ON STABILITY ZONES FOR DISCRETE-TIME PERIODIC LINEAR HAMILTONIAN SYSTEMS

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The main purpose of the paper is to give discrete-time counterpart for some strong (robust) stability results concerning periodic linear Hamiltonian systems. In the continuous-time version, these results go back to Liapunov and Žukovskii; their deep generalizations are due to Kreĭn, Gel'fand, and Jakubovič and obtaining the discrete version is not an easy task since not all results migrate *mutatis-mutandis* from continuous time to discrete time, that is, from ordinary differential to difference equations. Throughout the paper, the theory of the stability zones is performed for scalar (2nd-order) canonical systems. Using the characteristic function, the study of the stability zones is made in connection with the characteristic numbers of the periodic and skew-periodic boundary value problems for the canonical system. The multiplier motion ("traffic") on the unit circle of the complex plane is analyzed and, in the same context, the Liapunov estimate for the central zone is given in the discrete-time case.

1. Introduction, motivation, and problem statement

(A) Stability analysis of linear Hamiltonian systems with periodic coefficients goes back to Liapunov [21] and Žukovskii [27]. If the simplest case of the second-order scalar equation is considered

$$y'' + \lambda^2 p(t) y = 0, \tag{1.1}$$

where $p(t)$ is T-periodic, then we call λ_0 a λ-*point of stability* of (1.1) if for $\lambda = \lambda_0$ all solutions of (1.1) are bounded on $\mathbb{R}$. If moreover all solutions of any equation of (1.1) type but with $p(t)$ replaced by $p_1(t)$ sufficiently close to $p(t)$ (in some sense) are also bounded for $\lambda = \lambda_0$, then λ_0 is called a λ-*point of strong (robust) stability*.

Remark that we might take $p_1(t) = \lambda p(t)$ with $\lambda \neq \lambda_0$. In this case it was established by Liapunov himself [21] that the set of the λ-points of strong stability of (1.1) is open and if it is nonempty, it decomposes into a system of disjoint open intervals called λ-*zones of strong stability*.

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Equation (1.1) belongs to the more general class of linear periodic Hamiltonian systems described by

$$\dot{x} = \lambda J H(t) x, \tag{1.2}$$

with $H(t)$ a T-periodic Hermitian $2m \times 2m$ matrix and

$$J = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}. \tag{1.3}$$

For this system, the results of Liapunov have been generalized by Kreĭn [19], Gel'fand and Lidskiĭ [11], Yakubovich, and many others; the final part of this long line of research was the book of Yakubovich and Staržinskii [26]. As pointed out by Kreĭn and Jakubovič [20], this research is motivated by various problems in contemporary physics and engineering (e.g., dynamic stability of structures, parametric resonance both in mechanical and electrical engineering, quantum-mechanical treatment of the motion of the electron in a periodic field—see the book of Eastham [5]—and others).

(B) The discrete-time Hamiltonian systems represent, from several points of view, a more recent field of research, emerging from various sources. If, for instance, in the book of Kratz [17] the first paper on discrete-time Hamiltonian systems is considered to be that of Hartman [16] (because it deals with disconjugacy, principal solutions, etc., which are directly connected with book's topics), such systems are known earlier with particular reference to linear quadratic optimization problems: we may cite here the genuine pioneering paper of Halanay [12] and the book of Tou [25]—a reference book that used to be very popular among engineers of that time. Linear periodic discrete-time Hamiltonian systems are met in the existence problem for forced oscillations (periodic and almost periodic) in discrete-time periodic systems with sector restricted nonlinearities (see the paper of Halanay and Răsvan [15]). A good reference on discrete-time Hamiltonian systems in optimization and control is the book of Halanay and Ionescu [13]. As we already mentioned, another line of research in the field is that represented by disconjugacy, oscillation, and associated boundary value problems. A good reference is the book of Ahlbrandt and Peterson [1], the papers of Erbe and Yan [7–10] and the long list of papers by Bohner et al. among which we cite the more recent ones [2–4].

It is worth mentioning that disconjugacy is a basic property of the Hamiltonian systems both in the case of linear quadratic optimization and in the studies of Erbe and Yan, Bohner, Došlý, Kratz a.s.o. This shows the "calculus of variations flavor" of all this line of research.

(C) When such problems as stability and oscillations for systems with sector restricted nonlinearities or linear quadratic stabilization are considered, the associated linear discrete-time periodic Hamiltonian systems have to be not only (strongly) disconjugate but also totally unstable (exponentially dichotomic, i.e., of hyperbolic type). This last property is robust with respect to structural perturbations of the Hamiltonian. On the contrary, the total stability discussed earlier is not robust—generally speaking—but, as already mentioned, it is preserved against such perturbations that do not affect the Hamiltonian structure; this is the strong stability introduced by Kreĭn (e.g., [19]).

The results that will be presented in this paper deal with strong stability (in the sense of Kreĭn) of discrete-time Hamiltonian systems. We will consider here the discretized (sampled) version of (1.2). *Since stability is, generally speaking, not preserved by sampling* (not always), considering strong stability for discrete-time Hamiltonian systems is not without interest. On the other hand, not all results of the continuous-time fields may migrate, *mutatis mutandis*, to the discrete-time field. In order to illustrate this last statement, consider the sampled version of (1.2) with

$$H(t)=\begin{pmatrix} A(t) & B^*(t) \\ B(t) & D(t) \end{pmatrix}, \tag{1.4}$$

that is,

$$y_{k+1}-y_k=\lambda\left(B_k y_k+D_k z_{k+1}\right), \tag{1.5}$$

$$z_{k+1}-z_k=-\lambda\left(A_k y_k+B_k^* z_{k+1}\right). \tag{1.6}$$

Here some details and comments are necessary. First of all, the above structure of $H(t)$ in the continuous-time case combined with the fact that $H(t)$ is Hermitian—see the explanation for system (1.2)—will imply $A(t)$ and $D(t)$ to be also Hermitian (symmetric if the entries of the matrices are real). Also the discretization is such that the periodicity and the Hamiltonian character migrate in the discrete-time case: this may be achieved if the discretization step is chosen as T/N, where T is the period in the continuous-time case and N is a (sufficiently large) positive integer; the Hamiltonian character is preserved by forward discretization in one equation and backward in the other. Consequently system (1.5) results as Hamiltonian—see [2–4, 17] and other texts where systems with such structure are defined as discrete-time Hamiltonian; in fact this follows from several of their properties which in the continuous-time case are known as characterizing Hamiltonian systems, an important one being the J-unitary character or symplecticity. Indeed, system (1.5) may be written also as follows:

$$x_{k+1}=C_k(\lambda)x_k, \tag{1.7}$$

where

$$x=\begin{pmatrix} y \\ z \end{pmatrix}, \qquad C_k(\lambda)=\begin{pmatrix} I & -\lambda D_k \\ 0 & I+\lambda B_k^* \end{pmatrix}^{-1}\begin{pmatrix} I+\lambda B_k & 0 \\ -\lambda Ak & I \end{pmatrix}, \tag{1.8}$$

for those λ for which $C_k(\lambda)$ exists, that is, the matrix

$$\begin{pmatrix} I & -\lambda D_k \\ 0 & I+\lambda B_k^* \end{pmatrix} \tag{1.9}$$

is invertible; this happens if the matrix $I+\lambda B_k^*$ is nonsingular, that is, for all $\lambda\in\mathbb{C}$ except those for which $\det(I+\lambda B_k^*)=0$: these are the symmetric with respect to the unit circle of the complex plane (in the sense of inversion) of the eigenvalues of $-B_k$. Indeed, if μ is an eigenvalue of $-B_k$, then

$$\det\left(\mu I+B_k\right)=0. \tag{1.10}$$

The symmetric of μ with respect to the unit circle is $\lambda = \overline{\mu}^{-1}$, where the bar denotes the complex conjugate, hence

$$\det\left(I + \lambda B_k^*\right) = \det\left(I + \overline{\mu}^{-1} B_k^*\right) = (\overline{\mu})^{-m} \overline{\det\left(\mu I + B_k\right)} = 0. \tag{1.11}$$

In this way, the solution of (1.7) can be constructed forward for both y_k and z_k, that is, the initial value (Cauchy) problem has a well-defined solution. Further, it is easily shown that $C_k^*(\lambda) J C_k(\lambda) = J$ for real λ, that is, $C_k(\lambda)$ is in this case J-unitary. If besides λ all matrices are real, we deduce that $C_k(\lambda)$ is symplectic. As pointed out in [2–4], in the discrete-time case Hamiltonian systems are a subset of the symplectic systems; if we refer to [26] where systems (1.5) with real coefficients are called canonical, we may say that *in the discrete-time case canonical systems are a subset of the symplectic systems and Hamiltonian systems (with complex coefficients) are a subset of the J-unitary systems*. On the contrary, symplectic (or J-unitary) and canonical (or Hamiltonian) systems coincide in the continuous-time case.

We will mention here also another argument for the assertion that not all results from the continuous-time case may migrate automatically to the discrete-time one.

The results on λ stability in the continuous-time case, more precisely the estimates of the central zone, strongly rely on the fact that only entire functions of λ are met (starting with the transition matrix and going on with the monodromy and the matrices in the boundary value problem). In the discrete-time case we may see from (1.5) that this is no longer true: in fact the assumption on invertibility of $I + \lambda B_k^*$ speaks for that. There are, nevertheless, notable exceptions. For instance, in [14] we considered the discretized version of

$$y'' + \lambda P(t) y = 0 \tag{1.12}$$

which leads to a system (1.5) with $B_k = 0$, $D_k = I$, $A_k = P_k$. Since $B_k = 0$, the above-mentioned assumption is automatically fulfilled. Moreover $C_k(\lambda)$ is a polynomial matrix function, hence it is of entire type.

Another case is suggested by [4]: starting from the Sturm-Liouville equations, the following symplectic system is considered:

$$x_{k+1} = \left(S_k - \lambda \widehat{S}_k\right) x_k, \tag{1.13}$$

where

$$S_k = \begin{pmatrix} A_k & B_k \\ C_k & D_k \end{pmatrix} \tag{1.14}$$

is symplectic and

$$\widehat{S}_k = \begin{pmatrix} 0 & 0 \\ W_k A_k & W_k B_k \end{pmatrix}, \quad W_k \geq 0. \tag{1.15}$$

The two cases cannot be reduced one to another because the structures of matrices are different. Nevertheless, if we want to obtain results on λ-stability for (1.13), the approach to be taken is exactly that of [14].

(D) With all these facts in mind, a research programme started, aiming to extend the results of Kreĭn type to the discrete case with the final outcome the migration of the Liapunov programme (announced or suggested in his early paper) to discrete-time systems. Besides the already cited reference of Halanay and Răsvan [14], we mention here [23, 24] where the line of Kreĭn [19, 18] is followed and attempts are made to adapt those techniques borrowed from the continuous-time field that cannot migrate *mutatis-mutandis* to the discrete-time one.

In this paper, we will perform a rather complete analysis of the real scalar discrete-time case and show how the obtained results are connected to Liapunov and Kreĭn programmes.

2. Stability zones for discrete-time 2nd-order canonical systems

We will consider here canonical systems of the form

$$y_{k+1} - y_k = \lambda(b_k y_k + d_k z_{k+1}), \tag{2.1}$$

$$z_{k+1} - z_k = -\lambda(a_k y_k + b_k z_{k+1}), \tag{2.2}$$

the scalar version of (1.5) with a_k, b_k, d_k being real and N-periodic. This canonical system is defined by

$$H_k = \begin{pmatrix} A_k & B_k^* \\ B_k & D_k \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \tag{2.3}$$

and may be written as (1.7) with

$$C_k(\lambda) = \begin{pmatrix} 1 & -\lambda d_k \\ 0 & 1+\lambda b_k \end{pmatrix}^{-1} \begin{pmatrix} 1+\lambda b_k & 0 \\ -\lambda a_k & 1 \end{pmatrix} = \frac{1}{1+\lambda b_k} \begin{pmatrix} (1+\lambda b_k)^2 - \lambda^2 d_k a_k & \lambda d_k \\ -\lambda a_k & 1 \end{pmatrix}. \tag{2.4}$$

Obviously this is a matrix with rational items, having a real pole at $\lambda = -1/b_k$. At the same time $\det C_k(\lambda) \equiv 1$, hence it is *an unimodular matrix*. As known, for periodic systems the structure and the stability properties are given by *system's multipliers—the eigenvalues of the monodromy matrix* $U_N(\lambda) = C_{N-1}(\lambda) \cdots C_1(\lambda) C_0(\lambda)$. As a product of rational unimodular matrices, $U_N(\lambda)$ is also *rational and unimodular (unlike the continuous-time case when it is an entire matrix function)*. It follows that the characteristic equation of $U_N(\lambda)$ in this case is

$$\varrho^2 - 2A(\lambda)\varrho + 1 = 0, \tag{2.5}$$

where $2A(\lambda) = \operatorname{tr}(U_N(\lambda))$—the trace of the unimodular monodromy matrix of (2.1); the function $A(\lambda)$ is called *characteristic function* of the canonical system. Its properties are essential for defining and computing the λ-zones. In the continuous-time case, $A(\lambda)$ is an entire function while in *the case of (2.1), it is a rational function* with its poles are the real numbers $-1/b_k$, $k = \overline{0, N-1}$ (these poles may not be distinct). In the following we will see, once more, that not all properties of $A(\lambda)$ in the continuous-time case are valid *mutatis mutandis* in the discrete-time case.

In the following, we will assume that (2.1) *is of positive type in the sense of Kreĭn* [19], that is, $H_k \geq 0$, $\forall k$, $\sum_0^{N-1} H_k > 0$. We will start with some basic properties of $A(\lambda)$.

Proposition 2.1. *All zeros of $A(\lambda) - \alpha$, where $|\alpha| \leq 1$, are real.*

The proof follows the line of [18, 26]. Let λ_* be some zero of the rational function $A(\lambda) - \alpha$ with $|\alpha| \leq 1$. We deduce that system's multipliers $\varrho_1(\lambda_*)$ and $\varrho_2(\lambda_*)$ are given by $\varrho_{1,2}(\lambda_*) = \alpha \pm \imath\sqrt{1-\alpha^2}$ and are located on the unit disk, that is, $|\varrho_i(\lambda_*)| = 1$, $i = 1,2$. Consider the boundary value problem for (1.7) defined by $x_N = \varrho_i(\lambda_*)x_0$. As known from the more general results of Kreĭn [19] for continuous-time systems and of [14, 23] for discrete-time systems, the characteristic numbers of the boundary value problem for the Hamiltonian systems (1.5) of positive type, defined by $x_N = Gx_0$ with J-unitary G, *are real*. If $G = \varrho I$ with $\varrho\overline{\varrho} = 1$, it is obviously J-unitary and the boundary value problem has a nontrivial solution if and only if

$$\det\left(U_N(\lambda) - \varrho I\right) = \varrho^2 - 2A(\lambda)\varrho + 1 = 0, \tag{2.6}$$

hence if and only if $\varrho = \varrho_i(\lambda)$ is a multiplier. Substituting $\varrho_i(\lambda_*)$ in the above equation, we obtain $A(\lambda_*) - \alpha = 0$ hence λ_* is a characteristic number of the boundary value problem, being thus real.

In the following we will need also the following result of a rather general character

Lemma 2.2. *Let λ be some real number and let u be an eigenvector of $U_N(\lambda)$, the monodromy matrix of (2.1), corresponding to some nonreal root of (2.5) such that $|\varrho| = 1$ (but $\varrho \neq \pm 1$). Then the scalar product $(Ju,u) \neq 0$.*

Proof. Since $U_N(\lambda)$ is real, we will have

$$U_N(\lambda)u = \varrho(\lambda)u, \qquad U_N(\lambda)\overline{u} = \overline{\varrho(\lambda)}\overline{u} \tag{2.7}$$

hence $\overline{u}$ is the eigenvalue associated to $\overline{\varrho}$ and is linearly independent of u. Therefore the matrix $(u\,\overline{u})$ is nonsingular; we have, by direct computation

$$(u\,\overline{u})^* J(u\,\overline{u}) = \begin{pmatrix} (Ju,u) & 0 \\ 0 & \overline{(Ju,u)} \end{pmatrix}. \tag{2.8}$$

Since the left-hand side of the above equality is a nonsingular matrix, the right-hand side matrix is such and the lemma is proved. □

According to the definition of [19], the multipliers having this property are called *definite. Using the terminology of [6], the multiplier is called K-positive if $\imath(Ju,u) > 0$ and K-negative if $\imath(Ju,u) < 0$.*

Proposition 2.3. *All zeros of the rational function $A(\lambda) - \alpha$, $|\alpha| \leq 1$, are simple, that is, $A'(\lambda) \neq 0$ for those λ such that $|A(\lambda)| < 1$.*

Outline of proof. Let λ_* be some zero of $A(\lambda) - \alpha$ for some α such that $|\alpha| < 1$; according to Proposition 2.1, λ_* is real. The multipliers of the system will be

$$\varrho_{1,2}(\lambda) = A(\lambda) \pm \sqrt{A^2(\lambda) - 1} = \alpha \pm \imath\sqrt{1-\alpha^2} \tag{2.9}$$

and are nonreal, simple and of modulus 1; according to Proposition 2.1 the multipliers are definite. Therefore, as showed in [19, 26], $\varrho_j(\lambda)$ are analytic in a neighborhood of λ_* and

$$\varrho_j(\lambda)=\varrho_j(\lambda_*)\left[1+\delta_j(\lambda-\lambda_*)+o(\lambda-\lambda_*)\right], \tag{2.10}$$

where it can be shown, using the properties of discrete-time Hamiltonian systems, that

$$\begin{aligned}\delta_j=-\frac{1}{\imath(Ju^j,u^j)}\sum_0^{N-1}\Big[&(y_k^j(\lambda_*))^*(A_k y_k^j(\lambda_*)+B_k^* z_{k+1}^j(\lambda_*))\\ &+(z_{k+1}^j(\lambda_*))^*(B_k y_k^j(\lambda_*)+D_k^* z_{k+1}^j(\lambda_*))\Big]\neq 0,\end{aligned} \tag{2.11}$$

where u^j is an eigenvector of $\varrho_j(\lambda_*)$ and $(y_k^j(\lambda_*),\ z_k^j(\lambda_*))$ is a solution of the Hamiltonian system with $\lambda=\lambda_*$ and having u^j as initial condition.

From the symmetry properties of the multipliers, we deduce

$$2A(\lambda)=\varrho_1(\lambda)+\varrho_2(\lambda)=\varrho_j(\lambda)+\frac{1}{\varrho_j(\lambda)}, \tag{2.12}$$

$$2A'(\lambda)=\left(1-\frac{1}{\varrho_j^2(\lambda)}\right)\varrho_j'(\lambda)\neq 0 \tag{2.13}$$

in some neighborhood of λ_*. The proof is complete. □

We have thus shown that in the band $(-1,1)$, the function $A(\lambda)$ has no critical points and the zeros of $A(\lambda)-\alpha$ are simple for all α, $|\alpha|<1$.

As already mentioned, stability of the canonical system means boundedness on $\mathbb{Z}$ of all its solutions. We deduce in our case that the multipliers have to be located on the unit circle and be simple. This requires $|A(\lambda)|<1$. Therefore, we may define a stability zone as an interval where λ is confined in order to have $-1<A(\lambda)<1$. *In this simple case, we may describe stability and instability zones using the properties of the characteristic function $A(\lambda)$ discussed above and some additional ones.* Its general form as a rational function is as follows:

$$A(\lambda)=\frac{(1-\lambda/\lambda_1)^{\nu_1}\cdots(1-\lambda/\lambda_q)^{\nu_q}}{(1+\lambda b_1)^{\mu_1}\cdots(1+\lambda b_r)^{\mu_r}}, \tag{2.14}$$

with $\sum\nu_i$ and $\sum\mu_i$ equal to the degree of the numerator and of the denominator of $A(\lambda)$, respectively. A straightforward computation gives

$$\frac{d}{d\lambda}\left(\frac{A'(\lambda)}{A(\lambda)}\right)=(\ln A(\lambda))''=-\sum_1^q\frac{\nu_i}{(\lambda-\lambda_i)^2}+\sum_1^r\frac{\mu_j b_j^2}{(1+\lambda b_j)^2}. \tag{2.15}$$

From now on, we have to consider two cases.

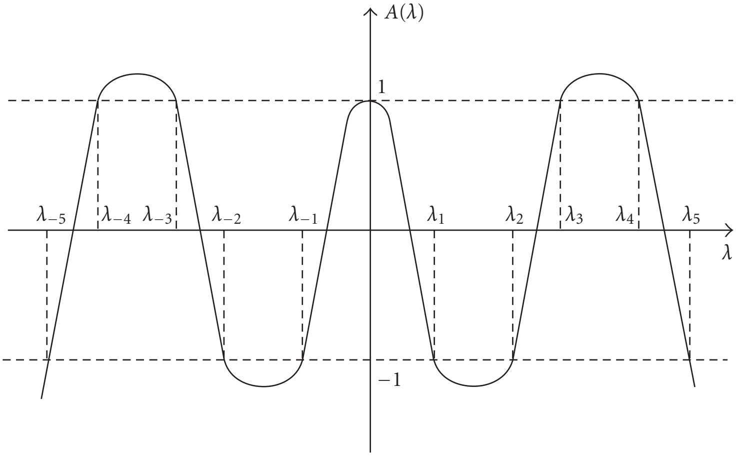

Figure 2.1. The graphic of an entire $A(\lambda)$.

(A) Let $b_k = 0$, for all $k = \overline{0, N-1}$; in this case the denominator is identically equal to 1 and $A(\lambda)$ is a polynomial, that is, of entire type. The required properties are as in [19, 26]. Indeed, it follows from (2.15) that $(\ln A(\lambda))'' < 0$ which gives $A(\lambda_*)A''(\lambda_*) < 0$ for each critical point. Consequently, the following geometric and analytic properties of $A(\lambda)$ may be deduced:

(i) the zeros of $A(\lambda) - 1$ and $A(\lambda) + 1$ have their multiplicities at most 2;

(ii) each critical point of $A(\lambda)$ is an extremum: more precisely, it is a local maximum if $A(\lambda_*) > 1$ and it is a local minimum if $A(\lambda_*) < -1$.

We deduce the representation of $A(\lambda)$ as in Figure 2.1. Note that a stability zone is delimited by those parts of function's representation where $|A(\lambda)| < 1$ while the instability zones are delimited by those parts where either $A(\lambda) > 0$ or $A(\lambda) < -1$. The extrema are enclosed in the instability zone, except, possibly, a maximum at $\lambda = 0$ representing a double root of $A(\lambda) = 1$. The fact that $(\lambda_{-1}, \lambda_1)$ with $\lambda_{-1} < 0$, $\lambda_1 > 0$, is a (central) stability zone is ensured by a general theorem which ensures existence of the central stability zone for Hamiltonian systems of positive type (see [19] also [14] in the discrete-time case).

(B) Assume now that at least one $b_k \neq 0$. Under these circumstances, $A(\lambda)$ is rational and (2.15) shows that $(\ln A(\lambda))$" may change the sign. Also existence of vertical asymptotes shows that a representation of the type of Figure 2.1 is no longer valid. On the other hand, an asymptote at $\lambda = 0$ is not possible which confirms once more existence of the central stability zone; here the graphic is exactly as in Figure 2.1. Also any stability zone is delimited as in the previous case. The instability zones are nevertheless more complicated from the point of view of the representation of $A(\lambda)$ there. An instability zone may contain *asymptotic points* and *more than one critical point* of $A(\lambda)$. Moreover an asymptote coordinate ($\lambda = -1/b_k$) *belongs only to an instability zone* and it may happen to a whole interval $(-1/b_k, -1/b_{k+1})$ to be included in some instability zone. All these properties follow from specific features of $A(\lambda)$ in each case and we will not insist on this topic (see Figure 2.2).

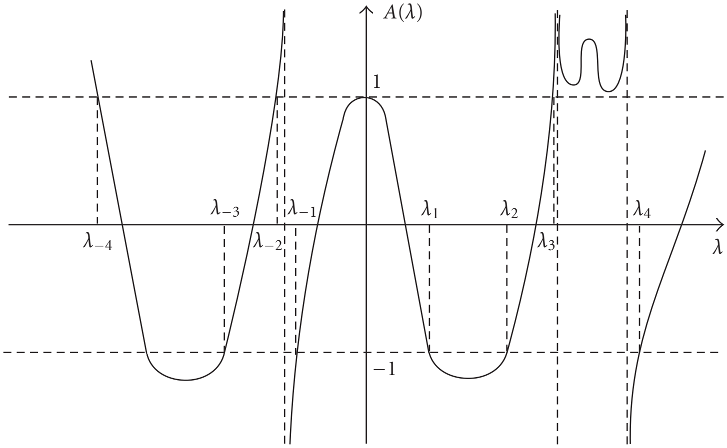

Figure 2.2. The graphic of $A(\lambda)$ having real poles.

3. Multiplier traffic rules

We have already mentioned that strong (robust) stability of Hamiltonian systems in the case of total stability (boundedness on $\mathbb{R}$) means stability preservation against structural perturbations that do not affect the Hamiltonian structure. In this case, system's multipliers do not always leave the unit circle but rather "move" on it for a while. For instance, in the 2nd-order case, if the perturbation is the modification of λ within a stability zone, the multipliers will move on the circle and remain simple up to the point when λ will enter an instability zone. The fact that the multipliers are of definite type but of different kinds allowed Kreĭn [19] to formulate his famous "traffic rules"; these rules are valid in the discrete-time case also [14, 23] and in the present case when there are only two multipliers, these rules are particularly simple [26]. Let first $|A(\lambda)| < 1$. In this case, the multipliers are complex conjugate, of modulus 1.

$$\varrho_1(\lambda) = \exp(\imath\varphi(\lambda)) = \overline{\varrho_2(\lambda)}, \quad 0 < \varphi(\lambda) < \pi. \tag{3.1}$$

If we take into account (2.11) and compute $\varphi'(\lambda)$, we find

$$\varphi'(\lambda) = \frac{1}{\imath(Ju^1(\lambda), u^1(\lambda))} \sum_0^{N-1} \left[a_k \left| y_k^1(\lambda) \right|^2 + 2b_k \Re\left(\overline{y_k^1(\lambda)} z_{k+1}^1(\lambda)\right) + d_k \left| z_{k+1}^1(\lambda) \right|^2\right] \tag{3.2}$$

which has a strictly positive numerator. The sign of $\varphi'(\lambda)$ is given by the sign of the denominator. For a positive denominator, the multiplier is of 1st kind (K-positive); for λ increasing within a stability zone, it moves on the upper semicircle, counterclockwise, from the point $(1,0)$ to the point $(-1,0)$; the other multiplier is of 2nd kind (K-negative) and it moves on the lower semicircle, clockwise, also from the point $(1,0)$ to the point $(-1,0)$. Note that in $(1,0)$ and $(-1,0)$ there are encounters of multipliers of different kinds: this means ending of a λ-stability zone and splitting of the double multiplier in two multipliers: a K-positive one (outside the unit disk) and a K-negative one (inside the unit disk), respectively.

Indeed, if $|A(\lambda)| > 1$, the multipliers given by (2.9) are real. Moreover,

$$\frac{d\varrho_1}{dA} = 1 + \frac{A}{\sqrt{A^2-1}} > 0, \qquad \frac{d\varrho_2}{dA} = 1 - \frac{A}{\sqrt{A^2-1}} < 0. \tag{3.3}$$

These equations show that the multipliers move on the real axis outside or inside the unit disk, keeping the well-known symmetry with respect to the unit circle. In the case of Figure 2.1, they will move up to some extremal positions on the real axis and further will recover the critical point where they originated, thus meeting a new stability zone. In the case of Figure 2.2, the extremal positions might be also $\pm\infty$ and the origin which correspond to asymptote value crossing.

4. Some Liapunov-like results in the discrete-time case

It has been shown in the previous section that the stability and instability zones of (2.1) alternate. As seen from Figures 2.1 and 2.2, $(\lambda_{\pm 2},\lambda_{\pm 3})$, $(\lambda_{\pm 4},\lambda_{\pm 5}),\dots,(\lambda_{\pm 2k},\lambda_{\pm(2k+1)}),\dots$ are stability zones while $(\lambda_{\pm 1},\lambda_{\pm 2}),(\lambda_{\pm 3},\lambda_{\pm 4}),\dots,(\lambda_{\pm(2k-1)},\lambda_{\pm 2k}),\dots$ are instability zones: also (λ_{-1},λ_1) defines the central stability zone.

Now let λ_* be such that $\varrho(\lambda_*) = 1$, that is, $A(\lambda_*) = 1$ which defines a "border" between a stability and an instability zone. But in this case, we deduce that for this λ_* we have

$$\det\left(U_N(\lambda_*) - I\right) = 0, \tag{4.1}$$

hence the periodic boundary value problem defined by (2.1) and

$$y_N = y_0, \qquad z_N = z_0 \tag{4.2}$$

have a nontrivial solution, that is, λ_* is a characteristic number of the *periodic boundary value problem*.

If λ_{**} is such that $\varrho(\lambda_{**}) = -1$, that is, $A(\lambda_{**}) = -1$, then we have

$$\det\left(U_N(\lambda_{**}) + I\right) = 0 \tag{4.3}$$

and λ_{**} is a characteristic number of the *skew-periodic boundary value problem* defined by (2.1) and

$$y_N = -y_0, \qquad z_N = -z_0. \tag{4.4}$$

It is now obvious that the characteristic numbers of the boundary value problems defined by (2.1) and (4.2), (4.4), respectively, *alternate in pairs*. An open interval $(\lambda_i,\lambda_{i+1})$ is a *stability zone* if and only if its endpoints are *characteristic numbers of distinct boundary value problems*.

If we consider now the 2nd-order scalar equation

$$y_{k+1} - 2y_k + y_{k-1} + \lambda^2 p_k y_k = 0, \tag{4.5}$$

we may introduce

$$y_{k+1} - y_k = \lambda z_{k+1} \tag{4.6}$$

to obtain the system

$$y_{k+1} - y_k = \lambda z_{k+1}, \tag{4.7}$$
$$z_{k+1} - z_k = -\lambda p_k y_k, \tag{4.8}$$

which is alike (2.1) but with $b_k = 0$; in this case $A(\lambda)$ is polynomial and we may refer to Figure 2.1 and to considerations made at Section 2, Case (A). Moreover, as pointed out in [24], the endpoints of the central stability zone being the first (largest) negative and the first (smallest) positive characteristic numbers of the skew-periodic boundary value problem defined by (4.7) and (4.4), the estimates for the width of the central stability zone of Kreĭn type given in [24] are valid. Among them, we would like to mention the discrete version of the well-known Liapunov criterion formulated for (1.1) [21].

Proposition 4.1 [24]. *All solutions of (4.5) are bounded provided* $p_k \geq 0$, $\sum_0^{N-1} p_k > 0$ *and* $\lambda^2 < 4/(\sum_0^{N-1} p_k)$.

In this way, all assertions of Liapunov's paper [21] have been extended to the discrete-time case using the general framework developed by Kreĭn [19]. Worth mentioning that even in this case the Liapunov criterion is only a sufficient estimate of the stability zone while not very conservative. The exact width of the central stability zone is given by the inequality [14]

$$\lambda^2 < \frac{\pi^2}{\left(\sum_0^{N-1} p_k\right)}. \tag{4.9}$$

As pointed out by Kreĭn [19], the results of Liapunov for the central stability zone of (1.1) have been extended to the case when $p(t)$ has values of both signs [22] but the cited reference contained no proofs. The proofs are to be found following the line of [19] (see Section 9 of this reference or [26]); the discrete version can be obtained in an analogous way following the hints contained in the cited references and using the results of [14].

5. Conclusions and some further research problems

Following the programme announced in [14, 23], we obtained in this paper some discrete-time counterparts of the results of Liapunov and Kreĭn about λ-zones of stability for linear periodic Hamiltonian systems of positive type. Within the programme mentioned above, several new steps may be foreseen. One of them could be the counterpart of the results of Yakubovich about asymptotics of the characteristic numbers of the periodic and skew-periodic boundary value problems and, further, the *discrete-time parametric resonance*.

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