1]}. \tag{2.9}$$

Using the fact $u_{n-1}^{[2]} = u_{n-2}^{[2]} + \Delta u_{n-2}^{[2]}$, we obtain

$$\begin{aligned}D_n^{[1]} &= r_n\Delta D_n = u_n v_{n-1}^{[2]} - v_n u_{n-1}^{[2]} \\ &= u_n\left(v_{n-2}^{[2]} + \Delta v_{n-2}^{[2]}\right) - v_n\left(u_{n-2}^{[2]} + \Delta u_{n-2}^{[2]}\right) \\ &= u_n\left(v_{n-2}^{[2]} + q_{n-1}v_n\right) - v_n\left(u_{n-2}^{[2]} + q_{n-1}u_n\right) \\ &= u_n v_{n-2}^{[2]} - v_n u_{n-2}^{[2]}.\end{aligned} \tag{2.10}$$

Using the same argument as before, we get

$$\begin{aligned}\Delta D_n^{[1]} &= v_{n-1}^{[2]}\Delta u_n + u_n\Delta v_{n-2}^{[2]} - u_{n-1}^{[2]}\Delta v_n - v_n\Delta u_{n-2}^{[2]} \\ &= v_{n-1}^{[2]}\Delta u_n - u_{n-1}^{[2]}\Delta v_n.\end{aligned} \tag{2.11}$$

Hence

$$D_n^{[2]} = p_n\Delta D_n^{[1]} = u_n^{[1]}v_{n-1}^{[2]} - v_n^{[1]}u_{n-1}^{[2]}. \tag{2.12}$$

Finally

$$\begin{aligned}\Delta D_n^{[2]} &= u_n^{[1]}\Delta v_{n-1}^{[2]} + v_n^{[2]}\Delta u_n^{[1]} - v_n^{[1]}\Delta u_{n-1}^{[2]} - u_n^{[2]}\Delta v_n^{[1]} \\ &= u_n^{[1]}q_n v_{n+1} + r_{n+1}\Delta v_n^{[1]}\Delta u_n^{[1]} - v_n^{[1]}q_n u_{n+1} - r_{n+1}\Delta u_n^{[1]}\Delta v_n^{[1]} \\ &= -q_n\left[v_n^{[1]}(\Delta u_n + u_n) - u_n^{[1]}(\Delta v_n + v_n)\right] \\ &= -q_n\left(p_n\Delta v_n\Delta u_n + u_n v_n^{[1]} - p_n\Delta u_n\Delta v_n - v_n u_n^{[1]}\right) = -q_n D_{n+1},\end{aligned} \tag{2.13}$$

that is, D_n is a solution of (E). □

Relationship between solutions of (E) and (E^A) described in Theorem 2.1 is a discrete analogue of the relationship valid for the differential (1.2) and its adjoint (1.3). For this reason, we call (E^A) the *adjoint equation* to (E). This is in accordance with the definition of the adjoint system to the difference system as the following remark shows.

Remark 2.2. According to [1, page 60], if $X = \{X_n\}$ is a nontrivial solution of the system

$$X_{n+1} = A_n X_n, \tag{2.14}$$

then $U = \{U_n\}$, where $U_n = (X_n^T)^{-1}$ is a solution of the system

$$U_n = A_n^T U_{n+1}. \tag{2.15}$$

System (2.15) is called the *adjoint system* of (2.14).

Equation (E) can be written as a first-order difference system

$$\begin{aligned} \Delta x_n^{[0]} &= \frac{1}{r_n} x_n^{[1]}, \\ \Delta x_n^{[1]} &= \frac{1}{p_n} x_n^{[2]}, \\ \Delta x_n^{[2]} &= -q_n x_{n+1}^{[0]}, \end{aligned} \tag{2.16}$$

for the vector $X_n = (x_n^{[0]}, x_n^{[1]}, x_n^{[2]})$. Since $x_{n+1}^{[0]} = x_n^{[0]} + \Delta x_n^{[0]}$, we have

$$\Delta x_n^{[2]} = -q_n \left(x_n^{[0]} + \frac{1}{r_n} x_n^{[1]} \right). \tag{2.17}$$

Using the usual convention that no index actually means the index n, otherwise the index is explicitly specified, we obtain

$$\begin{pmatrix} l x_{n+1}^{[0]} \\ x_{n+1}^{[1]} \\ x_{n+1}^{[2]} \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{r} & 0 \\ 0 & 1 & \frac{1}{p} \\ -q & -\frac{q}{r} & 1 \end{pmatrix} \begin{pmatrix} l x^{[0]} \\ x^{[1]} \\ x^{[2]} \end{pmatrix}. \tag{2.18}$$

Hence (E) can be interpreted as the system of the form (2.14). Its adjoint system is

$$\begin{pmatrix} l u^{[0]} \\ u^{[1]} \\ u^{[2]} \end{pmatrix} = \begin{pmatrix} 1 & 0 & -q \\ \frac{1}{r} & 1 & -\frac{q}{r} \\ 0 & \frac{1}{p} & 1 \end{pmatrix} \begin{pmatrix} l u_{n+1}^{[0]} \\ u_{n+1}^{[1]} \\ u_{n+1}^{[2]} \end{pmatrix}. \tag{2.19}$$

From here we get

$$\begin{aligned} \Delta u_n^{[0]} &= q_n u_{n+1}^{[2]}, \\ \Delta u_n^{[1]} &= -\frac{1}{r_n} u_{n+1}^{[0]} + \frac{q_n}{r_n} u_{n+1}^{[2]}, \\ \Delta u_n^{[2]} &= -\frac{1}{p_n} u_{n+1}^{[1]}, \end{aligned} \tag{2.20}$$

and the last equation gives $\Delta u_{n+1}^{[1]} = -\Delta(p_n \Delta u_n^{[2]})$. Replacing the shift n by $n+1$ and substituting into the second equation, we have

$$-\Delta\left(p_n \Delta u_n^{[2]}\right) = -\frac{1}{r_{n+1}} u_{n+2}^{[0]} + \frac{q_{n+1}}{r_{n+1}} u_{n+2}^{[2]}. \tag{2.21}$$

Multiplying this equation by $-r_{n+1}$ and differentiating it, we obtain

$$\Delta\left(r_{n+1}\Delta\left(p_n \Delta u_n^{[2]}\right)\right) + \Delta\left(q_{n+1} u_{n+2}^{[2]}\right) = \Delta u_{n+2}^{[0]}. \tag{2.22}$$

Substituting from the first equation in (2.20), we get

$$\Delta\left(r_{n+1}\Delta\left(p_n \Delta u_n^{[2]}\right)\right) + q_{n+2} u_{n+3}^{[2]} - q_{n+1} u_{n+2}^{[2]} = q_{n+2} u_{n+3}^{[2]}, \tag{2.23}$$

which means that the sequence $v_n = u_n^{[2]}$ satisfies (E^A).

Notation 2.3. Let S denote the solution space of (E) and let $S^\star$ denote the solution space of (E^A). For $(x,u) \in S \times S^\star$, define $\mathcal{L} = (\mathcal{L}_n)$, where

$$\mathcal{L}_n = \mathcal{L}(x_n, u_n) = x_{n+1} u_n^{[2]} - x_{n+1}^{[1]} u_n^{[1]} + x_{n+1}^{[2]} u_{n+1}. \tag{2.24}$$

The functional $\mathcal{L}$ has the following properties.

Lemma 2.4. *The sequence $\mathcal{L} : S \times S^\star \to \mathbb{R}$ is a constant which depends only on the choice of solutions x and u, and not on n.*

Proof. By a direct computation we get

$$\begin{aligned}
\Delta\mathcal{L}_n &= \Delta\left(x_{n+1} u_n^{[2]} - x_{n+1}^{[1]} u_n^{[1]} + x_{n+1}^{[2]} u_{n+1}\right) \\
&= x_{n+2}\Delta u_n^{[2]} + u_n^{[2]}\Delta x_{n+1} - x_{n+1}^{[1]}\Delta u_n^{[1]} - u_{n+1}^{[1]}\Delta x_{n+1}^{[1]} \\
&\quad + u_{n+2}\Delta x_{n+1}^{[2]} + x_{n+1}^{[2]}\Delta u_{n+1} \\
&= x_{n+2} q_{n+1} u_{n+2} + r_{n+1}\Delta u_n^{[1]}\Delta x_{n+1} - r_{n+1}\Delta x_{n+1}\Delta u_n^{[1]} \\
&\quad - p_{n+1}\Delta u_{n+1}\Delta x_{n+1}^{[1]} - u_{n+2} q_{n+1} x_{n+2} + p_{n+1}\Delta x_{n+1}^{[1]}\Delta u_{n+1} = 0,
\end{aligned} \tag{2.25}$$

which completes the proof. □

Lemma 2.5. *Let x, y, z be solutions of (E). Let C and $\mathcal{L}$ be defined by (2.1) and (2.24), respectively. Then the sequence $R = (R_n)$, where*

$$R_n = \begin{vmatrix} x_n & y_n & z_n \\ x_n^{[1]} & y_n^{[1]} & z_n^{[1]} \\ x_n^{[2]} & y_n^{[2]} & z_n^{[2]} \end{vmatrix} \tag{2.26}$$

satisfies

$$R_n = \mathcal{L}(z_{n-1}, C_{n-1}). \tag{2.27}$$

Proof. Expanding R_n along its third column, we obtain

$$R_n = z_n \begin{vmatrix} x_n^{[1]} & y_n^{[1]} \\ x_n^{[2]} & y_n^{[2]} \end{vmatrix} - z_n^{[1]} \begin{vmatrix} x_n & y_n \\ x_n^{[2]} & y_n^{[2]} \end{vmatrix} + z_n^{[2]} \begin{vmatrix} x_n & y_n \\ x_n^{[1]} & y_n^{[1]} \end{vmatrix}. \tag{2.28}$$

Using (2.24), we have

$$\mathcal{L}(z_{n-1}, C_{n-1}) = z_n C_{n-1}^{[2]} - z_n^{[1]} C_{n-1}^{[1]} + z_n^{[2]} C_n. \tag{2.29}$$

From here, (2.1), (2.5), and (2.7) show that (2.27) holds. □

3. Nonoscillatory solutions of adjoint equations

In this section, we study nonoscillatory solutions. We start with the following auxiliary results.

Lemma 3.1. *There always exists nonoscillatory solution u of (E^A) with the property*

$$u_n > 0, \quad u_n^{[1]} > 0, \quad u_n^{[2]} > 0 \quad \text{for } n \in \mathbb{N}, \tag{3.1}$$

that is, (E^A) has a strongly monotone solution.

For the proof, see [4, Theorem 3.2].

Lemma 3.2. *If a solution y of (E) satisfies for some integer $m > 1$ that*

$$y_m \geq 0, \qquad y_m^{[1]} \leq 0, \qquad y_m^{[2]} > 0, \tag{3.2}$$

then

$$y_k > 0, \qquad y_k^{[1]} < 0, \qquad y_k^{[2]} > 0 \tag{3.3}$$

for each $k \in \mathbb{N}$ such that $1 \leq k < m$.

The proof follows from the proof of [3, Proposition 2].

The existence of Kneser solutions of (E) is ensured by the following result.

Theorem 3.3. *There always exists nonoscillatory solution x of (E) with the property*

$$x_n > 0, \quad x_n^{[1]} < 0, \quad x_n^{[2]} > 0 \quad \text{for } n \in \mathbb{N}, \tag{3.4}$$

that is, (E) has a Kneser solution.

Proof. Let $x = (x(n))$, $y = (y(n))$, $z = (z(n))$ be a basis of the solution space S of (E). For $k \in \mathbb{N}$, define

$$\omega_k(n) = a_k x(n) + b_k y(n) + c_k z(n), \tag{3.5}$$

where a_k, b_k, c_k are chosen such that

$$\omega_k(k) = 0, \qquad \omega_k(k+1) = 0, \qquad a_k^2 + b_k^2 + c_k^2 = 1. \tag{3.6}$$

Then $\omega_k^{[1]}(k) = 0$. By [3, Lemma 1], $\omega_k(k+2) \neq 0$. Without loss of generality, assume that $\omega_k(k+2) > 0$. Then

$$\omega_k^{[1]}(k+1) = r_{k+1}\Delta\omega_k(k+1) = r_{k+1}\left(\omega_k(k+2) - \omega_k(k+1)\right) > 0, \tag{3.7}$$

hence

$$\omega_k^{[2]}(k) = p_k\Delta\omega_k^{[1]}(k) = p_k\left(\omega_k^{[1]}(k+1) - \omega_k^{[1]}(k)\right) > 0. \tag{3.8}$$

Since

$$\omega_k(k) = 0, \qquad \omega_k^{[1]}(k) = 0, \qquad \omega_k^{[2]}(k) > 0, \tag{3.9}$$

by Lemma 3.2

$$\omega_k(n) > 0, \quad \omega_k^{[1]}(n) < 0, \quad \omega_k^{[2]}(n) > 0, \quad \text{for } 1 \leq n < k. \tag{3.10}$$

Put $A_k = (a_k, b_k, c_k)$. Then $\|A_k\| = 1$ for each k. The unit ball is compact in $\mathbb{R}^3$, so (A_k) has a convergent subsequence (A_{k_i}). Denote

$$A = \lim_{i\to\infty} A_{k_i} = (a,b,c). \tag{3.11}$$

Then $a^2+b^2+c^2 = 1$ and

$$\omega(n) = \lim_{i\to\infty} \omega_{k_i} = \lim\left(a_{k_i}x(n) + b_{k_i}y(n) + c_{k_i}z(n)\right) \tag{3.12}$$

is a nontrivial solution of (E). Then in view of (3.10) and the fact that k is arbitrary integer, we get

$$\omega(n) \geq 0, \quad \omega^{[1]}(n) \leq 0, \quad \omega^{[2]}(n) \geq 0 \quad \text{for } n \geq 1. \tag{3.13}$$

If $\omega(n_0) = 0$ for some $n = n_0$, then $\omega(n) = 0$ for all $n \geq n_0$ which is a contradiction with the fact that ω is a nontrivial solution. Thus $\omega(n) > 0$ for every $n \geq 1$, and so

$$\Delta\left(\omega^{[2]}(n)\right) = -q(n)\omega(n+1) < 0 \quad \text{for } n \geq 1. \tag{3.14}$$

Hence, $\omega^{[2]}$ is decreasing and so $\omega^{[2]}(n) > 0$ for $n \in \mathbb{N}$. From here $\Delta(\omega^{[1]}(n)) > 0$ for $n \in \mathbb{N}$, which implies that $\omega^{[1]}$ is increasing and $\omega^{[1]}(n) < 0$ for $n \in \mathbb{N}$. □

Theorem 3.4. *Every nonoscillatory solution of* (E^A) *is strongly monotone if and only if every nonoscillatory solution of (E) is a Kneser solution.*

Proof. Let every nonoscillatory solution of (E^A) be strongly monotone. Assume by contradiction that there exists solution y of (E) which belongs to the class N_i, where $i \in \{1,2,3\}$. Let x be a Kneser solution of (E). Without loss of generality, we may suppose that $x_n > 0$ and $y_n > 0$ for large n. Then the sequence C defined by (2.1) is according to

Theorem 2.1 solution of (E^A) and in view of (2.5) and (2.7) it satisfies, for large n,

$$\begin{aligned} C_{n-1} &> 0, & C_{n-1}^{[1]} &< 0 \quad (\text{if } i = 1) \\ C_{n-1} &> 0, & C_{n-1}^{[2]} &< 0 \quad (\text{if } i = 2) \\ C_{n-1}^{[1]} &< 0, & C_{n-1}^{[2]} &> 0 \quad (\text{if } i = 3). \end{aligned} \tag{3.15}$$

This is a contradiction with the fact that C is strongly monotone solution.

Now suppose that every solution of (E) is a Kneser solution. Assume by contradiction that there exists solution v of (E^A) which belongs to the class N_i, where $i \in \{0,1,3\}$. Let u be a strongly monotone solution of (E^A). Without loss of generality, we may suppose that $u_n > 0$ and $v_n > 0$ for large n. Then the sequence D defined by (2.2) is according to Theorem 2.1 solution of (E) and it satisfies, for large n,

$$\begin{aligned} D_n &< 0, & D_n^{[2]} &> 0 \quad (\text{if } i = 0) \\ D_n^{[1]} &< 0, & D_n^{[2]} &< 0 \quad (\text{if } i = 1) \\ D_n &< 0, & D_n^{[1]} &< 0 \quad (\text{if } i = 3). \end{aligned} \tag{3.16}$$

This is a contradiction with the fact that D is a Kneser solution. □

4. Oscillatory properties of adjoint equations

Lemma 4.1. *Let u be a strongly monotone solution and v an oscillatory solution of (E^A). Then their Casoratian D defined by (2.2) is an oscillatory solution of (E).*

Proof. By Theorem 2.1, D is a solution of (E). We will show that D is an oscillatory solution. Without loss of generality, we may suppose that u satisfies (3.1). Since v is an oscillatory solution, there exist increasing sequences of positive integers (i_n) and (j_n), with properties

$$\begin{aligned} v_{i_n} &\leq 0, & v_{i_n}^{[1]} &> 0 \quad \text{for } n \in \mathbb{N}, \\ v_{j_n} &\geq 0, & v_{j_n}^{[1]} &< 0 \quad \text{for } n \in \mathbb{N}. \end{aligned} \tag{4.1}$$

From the above inequalities, (2.2), and (3.1), we have

$$D_{i_n+1} = u_{i_n} v_{i_n}^{[1]} - v_{i_n} u_{i_n}^{[1]} > 0 \quad \text{for } n \in \mathbb{N}, \tag{4.2}$$

and similarly $D_{j_n+1} < 0$ for $n \in \mathbb{N}$. Hence the sequence D is an oscillatory solution of (E). □

Lemma 4.2. *Let x be a Kneser solution and y an oscillatory solution of (E). Then their Casoratian C defined by (2.1) is an oscillatory solution of (E^A).*

Proof. By Theorem 2.1, C is a solution of (E^A). We will show that C is an oscillatory solution. Without loss of generality, we may suppose that x satisfies (3.4). Because y is an oscillatory solution, there exist increasing sequences of positive integers $(i_n)_1^\infty$ and $(j_n)_1^\infty$

with properties

$$\begin{aligned} &i_1 > M, \qquad y_{i_n} \le 0, \quad y_{i_n}^{[1]} > 0 \quad \text{for } n \in \mathbb{N}, \\ &j_1 > M, \qquad y_{j_n} \ge 0, \quad y_{j_n}^{[1]} < 0 \quad \text{for } n \in \mathbb{N}, \end{aligned} \tag{4.3}$$

where $M = \min\{n \in \mathbb{N} : y_n y_{n+1} \le 0\}$.

Assume that $y_{j_n}^{[2]} > 0$ for some $n \in \mathbb{N}$. Then by Lemma 3.2, we get

$$y_k > 0, \quad y_k^{[1]} < 0 \quad \text{for } 1 \le k < j_n, \tag{4.4}$$

which is a contradiction with $j_1 > M$. Hence $y_{j_n}^{[2]} \le 0$ for $n \in \mathbb{N}$. From here and using (2.7) follows

$$C_{j_n-1}^{[2]} = x_{j_n}^{[1]} y_{j_n}^{[2]} - y_{j_n}^{[1]} x_{j_n}^{[2]} > 0 \quad \text{for } n \in \mathbb{N}. \tag{4.5}$$

By similar argument as before, we obtain $y_{i_n}^{[2]} \ge 0$ for $n \in \mathbb{N}$, which implies that

$$C_{i_n-1}^{[2]} = x_{i_n}^{[1]} y_{i_n}^{[2]} - y_{i_n}^{[1]} x_{i_n}^{[2]} < 0 \quad \text{for } n \in \mathbb{N}. \tag{4.6}$$

By [3, Lemma 2], it follows from inequalities (4.5) and (4.6) that C is an oscillatory solution of (E^A). The proof is now complete. □

Our next result characterizes the existence of oscillatory solutions of the adjoint equations.

Theorem 4.3. *Equation (E^A) is oscillatory if and only if (E) is oscillatory.*

The proof follows from Theorem 3.3 and Lemmas 3.1, 4.1, and 4.2.

In the sequel, we study the existence of an oscillatory solution in terms of second-order equations.

Theorem 4.4. (a) *If u is a nonoscillatory solution of (E^A), then two linearly independent solutions of (E) satisfy the second-order difference equation*

$$p_{n+1}\Delta\left(\frac{r_{n+1}\Delta x_{n+1}}{u_{n+1}}\right) + \frac{u_n^{[2]}}{u_{n+1}u_{n+2}} x_{n+2} = 0. \tag{4.7}$$

(b) *If x is a nonoscillatory solution of (E), then two linearly independent solutions of (E^A) satisfy the second-order difference equation*

$$r_{n+1}\Delta\left(\frac{p_n\Delta u_n}{x_{n+1}}\right) + \frac{x_{n+1}^{[2]}}{x_{n+1}x_{n+2}} u_{n+1} = 0. \tag{4.8}$$

Proof. Claim (a). Let u be a fixed nonoscillatory solution of (E^A) such that $u_n > 0$ for $n \ge N$. Let $L : S \to \mathbb{R}$ be the functional on S defined by $L(x) = \mathcal{L}(x_n, u_n)$. The set

$$K = \{x \in S : L(x) = 0\} \tag{4.9}$$

is the kernel of linear functional L defined on S. Then $x \in K$ satisfies

$$u_{n+1}x_{n+1}^{[2]} - u_n^{[1]}x_{n+1}^{[1]} + u_n^{[2]}x_{n+1} = 0. \tag{4.10}$$

Multiplying the last equation by $(u_{n+1}u_{n+2})^{-1}$, we get

$$\frac{u_{n+1}x_{n+1}^{[2]} - u_{n+1}^{[1]}x_{n+1}^{[1]}}{u_{n+1}u_{n+2}} + \frac{u_{n+1}^{[1]}x_{n+1}^{[1]} - u_n^{[1]}x_{n+1}^{[1]}}{u_{n+1}u_{n+2}} + \frac{u_n^{[2]}x_{n+1}}{u_{n+1}u_{n+2}} = 0. \tag{4.11}$$

From here and using

$$\begin{aligned} p_{n+1}\Delta\left(\frac{x_{n+1}^{[1]}}{u_{n+1}}\right) &= \frac{u_{n+1}p_{n+1}\Delta x_{n+1}^{[1]} - x_{n+1}^{[1]}p_{n+1}\Delta u_{n+1}}{u_{n+1}u_{n+2}} \\ &= \frac{u_{n+1}x_{n+1}^{[2]} - u_{n+1}^{[1]}x_{n+1}^{[1]}}{u_{n+1}u_{n+2}}, \end{aligned} \tag{4.12}$$

we obtain

$$p_{n+1}\Delta\left(\frac{x_{n+1}^{[1]}}{u_{n+1}}\right) + \frac{x_{n+1}^{[1]}\Delta u_n^{[1]}}{u_{n+1}u_{n+2}} + \frac{u_n^{[2]}x_{n+1}}{u_{n+1}u_{n+2}} = 0. \tag{4.13}$$

In view of the identity

$$x_{n+1}^{[1]}\Delta u_n^{[1]} + u_n^{[2]}x_{n+1} = u_n^{[2]}\Delta x_{n+1} + u_n^{[2]}x_{n+1} = u_n^{[2]}x_{n+2}, \tag{4.14}$$

(4.13) can be rewritten in the form (4.7). Since $\dim K = \dim S - 1 = 2$, we get the conclusion.

Claim (b). Let x be a fixed nonoscillatory solution of (E) such that $x_n > 0$ for $n \geq N$. Let $L^\star : S^\star \to \mathbb{R}$ be the functional on S defined by $L^\star(u) = \mathscr{L}(x_n, u_n)$. The set

$$K^\star = \{u \in S^\star : L^\star(u) = 0\} \tag{4.15}$$

is the kernel of linear functional defined on $S^\star$. Then $u \in K^\star$ satisfies

$$x_{n+1}u_n^{[2]} - x_{n+1}^{[1]}u_n^{[1]} + x_{n+1}^{[2]}u_{n+1} = 0. \tag{4.16}$$

Multiplying the last equation by $(x_{n+1}x_{n+2})^{-1}$, we get

$$\frac{x_{n+1}u_n^{[2]} - x_{n+1}^{[1]}u_n^{[1]}}{x_{n+1}x_{n+2}} + \frac{x_{n+1}^{[2]}u_{n+1}}{x_{n+1}x_{n+2}} = 0. \tag{4.17}$$

From here using

$$\begin{aligned} r_{n+1}\Delta\left(\frac{u_n^{[1]}}{x_{n+1}}\right) &= \frac{x_{n+1}r_{n+1}\Delta u_n^{[1]} - u_n^{[1]}r_{n+1}\Delta x_{n+1}}{x_{n+1}x_{n+2}} \\ &= \frac{x_{n+1}u_n^{[2]} - x_{n+1}^{[1]}u_n^{[1]}}{x_{n+1}x_{n+2}}, \end{aligned} \tag{4.18}$$

we get (4.8). Since $\dim K^\star = \dim S^\star - 1 = 2$, we get the conclusion. □

Corollary 4.5. (a) *If (E) is oscillatory, then there exists a basis for S consisting of one nonoscillatory solution and two oscillatory solutions.*

(b) *If* (E^A) *is oscillatory, then there exists a basis for* $S^\star$ *consisting of one nonoscillatory solution and two oscillatory solutions.*

Proof. By Theorem 3.3 and Lemma 3.1, there exist Kneser solution of (E) and strongly monotone solution of (E^A). Assume that (E) and (E^A) are oscillatory. In view of Theorem 4.4 and its proof, (E) has two independent solutions which must be oscillatory. Similarly, there exist two independent oscillatory solutions of (E^A). Because $\dim S = \dim S^\star = 3$, the proof is complete. □

Theorem 4.6. (a) *If (4.7), where u is a nonoscillatory solution of* (E^A)*, is oscillatory, then (E) and* (E^A) *are oscillatory.*

(b) *If (4.8), where x is a nonoscillatory solution of (E), is oscillatory, then (E) and* (E^A) *are oscillatory.*

Proof. Assume that (4.7) is oscillatory, that is, there exists an oscillatory solution x of (4.7). Using the same argument as in the proof of Theorem 4.4, we obtain that $x \in K$, where K is defined by (4.9). Hence x is an oscillatory solution of (E). By Theorem 4.3, (E^A) is oscillatory, too.

Similarly, if u is an oscillatory solution of (4.8), then $u \in K^\star$. Hence u is oscillatory solution of (E^A) and this implies that (E) is oscillatory too. □

Theorem 4.7. *Equation (E) is oscillatory if and only if (4.8) is oscillatory.*

Proof. Assume that (E) is oscillatory, that is, there exists an oscillatory solution y of (E). By Theorem 3.3, there exists a Kneser solution x of (E). According to (2.27),

$$0 = \begin{vmatrix} x_n & y_n & x_n \\ x_n^{[1]} & y_n^{[1]} & x_n^{[1]} \\ x_n^{[2]} & y_n^{[2]} & x_n^{[2]} \end{vmatrix} = \mathcal{L}(x_{n-1}, C_{n-1}), \tag{4.19}$$

where C is defined by (2.1). By Theorem 2.1, C is solution of (E^A). From Lemma 4.2 follows that C is an oscillatory solution of (E^A). In view of Lemma 2.4,

$$L^\star(C) = \mathcal{L}(x_n, C_n) = 0, \tag{4.20}$$

hence $C \in K^\star$, where $K^\star$ is defined by (4.15). From here and the proof of Theorem 4.4, we get the fact that C is an oscillatory solution of (4.8). The opposite statement follows from Theorem 4.6. □

Open problems.

(1) It is an open problem whether the existence of an oscillatory solution of (E^A) implies the oscillation of (4.7). To solve this problem, it would be useful to find the functional $\mathcal{L}^\star$ defined on $S^\star \times S$ with similar properties to those of $\mathcal{L}$ described in Lemmas 2.4 and 2.5.

(2) To generalize results of this paper to the linear nth-order difference equations involving quasi-differences.

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