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Research Article

Relations between Limit-Point and Dirichlet Properties of Second-Order Difference Operators

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Dedicated to Professor W. D. Evans on the occasion of his 65th birthday

Recommended by Martin J. Bohner

We consider second-order difference expressions, with complex coefficients, of the form $w_n^{-1}[-\Delta(p_{n-1}\Delta x_{n-1})+q_nx_n]$ acting on infinite sequences. The discrete analog of some known relationships in the theory of differential operators such as *Dirichlet*, *conditional Dirichlet*, *weak Dirichlet*, and *strong limit-point* is considered. Also, connections and some relationships between these properties have been established.

1. Introduction

In this paper, we will deal with the second-order formally symmetric difference expression M acting on complex valued sequences $x=\{x_n\}_{-1}^{\infty}$ defined by

$$Mx_n := \begin{cases} \dfrac{1}{w_n}\big[-\Delta(p_{n-1}\Delta x_{n-1})+q_nx_n\big], & n\geq 0,\\ -\dfrac{p_{-1}}{w_{-1}}\Delta x_n, & n=-1, \end{cases} \tag{1.1}$$

with complex coefficients $p=\{p_n\}_{-1}^{\infty}$, $q=\{q_n\}_{-1}^{\infty}$ and weight $w=\{w_n\}_{-1}^{\infty}$. In differential operators case, when the coefficients p and q are real-valued, the terms *limit-point* (LP), *strong limit-point* (SLP), *Dirichlet* (D), *conditional Dirichlet* (CD), and *weak Dirichlet* (WD) at the regular endpoint are often used to describe certain properties associated with the differential expression under consideration, see [1–10]. Here, we introduce the discrete analogue of these properties and some relations between them. In studying inequalities involving expression (1.1), such as HELP (after Hardy, Everitt, Littlewood and Polya) and Kolmogorov-type inequalities, these properties and the relationships between them are crucial. The work we present here is the discrete analogue of the work by Race [9] for differential expressions.

2. Preliminaries

We use the following notation throughout: $\mathbb{R}$ and $\mathbb{C}$ denote the real and complex number fields, and $\mathbb{N}$ is the set of nonnegative integers. $\overline{z}$ denotes the complex conjugate of $z \in \mathbb{C}$. $\Im(\cdot)$ and $\Re(\cdot)$ represent the imaginary and real part of a complex number. ℓ^1 is the space of all absolutely summable complex sequences. ℓ^2 and ℓ^2_w are the Hilbert spaces

$$\ell^2 = \left\{ x = \{x_n\}_{-1}^{\infty} : \sum_{n=-1}^{\infty} |x_n|^2 < \infty \right\},$$
$$\ell^2_w = \left\{ x = \{x_n\}_{-1}^{\infty} : \sum_{n=-1}^{\infty} |x_n|^2 w_n < \infty \right\} \tag{2.1}$$

with $w_n > 0$ for all n and the inner products

$$(x,y) = \sum_{n=-1}^{\infty} x_n \overline{y}_n, \qquad (x,y) = \sum_{n=-1}^{\infty} x_n \overline{y}_n w_n, \tag{2.2}$$

respectively. If $\{x_n\}_{-1}^{\infty} \notin \ell^1$ but $\sum_{n=-1}^{\infty} x_n < \infty$, then we say that the sum $\sum_{n=-1}^{\infty} x_n$ is conditionally convergent. We associate a maximal operator, $T(M)$, in ℓ^2_w with the linear difference expression

$$Mx_n := \begin{cases} \dfrac{1}{w_n}\left[-\Delta(p_{n-1}\Delta x_{n-1}) + q_n x_n\right], & n \geq 0, \\ -\dfrac{p_{-1}}{w_{-1}}\Delta x_n, & n = -1, \end{cases} \tag{2.3}$$

where $\Delta x_n = x_{n+1} - x_n$, the forward difference, and the coefficients $\{p_n\}_{-1}^{\infty}$ and $\{q_n\}_{-1}^{\infty}$ are complex valued with

$$p_n \neq 0, \quad q_{-1} = 0, \quad w_n > 0, \quad \forall n = -1, 0, 1, \ldots. \tag{2.4}$$

Note that defining M by (2.3) makes the difference equation

$$Mx_n = \lambda x_n, \quad n = 0, 1, 2, \ldots \ (\lambda \in \mathbb{C}), \tag{2.5}$$

a three-term recurrence relation. The operator $T(M)$ is defined on $D_{T(M)}$ into ℓ^2_w as

$$[T(M)x]_n = T(M)x_n := Mx_n, \quad n = -1, 0, 1, \ldots, \tag{2.6}$$

$$D_{T(M)} := \left\{ x = \{x_n\}_{-1}^{\infty} \in \ell^2_w : \sum_{n=-1}^{\infty} |T(M)x_n|^2 w_n < \infty \right\}. \tag{2.7}$$

The summation-by-parts formula

$$\sum_{n=k}^{m} x_n \Delta y_n = x_{m+1} y_{m+1} - x_k y_k - \sum_{n=k}^{m} y_{n+1} \Delta x_n, \quad k \leq m,\ k, m \in \mathbb{N}, \tag{2.8}$$

gives rise to the equalities

$$\sum_{n=0}^{m} \overline{x_n} M y_n w_n = \sum_{n=0}^{m} q_n y_n \overline{x_n} + \sum_{n=0}^{m} (p_n \Delta y_n) \overline{\Delta x_n} - p_m \Delta y_m \overline{x_{m+1}} + p_{-1} \Delta y_{-1} \overline{x_0} \tag{2.9}$$

and, for all $x, y \in D_{T(M)}$,

$$\sum_{n=0}^{\infty} (p_n \Delta y_n \overline{\Delta x_n} + q_n y_n \overline{x_n}) = \sum_{n=0}^{\infty} (\overline{x_n} T(M) y_n) w_n + \lim_{m\to\infty} p_m \Delta y_m \overline{x_{m+1}} - p_{-1} \Delta y_{-1} \overline{x_0}. \tag{2.10}$$

The left-hand side of (2.10) is called the *Dirichlet sum*, and (2.10) is called the *Dirichlet formula*. The following also holds for all $x, y \in D_{T(M)}$:

$$\sum_{n=0}^{\infty} (x_n T(M) y_n - y_n T(M) x_n) w_n = \lim_{m\to\infty} p_m (\Delta x_m y_{m+1} - \Delta y_m x_{m+1}) - p_{-1} (\Delta x_{-1} y_0 - \Delta y_{-1} x_0). \tag{2.11}$$

Following (2.10) we have, for $x \in D_{T(M)}$,

$$\sum_{n=0}^{\infty} (p_n |\Delta x_n|^2 + q_n |x_n|^2) = \sum_{n=0}^{\infty} (\overline{x_n} T(M) x_n) w_n + \lim_{m\to\infty} p_m \Delta x_m \overline{x_{m+1}} - p_{-1} \Delta x_{-1} \overline{x_0}. \tag{2.12}$$

An immediate consequence of (2.11) together with (2.7) is that

$$\lim_{m\to\infty} p_m (\Delta x_m y_{m+1} - \Delta y_m x_{m+1}) \quad \text{exists and is finite } \forall x, y \in D_{T(M)}. \tag{2.13}$$

Moreover, the expression in (2.13) is a constant for all $m \in \mathbb{N}$ when x, y are the solutions of (2.5), which is easy to prove. We also have the following *variation of parameters formula*: let $\phi = \{\phi_n\}_{-1}^{\infty}$ and $\psi = \{\psi_n\}_{-1}^{\infty}$ be linearly independent solutions of (2.5) and suppose that $[\phi, \psi]_n := p_n[(\Delta\phi_n)\psi_{n+1} - (\Delta\psi_n)\phi_{n+1}] = 1$ for all n. Then, $\Phi = \{\Phi_n\}_{-1}^{\infty}$ defined by

$$\begin{gathered} \Phi_n = \sum_{m=0}^{n} (-\psi_m \phi_n + \phi_m \psi_n) w_m f_m \quad (n \in \mathbb{N}), \\ \Phi_{-1} = 0 \end{gathered} \tag{2.14}$$

satisfies

$$M\Phi_n = \lambda \Phi_n + f_n, \quad n \in \mathbb{N},\ \lambda \in \mathbb{C}, \tag{2.15a}$$

$$\Phi_{-1} = \Phi_0 = 0. \tag{2.15b}$$

Any solution of (2.15a) is of the form

$$\Psi = \Phi + A\phi + B\psi \tag{2.16}$$

for some constants $A, B \in \mathbb{C}$.

Definition 2.1. If there is precisely one ℓ_w^2 solution (up to constant multiples) of (2.5) for $\Im(\lambda) \neq 0$, then the expression M is said to be in the *limit-point* (LP) case; otherwise all solutions of (2.5) are in ℓ_w^2 for all $\lambda \in \mathbb{C}$ and M is said to be in the *limit-circle* (LC) case, see Atkinson [11] and Hinton and Lewis [6]. Note that in the limit-circle (LC) case, the defect numbers are equal and the limit-point case does not hold. An alternative but equivalent characterization of M being LP is that

$$\lim_{m\to\infty} p_m\left(\Delta\overline{x_m}y_{m+1} - \Delta y_m\overline{x_{m+1}}\right) = 0 \tag{2.17}$$

or

$$\lim_{m\to\infty} p_m\left(y_m\overline{x_{m+1}} - y_{m+1}\overline{x_m}\right) = 0 \tag{$*_1$}$$

for all $x, y \in D_{T(M)}$, see Hinton and Lewis [6, page 425]. It may also be observed that this condition is equivalent to saying that

$$\lim_{m\to\infty} p_m\left(\Delta\overline{x_m}x_{m+1} - \Delta x_m\overline{x_{m+1}}\right) = 0 \tag{2.18}$$

or

$$\lim_{m\to\infty} p_m\left(x_m\overline{x_{m+1}} - x_{m+1}\overline{x_m}\right) = 0 \tag{$*_2$}$$

for all $x \in D_{T(M)}$. To see that, take $x = y$ in $(*_1)$ to get the implication in one direction. For the implication on the other side, take x to be the linear combination of z and y, that is, $x = z + \alpha y$ in $(*_2)$, and then choose the complex number α as $\alpha = 1$ and $\alpha = i$ to get $(*_1)$.

Definition 2.2. M is said to be *strong limit-point* (SLP) on $D_{T(M)}$ if

$$\lim_{m\to\infty} p_m\Delta y_m\overline{x_{m+1}} = 0 \quad \forall x, y \in D_{T(M)}. \tag{2.19}$$

Definition 2.3. M is said to be

(i) *Dirichlet* (D) on $D_{T(M)}$ if

$$\left\{\left|p_n\right|^{1/2}\Delta x_n\right\}_{-1}^{\infty}, \quad \left\{\left|q_n\right|^{1/2}x_n\right\}_{-1}^{\infty} \in \ell^2 \quad \forall x \in D_{T(M)}; \tag{2.20}$$

(ii) *conditional Dirichlet* (CD) on $D_{T(M)}$ if

$$\left\{\left|p_n\right|^{1/2}\Delta x_n\right\}_{-1}^{\infty} \in \ell^2, \quad \sum_{n=0}^{\infty} q_n\left|x_n\right|^2 \text{ is convergent } \forall x \in D_{T(M)}, \tag{2.21}$$

(iii) *weak Dirichlet* (WD) on $D_{T(M)}$ if

$$\sum_{n=0}^{\infty}\left(p_n\overline{\Delta x_n}\Delta y_n + q_n\overline{x_n}y_n\right) \quad \text{is convergent } \forall x, y \in D_{T(M)}. \tag{2.22}$$

Observe that (2.19) is equivalent to

$$\lim_{m\to\infty} p_m \Delta x_m \overline{x_{m+1}} = 0 \quad \text{or} \quad \lim_{m\to\infty} p_m \Delta x_m x_{m+1} = 0 \quad \forall x \in D_{T(M)}. \tag{2.23}$$

Also, by *Dirichlet* formula (2.10), it is seen that the WD property, (2.22), is equivalent to

$$\lim_{m\to\infty} p_m \Delta y_m \overline{x_{m+1}} \quad \text{exists and is finite} \ \ \forall x, y \in D_{T(M)}, \tag{2.24}$$

and this is equivalent to

$$\lim_{m\to\infty} p_m \Delta x_m x_{m+1} \quad \text{exists and is finite} \ \ \forall x \in D_{T(M)}. \tag{2.25}$$

Note also that in (iii), for all $x, y \in D_{T(M)}$,

$$\{ |p_n|^{1/2} \Delta x_n \}_{-1}^{\infty} \in \ell^2 \iff \{ p_n (\Delta x_n)^2 \}_{-1}^{\infty} \in \ell^1 \iff \{ p_n \Delta x_n \Delta y_n \}_{-1}^{\infty} \in \ell^1. \tag{2.26}$$

Following the above definitions and subsequent comments, we have the following.

Corollary 2.4. *The following implications hold for all $x, y \in D_{T(M)}$:*
(a) *$D \Rightarrow CD \Rightarrow WD$;*
(b) *$SLP \Rightarrow WD$;*
(c) *$SLP \Rightarrow LP$.*

3. Statement of results

In this section, we would like to obtain some implications additional to Corollary 2.4 by imposing conditions on p, q, and w which are as weak as possible. The motivation of the problem and parts (a) and (b) of the following theorem was previously presented at the *17th National Symposium of Mathematics, Bolu, Turkey* [12]. It is presented here for the sake of completeness.

Theorem 3.1. *Let p and q be complex-valued.*
(a) *If $1/p \notin l^1$, then $CD \Rightarrow SLP$ on $D_{T(M)}$.*
(b) *If $1/p \in l^1$ but $\sum_{n=0}^{\infty} q_n$ is not convergent, then $CD \Rightarrow SLP$ on $D_{T(M)}$.*
(c) *If w, $1/p$, $q \in l^1$, then M is both D and LC.*

Proof. (a) We assume that $1/p \notin \ell^1$ and M is CD on $D_{T(M)}$. Let $x, y \in D_{T(M)}$ then, by (2.10),

$$\alpha := \lim_{m\to\infty} p_m \Delta y_m \overline{x}_{m+1} < \infty. \tag{3.1}$$

We need to prove that $\alpha = 0$ under the conditions in the hypothesis. Suppose the contrary that $\alpha \neq 0$, then for some $m_0 \in \mathbb{N}$,

$$| p_m \Delta y_m x_{m+1} | \geq \frac{|\alpha|}{2} \quad \forall m \geq m_0, \tag{3.2}$$

which implies that

$$| p_m \Delta y_m \Delta x_m | \geq \frac{|\alpha|}{2} \left| \frac{\Delta x_m}{x_{m+1}} \right| \quad \forall m \geq m_0, \ \forall x, y \in D_{T(M)}. \tag{3.3}$$

However, M is CD and this implies that, summing over m, the left-hand side of (3.3) belongs to ℓ^1. Thus,

$$\sum_{n=-1}^{\infty}\left|\frac{\Delta x_n}{x_{n+1}}\right| < \infty, \tag{3.4}$$

and hence in particular $|\Delta x_n/x_{n+1}| \to 0$ as $n \to \infty$. So, as $n \to \infty$,

$$\left|\log\frac{x_{n+1}}{x_n}\right| = \left|-\log\left(1-\frac{\Delta x_n}{x_{n+1}}\right)\right| \sim \left|\frac{\Delta x_n}{x_{n+1}}\right| \tag{3.5}$$

since

$$\lim_{t\to 0}\frac{\log(1-t)}{t} = -1. \tag{3.6}$$

Hence,

$$\begin{gathered}\sum_{n=-1}^{\infty}\left|\log\frac{x_{n+1}}{x_n}\right| < \infty \Longrightarrow \sum_{n=-1}^{\infty}\log\frac{x_{n+1}}{x_n} \quad \text{is convergent,}\\ \lim_{N\to\infty}\sum_{n=m_0}^{N}\log\frac{x_{n+1}}{x_n} \quad \text{exists for } m_0 \in \mathbb{N}.\end{gathered} \tag{3.7}$$

This implies that

$$\lim_{N\to\infty}\sum_{n=m_0}^{N}\Delta(\log x_n) = \lim_{N\to\infty}\left(\log x_{N+1} - \log x_{m_0}\right) \text{ exists.} \tag{3.8}$$

So,

$$\beta := \lim_{N\to\infty} x_N \neq 0. \tag{3.9}$$

Thus, since $\alpha := \lim_{m\to\infty} p_m \Delta y_m \overline{x}_{m+1} < \infty$,

$$\lim_{m\to\infty} p_m \Delta y_m = \alpha\beta^{-1}, \tag{3.10}$$

and, for some $m_0 \in \mathbb{N}$,

$$\left|p_m(\Delta y_m)^2\right| \geq \frac{1}{4}\left|\alpha\beta^{-1}\right|^2\left|p_m^{-1}\right| \quad \forall m \geq m_0. \tag{3.11}$$

However, summing over m, the left-hand side of (3.11) belongs to ℓ^1 by the hypothesis that M is CD. Hence, so does the right-hand side of (3.11) which is a contradiction to saying that $1/p \notin \ell^1$. Hence $\alpha = 0$, proving M is SLP.

(b) Assume that $p^{-1} \in \ell^1$ but $\sum_{n=0}^{\infty} q_n$ is not convergent and M is CD. Let $x \in D_{T(M)}$ and, as in (a) above, suppose that

$$\alpha = \lim_{m\to\infty} p_m x_{m+1}\Delta x_m \neq 0. \tag{3.12}$$

Then, $\lim_{m\to\infty} x_m = \beta \neq 0$ exists and it follows that

$$\lim_{m\to\infty} p_m \Delta x_m = \alpha\beta^{-1} \neq 0 \Longrightarrow \lim_{m\to\infty} \Delta x_m = \lim_{m\to\infty} \alpha\beta^{-1} p_m^{-1}. \tag{3.13}$$

So, since $p^{-1} \in \ell^1$, we have

$$\sum_{m=-1}^{\infty} |\Delta x_m| < \infty, \quad \text{that is, } \{\Delta x_n\}_{-1}^{\infty} \in \ell^1 \ (x \in D_{T(M)}). \tag{3.14}$$

Now, since $x \in D_{T(M)}$, using Cauchy-Schwarz inequality in ℓ^2, we have

$$\begin{aligned} &\sum_{n=-1}^{\infty} |x_n w_n^{1/2}[-\Delta(p_{n-1}\Delta x_{n-1}) + q_n x_n] w_n^{-1/2}| \\ &\quad\leq \left(\sum_{n=-1}^{\infty} |x_n w_n^{1/2}|^2\right)^{1/2} \left(\sum_{n=-1}^{\infty} |[-\Delta(p_{n-1}\Delta x_{n-1}) + q_n x_n] w_n^{-1/2}|^2\right)^{1/2} \end{aligned} \tag{3.15}$$

which gives

$$\sum_{n=-1}^{\infty} |x_n[-\Delta(p_{n-1}\Delta x_{n-1}) + q_n x_n]| < \infty. \tag{3.16}$$

Also, since $\lim_{m\to\infty} x_m = \beta \neq 0$, we have that

$$\sum_{n=-1}^{\infty} |[-\Delta(p_{n-1}\Delta x_{n-1}) + q_n x_n]| < \infty. \tag{3.17}$$

Now,

$$\sum_{n=0}^{\infty} [-\Delta(p_{n-1}\Delta x_{n-1}) + q_n x_n] = -\lim_{m\to\infty} p_m \Delta x_m + p_{-1}\Delta x_{-1} + \sum_{n=0}^{\infty} q_n x_n \tag{3.18}$$

implies that

$$\sum_{n=0}^{\infty} q_n x_n = \lim_{m\to\infty} p_m \Delta x_m - p_{-1}\Delta x_{-1} + \sum_{n=0}^{\infty} [-\Delta(p_{n-1}\Delta x_{n-1}) + q_n x_n], \tag{3.19}$$

which proves the convergence of the sum $\sum_{n=0}^{\infty} q_n x_n$. Since $\beta = \lim_{m\to\infty} x_m \neq 0$, then $x_m \neq 0$ for all large $m \in \mathbb{N}$. On the other hand, using summation-by-parts formula and supposing $k \in \mathbb{N}$ is such that $x_n \neq 0$ for all $n \geq k$, we have

$$\begin{aligned} \sum_{n=k}^{m} q_n &= \sum_{n=k}^{m} \frac{1}{x_n}(q_n x_n) = \frac{1}{x_{m+1}} \sum_{s=k-1}^{m} q_s x_s - \frac{1}{x_k} \sum_{s=k-1}^{k-1} q_s x_s - \sum_{n=k}^{m} \left(\sum_{s=k-1}^{n} q_s x_s\right) \Delta\left(\frac{1}{x_n}\right) \\ &= \frac{\sum_{n=k-1}^{m} q_n x_n}{x_{m+1}} - \frac{q_{k-1}x_{k-1}}{x_k} + \sum_{n=k}^{m} \left(\sum_{s=k-1}^{n} q_s x_s\right) \left(\frac{\Delta x_n}{x_{n+1}x_n}\right). \end{aligned} \tag{3.20}$$

As $m \to \infty$, we see that the right-hand side of (3.20) tends to a finite limit since $\sum_{n=0}^{\infty} q_n x_n$ is convergent and $\lim_{n\to\infty} x_n = \beta \neq 0$, which contradicts the hypothesis that $\sum_{n=0}^{\infty} q_n$ is divergent. This proves $\alpha = 0$ which guarantees that M is *SLP*.

(c) If $1/p$, w, $q \in \ell^1$, then M is *LC* and *D*. For the proof, we need the matrix representation of (2.5); for $n \geq 0$, we have the recurrence relation

$$p_n(x_{n+1} - x_n) = (-\lambda w_n + q_n)x_n + p_{n-1}(x_n - x_{n-1}), \tag{3.21}$$

which is equivalent to (2.5). So, taking

$$X_n = \begin{pmatrix} x_n \\ y_n \end{pmatrix}, \qquad A_n = \begin{pmatrix} 0 & \dfrac{1}{p_{n-1}} \\ (-\lambda w_n + q_n) & \dfrac{-\lambda w_n + q_n}{p_{n-1}} \end{pmatrix}, \tag{3.22}$$

we get

$$X_n = (I + A_n)X_{n-1}, \quad n = 0, 1, 2, \ldots, \tag{3.23}$$

where I is the identity matrix and

$$\begin{aligned} x_n &= x_{n-1} + \frac{y_{n-1}}{p_{n-1}} \\ y_n &= \left(x_{n-1} + \frac{y_{n-1}}{p_{n-1}}\right)(-\lambda w_n + q_n) + y_{n-1}. \end{aligned} \tag{3.24}$$

We are going to give the proof for the *LC* and *D* cases separately.

(i) *The LC case.* We prove that, for some λ, say $\lambda = 0$, for all solutions of (3.21), $\sum_{n=-1}^{\infty} |x_n|^2 w_n < \infty$ holds. Moreover, since $\sum_{n=-1}^{\infty} w_n < \infty$, it is sufficient to prove that all solutions of (3.21), with $\lambda = 0$, are bounded. For this purpose, we make use of the following theorem due to Atkinson [11, page 447].

Theorem 3.2 (Atkinson). *Let the sequence of k-by-k matrices,*

$$A_n, \quad n = 0, 1, 2, 3, \ldots; \quad A_n = (a_{nrs}), \quad r, s = 1, 2, 3, \ldots, k, \tag{3.25}$$

satisfy

$$\sum_{n=0}^{\infty} |A_n| < \infty, \quad |A_n| := \sum_{r=1}^{k}\sum_{s=1}^{k} |a_{nrs}|. \tag{3.26}$$

Then, the solutions of the recurrence relation

$$X_n - X_{n-1} = A_{n-1}X_{n-1}, \quad n = 0, 1, 2, \ldots, \tag{3.27}$$

where X_n is a k-vector, converge as $n \to \infty$. If in addition the matrices $I + A_n$ are all nonsingular, then $\lim_{n\to\infty} X_n \neq 0$, unless all the X_n are zero vectors.

So, applying this theorem to our case, $\{X_n\}_0^\infty$ is convergent, that is, the entries of X_n,

$$\{X_{n1}\}_0^\infty = \{x_n\}_0^\infty, \qquad \{X_{n2}\}_0^\infty = \{y_n\}_0^\infty = \{p_n\Delta x_n\}_0^\infty, \tag{3.28}$$

are convergent, so they are bounded and hence (i) of condition (c) is proved.

(ii) *The D case.* We will state the proof for $\lambda = 0$ only, but the proof also applies to all $\lambda \in \mathbb{C}$. Let $x \in D_{T(M)}$ and define $f = \{f_n\}_{-1}^\infty$ by

$$f_n = Mx_n. \tag{3.29}$$

Then $\sum_{n=-1}^\infty |f_n|^2 w_n < \infty$. Also, by the variation of parameters formula, if $\varphi = \{\varphi_n\}_{-1}^\infty$ and $\psi = \{\psi_n\}_{-1}^\infty$ are linearly independent solutions of (2.5) with

$$[\varphi,\psi]_n := p_{n-1}(\varphi_n\Delta\psi_{n-1} - \psi_n\Delta\varphi_{n-1}) = 1 \quad \forall n \in \mathbb{N}, \tag{3.30}$$

then any solution of

$$Mx_n = \lambda x_n + f_n \tag{3.31}$$

is of the form

$$x_n = \Phi_n + A\varphi_n + B\psi_n \tag{3.32}$$

in which A and B are constants, and

$$\Phi_n = \sum_{m=0}^{n} (\psi_m\varphi_n - \varphi_m\psi_n) w_m f_m, \quad n \in \mathbb{N},\ \Phi_{-1} = 0. \tag{3.33}$$

Since $\{\varphi\}_{-1}^\infty$ and $\{\psi\}_{-1}^\infty$ are bounded by case (i) of condition (c), using also Cauchy-Schwarz inequality in ℓ^2, it follows that

$$|\Phi_n| \le C\sum_{m=0}^{n} w_m |f_m|, \tag{3.34}$$

where C is a positive constant. Hence, Φ is bounded. This implies that $\{x_n\}_{-1}^\infty$ is bounded from the fact that $\{A\varphi_n + B\psi_n\}_{-1}^\infty$ and $\{\Phi_n\}_{-1}^\infty$ are bounded in (3.32). So, since $q \in \ell^1$ and following the above result,

$$\sum_{n=0}^{\infty} |q_n| |x_n|^2 < \infty. \tag{3.35}$$

We also need to prove that $\sum_{n=0}^\infty |p_n||\Delta x_n|^2 < \infty$. For, from (3.32),

$$\begin{aligned} p_n\Delta x_n &= p_n\Delta\Phi_n + p_n\Delta(A\varphi_n + B\psi_n), \\ p_n\Delta\Phi_n &= \sum_{m=0}^{n} [\psi_m(p_n\Delta\varphi_n) - \varphi_m(p_n\Delta\psi_n)] w_m f_m; \end{aligned} \tag{3.36}$$

and since $\{p_n\Delta\varphi_n\}_{-1}^{\infty}$, $\{p_n\Delta\psi_n\}_{-1}^{\infty}$, $\{\varphi_n\}_{-1}^{\infty}$, and $\{\psi_n\}_{-1}^{\infty}$ are bounded by the theorem of Atkinson, $\{p_n\Delta\Phi_n\}_{-1}^{\infty}$ is also bounded, and so is $\{p_n\Delta x_n\}_{-1}^{\infty}$. By the hypothesis that $p^{-1} \in \ell^1$, we obtain

$$\sum_{n=0}^{\infty} |p_n| |\Delta x_n|^2 = \sum_{n=0}^{\infty} \frac{(|p_n| |\Delta x_n|)^2}{|p_n|} < \infty. \tag{3.37}$$

Hence, M is D and the proof of Theorem 3.1 is complete. □

Corollary 3.3. *(1) Following the Dirichlet formula, (2.23), and Theorem 3.1(a)-(b), it may be deduced that if either $p^{-1} \notin \ell^1$ or $p^{-1} \in \ell^1$ but $\sum_{n=0}^{\infty} q_n$ is not convergent, then CD implies that the sum $\sum_{n=0}^{\infty}(p_n|\Delta x_n|^2 + q_n|x_n|^2)$ is convergent for all $x \in D_{T(M)}$. (2) Under the conditions of Theorem 3.1(a)-(b), $D \Rightarrow CD \Rightarrow SLP \Rightarrow LP$ on $D_{T(M)}$.*

Remarks 3.4. (1) When $w, p^{-1}, q \in \ell^1$, it is proved by Atkinson [11, page 134] that M is LC. We have additionally proved that M is also D. (2) The condition imposed on q in Theorem 3.1(a) is in general weaker than $q \notin \ell^1$. Indeed, in Example 3.5, we prove that $q \notin \ell^1$ is not sufficient to ensure that $CD \Rightarrow SLP$.

Example 3.5. In this example, we want to establish an expression M of the form (2.3) such that $\sum_{n=0}^{\infty} q_n$ is conditionally convergent and $w, 1/p \in \ell^1$ while M is CD and LC, hence not SLP, at the same time. This proves that $q \notin \ell^1$ is not sufficient to ensure that the implication $CD \Rightarrow SLP$. This example is a direct analogue of the example given in Kwong [7, page 332]. Let $\sum_{n=0}^{\infty} r_n$ be a conditionally convergent real series. Choose a constant C_1 so that the sequence

$$\{R_n\}_0^{\infty} = \left\{\sum_{k=0}^{n} r_k\right\}_0^{\infty} + C_1 \tag{3.38}$$

be positive, that is, $R_n > 0$ for all, $n = 0, 1, 2, \ldots$. Then $\{R_n\}_0^{\infty}$ is bounded, for $p_n > 0$ $n \in \mathbb{N}$ and given that $C_2 > 0$, the sequence

$$\{x_n\}_0^{\infty} = \left\{\sum_{k=0}^{n} \frac{R_{k-1}}{p_{k-1}}\right\}_0^{\infty} + C_2, \quad R_{-1} = 0,\ p_{n-1} > 0\ \forall n \in \mathbb{N},\ x_{-1} \geq x_0 \tag{3.39}$$

is also positive. Note that $\{x_n\}_{-1}^{\infty}$ is monotonic increasing, that is, $x_{n+1} \geq x_n$ for all n, from the fact that x_n are the sum of positive numbers. Now,

$$X = \lim_{n\to\infty} x_n \text{ exists} \tag{3.40}$$

since $\{R_n\}_{-1}^{\infty}$ is bounded and $p^{-1} = \{p_n^{-1}\}_{-1}^{\infty} \in \ell^1$. Moreover, $x \in \ell_w^2$ since $w \in \ell^1$ and $\{x_n\}_{-1}^{\infty}$ is bounded. We see that if $\{q_n\}_{-1}^{\infty}$ is given by

$$q_n = \frac{r_n}{x_n}, \quad n \geq 0,\ q_{-1} = 0, \tag{3.41}$$

then $\{x_n\}_{-1}^{\infty}$ is a solution of (2.5) with $\lambda = 0$. Note that, in

$$|q_n| = \frac{|r_n|}{x_n} \geq \frac{|r_n|}{X} \quad \forall n, \tag{3.42}$$

summing over n, we have $\{q_n\}_{-1}^{\infty} \notin \ell^1$ from the fact that $\sum_0^{\infty} r_n$ is conditionally convergent. Now, summation-by-parts formula gives, for all $N \in \mathbb{N}$,

$$\sum_{n=0}^{N} q_n = \sum_{n=0}^{N} \frac{r_n}{x_n} = \frac{R_N}{x_N} - \sum_{n=-1}^{N-1} \frac{R_n}{x_{n+1}} + \sum_{n=-1}^{N-1} \frac{R_n}{x_n}. \tag{3.43}$$

For the first expression on the right-hand side, the limits $\lim_{n\to\infty} R_n$ and $\lim_{n\to\infty} x_n$ exist and $X = \lim_{n\to\infty} x_n > 0$. For the sums on the right, since $\sum_{n=0}^{\infty} R_n$ is convergent and $\{1/x_n\}_{-1}^{\infty}$ is positive and decreasing, both $\sum_{n=-1}^{N}(R_n/x_{n+1})$ and $\sum_{n=-1}^{N}(R_n/x_n)$ are convergent, and therefore $\sum_{n=0}^{\infty} q_n$ is convergent. Now, let $\{y_n\}_{-1}^{\infty}$ be another solution of (2.5) together with (3.41) complementary to $\{x_n\}_{-1}^{\infty}$, that is, such that $[x,y]_n := p_{n-1}(y_n x_{n-1} - y_{n-1} x_n)$ is constant, or equivalently, $[x,y]_n = 1$. Then,

$$\Delta\left(\frac{y_{n-1}}{x_{n-1}}\right) = \frac{1}{p_{n-1}x_n x_{n-1}} \Longrightarrow y_n = x_n \sum_{k=0}^{n} \frac{1}{p_{k-1}x_k x_{k-1}}. \tag{3.44}$$

So, since $\{y_n\}_{-1}^{\infty}$ is bounded and increasing,

$$\lim_{n\to\infty} y_n \text{ exists.} \tag{3.45}$$

We note that $\sum_{k=0}^{\infty}(1/p_{k-1}x_k x_{k-1})$ is absolutely convergent since $\{x_n\}_{-1}^{\infty}$ is bounded and $p^{-1} \in \ell^1$. So, $y \in \ell_w^2$ since $w \in \ell^1$. We also see that $My_n = 0$. Hence, we have shown that M is *LC*, and hence not *SLP* since $x, y \in \ell_w^2$ and x, y are linearly independent solutions of $Mx_n = \lambda x_n$, $\lambda \in \mathbb{C}$. We now show that M is *CD*. Since, from the identity (2.12), the *CD* property is equivalent to

(a) $\{p_n|\Delta z_n|^2\}_{-1}^{\infty} \in \ell^1$,

(b) $\lim_{n\to\infty} p_n \Delta z_n \bar{z}_{n+1}$ exists $\forall z \in D_{T(M)}$,

and we will show both (a) and (b) above. So, let $z \in D_{T(M)}$. Then,

$$\{T(M)z_n\}_{-1}^{\infty} = \{Mz_n\}_{-1}^{\infty} = \{f_n\}_{-1}^{\infty} \in \ell_w^2, \quad w \in \ell^1. \tag{3.46}$$

The method of variation of parameters gives

$$z_n = Ax_n + By_n + \sum_{m=0}^{n} (x_n y_m - y_n x_m) f_m w_m \quad (z_{-1} = 0,\ n \in \mathbb{N}), \tag{3.47}$$

where A and B are constants. Note that $\lim_{n\to\infty} \sum_{m=0}^{n}(x_n y_m - y_n x_m) f_m w_m < \infty$, (3.40) and (3.45) together imply that

$$\lim_{n\to\infty} z_n \text{ exists.} \tag{3.48}$$

We see that $\{p_n^{1/2}\Delta x_n\}_{-1}^{\infty}$, $\{p_n^{1/2}\Delta y_n\}_{-1}^{\infty} \in \ell^2$ since $\{R_n\}_0^{\infty}$ is bounded and $\{p_n^{-1}\}_{-1}^{\infty} \in \ell^1$. Also, using the Cauchy-Schwarz inequality in $\ell^{2,n}$, we see that, for all $n \in \mathbb{N}$,

$$\sum_{m=0}^{n} [y_m(p_n^{1/2}\Delta x_n) - x_m(p_n^{1/2}\Delta y_n)] f_m w_m \le \frac{C}{p_n^{1/2}} \left(\sum_{m=0}^{n} w_m\right)^{1/2} \left(\sum_{m=0}^{n} w_m |f_m|^2\right)^{1/2}, \tag{3.49}$$

where C is a constant. Hence,

$$\{p_n^{1/2}\Delta z_n\}_{-1}^{\infty} \in \ell^2. \tag{3.50}$$

Finally,

(i) $\lim_{n\to\infty} p_n \Delta x_n = \lim_{n\to\infty} R_n < \infty$,

(ii) $\lim_{n\to\infty} p_n\Delta y_n = \lim_{n\to\infty} [1/x_n + (p_n\Delta x_n)\sum_{k=0}^{n}(1/p_{k-1}x_k x_{k-1})] < \infty$ since the limits $\lim_{n\to\infty} 1/x_n$ and $\lim_{n\to\infty} p_n\Delta x_n$ exist and $\sum_{k=0}^{\infty}(1/p_{k-1}x_k x_{k-1})$ is absolutely convergent,

(iii) For $K < \infty$,

$$\lim_{n\to\infty} \left| p_n\Delta x_n \sum_{m=0}^{n} y_m(w_m f_m) \right| \le K \lim_{n\to\infty} \left(\sum_{m=0}^{n} w_m\right)^{1/2} \left(\sum_{m=0}^{n} w_m |f_m|^2\right)^{1/2} < \infty, \tag{3.51}$$

(iv) $\lim_{n\to\infty} |p_n\Delta y_n \sum_{m=0}^{n} x_m(w_m f_m)| \le C\lim_{n\to\infty} |p_n\Delta y_n \sum_{m=0}^{n} w_m f_m| < \infty$.

A consequence of (i), (ii), (iii), and (iv) is that $\lim_{n\to\infty} p_n\Delta z_n$ exists. We know also that $\lim_{n\to\infty} z_n$ exists from (3.48). Therefore,

$$\lim_{n\to\infty} p_n\Delta z_n \overline{z}_{n+1} \text{ exists.} \tag{3.52}$$

It is a consequence of (3.50) and (3.52) that M is CD. This completes the desired example.

Theorem 3.6. *Suppose that $p_n > 0$ for all n, although $\{q_n\}_{-1}^{\infty}$ may still be complex. If either $\{w_m \sum_{n=-1}^{m} p_n^{-1}\}_{m=-1}^{\infty} \notin \ell^1$ or $\{q_n\}_{-1}^{\infty} \notin \ell^1$, then*

$$M \text{ is } D \text{ on } D_{T(M)} \iff \{|q_n|^{1/2} x_n\}_{-1}^{\infty} \in \ell^2, \quad x \in D_{T(M)}. \tag{3.53}$$

Proof. Since M is D on $D_{T(M)} \Rightarrow \{|q_n|^{1/2}x_n\}_{-1}^{\infty} \in \ell^2$ for all $x \in D_{T(M)}$, we only need to prove the other implication. So, suppose that $\{|q_n|^{1/2}x_n\}_{-1}^{\infty} \in \ell^2$ for all $x \in D_{T(M)}$. In the formula

$$\sum_{n=0}^{m} p_n |\Delta x_n|^2 = p_m\Delta x_m \overline{x}_{m+1} - p_{-1}\Delta x_{-1}\overline{x}_0 + \sum_{n=0}^{m} \overline{x}_n M x_n - \sum_{n=0}^{m} q_n |x_n|^2, \tag{3.54}$$

the sums on the right converge as $m \to \infty$. Thus, we see that $\{p_n^{1/2}|\Delta x_n|\}_{-1}^{\infty} \notin \ell^2$ only if $\lim_{m\to\infty} p_m\Delta x_m\overline{x}_{m+1} = \infty$. But,

$$p_m |\Delta x_m \overline{x}_{m+1}| \le p_m |\Delta x_m| (|x_{m+1}| + |x_m|) \le p_m \Delta(|x_m|^2), \tag{3.55}$$

and hence

$$\lim_{m\to\infty} p_m\Delta(|x_m|^2) = \infty. \tag{3.56}$$

This implies, since $p_m > 0$ for all $m \in \mathbb{N}$, that $\{|x_n|^2\}_{-1}^{\infty}$ is monotonic increasing, that is, $\Delta|x_n|^2 \geq 0$ for all large n. We now have two cases: either $\{q_n\}_{-1}^{\infty} \notin \ell^1$ or $\{q_n\}_{-1}^{\infty} \in \ell^1$. If $\{q_n\}_{-1}^{\infty} \notin \ell^1$, then we get a contradiction to the assumption since this would imply that $\{|q_n|^{1/2}x_n\}_{-1}^{\infty} \notin \ell^1$. So, $\{q_n\}_{-1}^{\infty}$ must be in ℓ^1. Then, $\Delta(|x_n|^2) > p_n^{-1}$ since, from (3.56), $p_n\Delta(|x_n|^2) > 1$ for large enough $n \in \mathbb{N}$. This implies, for some $m_0 \in \mathbb{N}$, that

$$|x_m|^2 \geq |x_m|^2 - |x_{m_0-1}|^2 > \sum_{n=m_0}^{m} p_{n-1}^{-1} \quad m \in \mathbb{N},\ m > m_0. \tag{3.57}$$

So,

$$\infty > \sum_{n=m_0}^{\infty} w_n|x_n|^2 > \sum_{n=m_0}^{\infty} w_n\left(\sum_{k=m_0}^{n} p_{k-1}^{-1}\right), \tag{3.58}$$

which is a contradiction to the assumption that $\{w_m\sum_{n=-1}^{m} p_n^{-1}\}_{m=-1}^{\infty} \notin \ell^1$, and hence $\{p_n^{1/2}|\Delta x_n|\}_{-1}^{\infty}$ is in ℓ^2, and M is D on $D_{T(M)}$ and the theorem is therefore proved. □

Remarks 3.7. (1) $w \notin \ell^1$ is a sufficient condition for Theorem 3.6 to hold. But, if $w \in \ell^1$, then the condition on p and w, that is,

$$\left\{w_m\sum_{n=-1}^{m} p_n^{-1}\right\}_{m=-1}^{\infty} \notin \ell^1, \tag{3.59}$$

is in general stronger than the requirement that $p^{-1} \notin \ell^1$.

(2) If $w \in \ell^1$, then, for any $m \in \mathbb{N} \cup \{-1\}$,

$$\sum_{n=-1}^{m} w_n\left(\sum_{k=-1}^{n} p_k^{-1}\right) = \sum_{n=-1}^{m} p_n^{-1}\left(\sum_{k=n}^{m} w_k\right), \quad n < m. \tag{3.60}$$

This follows by using the summation-by-parts formula. As $m \to \infty$, we see that the condition in Theorem 3.6 is equivalent to the condition that

$$\left\{p_n^{-1}\sum_{k=n}^{\infty} w_k\right\}_{n=-1}^{\infty} \notin \ell^1 \quad \text{when } w \in \ell^1. \tag{3.61}$$

For example, if $m < \infty$ and $w = 1$, this condition becomes

$$\sum_{n=-1}^{\infty} p_n^{-1}(m-n) = \infty. \tag{3.62}$$

Theorem 3.8. *Suppose that $p_n > 0$ for all n, $w/p \notin \ell^1$, and $\{w_n/w_{n+1}\}_{-1}^{\infty}$ is bounded above. Then, M is SLP on $D_{T(M)}$ if and only if M is WD on $D_{T(M)}$.*

Proof. Since *SLP* always implies *WD* by Corollary 2.4, we only need to prove that $WD \Rightarrow SLP$ under the conditions in the hypothesis. So, suppose that M satisfies the *WD* property, that is, $\beta = \lim_{m\to\infty} p_n \Delta x_n x_{n+1}$ exists and is finite for all $x \in D_{T(M)}$, but M is not *SLP*, that is, $\beta \neq 0$. We show that $\beta \neq 0$ leads to a contradiction under the hypothesis, and hence M is *SLP*. So, suppose that

$$\beta = \lim_{m\to\infty} p_m \Delta x_m x_{m+1} \neq 0 \quad \forall x \in D_{T(M)}. \tag{3.63}$$

Now, multiplying both sides of the following by $\overline{\beta}$ and w_m, and summing over m:

$$x_{m+1}\Delta x_m = x_{m+1}^2 - x_m x_{m+1}, \tag{3.64}$$

we have

$$\begin{aligned}\sum_{m=0}^{\infty} (\overline{\beta} p_m \Delta x_m x_{m+1}) w_m p_m^{-1} \\ = \overline{\beta}\left\{ \sum_{m=0}^{\infty} w_{m+1} x_{m+1}^2 \left(\frac{w_m}{w_{m+1}}\right) - \sum_{m=0}^{\infty} (w_m w_{m+1})^{1/2} x_m x_{m+1} \left(\frac{w_m}{w_{m+1}}\right)^{1/2} \right\}.\end{aligned} \tag{3.65}$$

Under the conditions of the hypothesis, the left-hand side of this equality is ∞ while the right-hand side is finite. This contradiction leads us to say that $\beta = 0$ and M is *SLP* on $D_{T(M)}$. Hence the theorem is proved. □

Remark 3.9. As a final remark, Theorem 3.1(c) demonstrates that when $w, p^{-1}, q \in \ell^1$ *WD* does not imply *SLP* or even *LP*. Thus, for the equivalency of *WD* and *SLP*, the hypothesis of Theorem 3.8 is needed. For example, when $w = 1$, the requirements for the result $SLP \iff WD$ become $\sum_{n=-1}^{\infty} p_n^{-1} = \infty$.

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