Hindawi Publishing Corporation
Advances in Difference Equations
Volume 2008, Article ID 247071, 11 pages
doi:10.1155/2008/247071

Research Article

Existence Theorems of Periodic Solutions for Second-Order Nonlinear Difference Equations

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Received 14 August 2007; Accepted 14 November 2007

Recommended by Patricia J. Y. Wong

The authors consider the second-order nonlinear difference equation of the type $\Delta(p_n(\Delta x_{n-1})^{\delta}) + q_n x_n^{\delta} = f(n, x_n), n \in \mathbb{Z}$, using critical point theory, and they obtain some new results on the existence of periodic solutions.

1. Introduction

We denote by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ the set of all natural numbers, integers, and real numbers, respectively. For $a, b \in \mathbb{Z}$, define $\mathbb{Z}(a) = \{a, a+1, \dots\}$, $\mathbb{Z}(a, b) = \{a, a+1, \dots, b\}$ when $a \le b$.

Consider the nonlinear second-order difference equation

$$\Delta\left(p_n\left(\Delta x_{n-1}\right)^{\delta}\right) + q_n x_n^{\delta} = f\left(n, x_n\right), \quad n \in \mathbb{Z}, \tag{1.1}$$

where the forward difference operator Δ is defined by the equation $\Delta x_n = x_{n+1} - x_n$ and

$$\Delta^2 x_{n-1} = \Delta\left(\Delta x_{n-1}\right) = \Delta x_n - \Delta x_{n-1}. \tag{1.2}$$

In (1.1), the given real sequences $\{p_n\}$, $\{q_n\}$ satisfy $p_{n+T} = p_n > 0$, $q_{n+T} = q_n$ for any $n \in \mathbb{Z}$, $f : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ is continuous in the second variable, and $f(n+T, z) = f(n, z)$ for a given positive integer T and for all $(n, z) \in \mathbb{Z} \times \mathbb{R}$. $(-1)^{\delta} = -1$, $\delta > 0$, and δ is the ratio of odd positive integers. By a solution of (1.1), we mean a real sequence $x = \{x_n\}$, $n \in \mathbb{Z}$, satisfying (1.1).

In [1, 2], the qualitative behavior of linear difference equations of type

$$\Delta\left(p_n \Delta x_n\right) + q_n x_n = 0 \tag{1.3}$$

has been investigated. In [3], the nonlinear difference equation

$$\Delta(p_n \Delta x_{n-1}) + q_n x_n = f(n, x_n) \tag{1.4}$$

has been considered. However, results on periodic solutions of nonlinear difference equations are very scarce in the literature, see [4, 5]. In particular, in [6], by critical point method, the existence of periodic and subharmonic solutions of equation

$$\Delta^2 x_{n-1} + f(n, x_n) = 0, \quad n \in \mathbb{Z}, \tag{1.5}$$

has been studied. Other interesting contributions can be found in some recent papers [7–11] and in references contained therein. It is interesting to study second-order nonlinear difference equations (1.1) because they are discrete analogues of differential equation

$$(p(t)\varphi(u'))' + f(t, u) = 0. \tag{1.6}$$

In addition, they do have physical applications in the study of nuclear physics, gas aerodynamics, infiltrating medium theory, and plasma physics as evidenced in [12, 13].

The main purpose here is to develop a new approach to the above problem by using critical point method and to obtain some sufficient conditions for the existence of periodic solutions of (1.1).

Let X be a real Hilbert space, $I \in C^1(\mathbf{X}, \mathbb{R})$, which implies that I is continuously Fréchet differentiable functional defined on X. I is said to be satisfying Palais-Smale condition (P-S condition) if any sequence $\{I(u_n)\}$ is bounded, and $I'(u_n) \to 0$ as $n \to \infty$ possesses a convergent subsequence in X. Let B_ρ be the open ball in X with radius ρ and centered at 0, and let ∂B_ρ denote its boundary.

Lemma 1.1 (mountain pass lemma, see [14]). *Let X be a real Hilbert space, and assume that $I \in C^1(X, \mathbb{R})$ satisfies the P-S condition and the following conditions:*

(I_1) *there exist constants $\rho > 0$ and $a > 0$ such that $I(x) \geq a$ for all $x \in \partial B_\rho$, where $B_\rho = \{x \in X : \|x\|_X < \rho\}$;*

(I_2) *$I(0) \leq 0$ and there exists $x_0 \overline{\in} B_\rho$ such that $I(x_0) \leq 0$.*

Then $c = \inf_{h\in\Gamma} \sup_{s\in[0,1]} I(h(s))$ is a positive critical value of I, where

$$\Gamma = \{h \in C([0,1], X) : h(0) = 0, \ h(1) = x_0\}. \tag{1.7}$$

Lemma 1.2 (saddle point theorem, see [14, 15]). *Let X be a real Banach space, $X = X_1 \oplus X_2$, where $X_1 \neq \{0\}$ and is finite dimensional. Suppose $I \in C^1(\mathbf{X}, \mathbb{R})$ satisfies the P-S condition and*

(I_3) *there exist constants $\sigma, \rho > 0$ such that $I|_{\partial B_\rho \bigcap X_1} \leq \sigma$;*

(I_4) *there is $e \in \mathbf{B}_p \bigcap \mathbf{X}_1$ and a constant $\omega > \sigma$ such that $I|_{e+X_2} \geq \omega$.*

Then I possesses a critical value $c \geq \omega$ and

$$c = \inf_{h\in\Gamma} \max_{u\in B_\rho \bigcap X_1} I(h(u)), \tag{1.8}$$

where $\Gamma = \{h \in C(\overline{\mathbf{B}}_\rho \bigcap \mathbf{X}_1, \mathbf{X}) | h|_{\partial B_\rho \bigcap X_1} = id\}$.

2. Preliminaries

In this section, we are going to establish the corresponding variational framework for (1.1).

Let Ω be the set of sequences

$$x = \{x_n\}_{n\in\mathbb{Z}} = (\ldots, x_{-n}, \ldots, x_{-1}, x_0, x_1, \ldots, x_n, \ldots), \tag{2.1}$$

that is,

$$\Omega = \{x = \{x_n\} : x_n \in \mathbb{R},\ n \in \mathbb{Z}\}. \tag{2.2}$$

For any $x, y \in \Omega,\ a, b \in \mathbb{R}$, $ax + by$ is defined by

$$ax + by := \{ax_n + by_n\}_{n=-\infty}^{+\infty}. \tag{2.3}$$

Then Ω is a vector space. For given positive integer T, E_T is defined as a subspace of Ω by

$$E_T = \{x = \{x_n\} \in \Omega : x_{n+T} = x_n,\ n \in \mathbb{Z}\}. \tag{2.4}$$

Clearly, E_T is isomorphic to $\mathbb{R}^T$, and can be equipped with inner product

$$\langle x, y\rangle = \sum_{i=1}^{T} x_i y_i, \quad \forall x, y \in E_T, \tag{2.5}$$

by which the norm $\|\cdot\|$ can be induced by

$$\|x\| := \left(\sum_{i=1}^{T} x_i^2\right)^{1/2}, \quad \forall x \in E_T. \tag{2.6}$$

It is obvious that E_T with the inner product defined by (2.5) is a finite-dimensional Hilbert space and linearly homeomorphic to $\mathbb{R}^T$. Define the functional J on E_T as follows:

$$J(x) = \frac{1}{\delta+1}\sum_{n=1}^{T} p_n(\Delta x_{n-1})^{\delta+1} - \frac{1}{\delta+1}\sum_{n=1}^{T} q_n x_n^{\delta+1} + \sum_{n=1}^{T} F(n, x_n), \quad \forall x \in E_T, \tag{2.7}$$

where $F(t, z) = \int_0^z f(t, s)ds$. Clearly, $J \in \mathbf{C}^1(E_T, \mathbb{R})$, and for any $x = \{x_n\}_{n\in\mathbb{Z}} \in \mathbf{E}_T$, by using $x_0 = x_T,\ x_1 = x_{T+1}$, we can compute the partial derivative as

$$\frac{\partial J}{\partial x_n} = -\Delta[p_n(\Delta x_{n-1})^{\delta}] - q_n x_n^{\delta} + f(n, x_n), \quad n \in \mathbb{Z}(1, T). \tag{2.8}$$

Thus $x = \{x_n\}_{n\in\mathbb{Z}}$ is a critical point of J on E_T (i.e., $J'(x) = 0$) if and only if

$$\Delta[p_n(\Delta x_{n-1})^{\delta}] + q_n x_n^{\delta} = f(n, x_n), \quad n \in \mathbb{Z}(1, T). \tag{2.9}$$

By the periodicity of x_n and $f(n, z)$ in the first variable n, we have reduced the existence of periodic solutions of (1.1) to that of critical points of J on E_T. In other words, the functional J is just the variational framework of (1.1). For convenience, we identify $x \in E_T$ with $x = (x_1, x_2, \ldots, x_T)^T$. Denote $\mathbf{W} = \{(x_1, x_2, \ldots, x_T)^T \in \mathbf{E}_T : x_i \equiv v,\ v \in \mathbb{R},\ i \in \mathbb{Z}(1, T)\}$ and $\mathbf{W}^{\perp} = Y$ such that $\mathbf{E}_T = \mathbf{Y} \oplus \mathbf{W}$. Denote other norm $\|\cdot\|_r$ on $\mathbf{E}_T$ as follows (see, e.g., [16]): $\|x\|_r = (\sum_{i=1}^{T} |x_i|^r)^{1/r}$, for all $x \in \mathbf{E}_T$ and $r > 1$. Clearly, $\|x\|_2 = \|x\|$. Due to $\|\cdot\|_{r_1}$ and $\|\cdot\|_{r_2}$ being equivalent when $r_1, r_2 > 1$, there exist constants c_1, c_2, c_3, and c_4 such that $c_2 \geq c_1 > 0$, $c_4 \geq c_3 > 0$, and

$$c_1\|x\| \leq \|x\|_{\delta+1} \leq c_2\|x\|, \tag{2.10}$$

$$c_3\|x\| \leq \|x\|_{\beta} \leq c_4\|x\|, \tag{2.11}$$

for all $x \in \mathbf{E}_T$, $\delta > 0$ and $\beta > 1$.

3. Main results

In this section, we will prove our main results by using critical point theorem. First, we prove two lemmas which are useful in the proof of theorems.

Lemma 3.1. *Assume that the following conditions are satisfied:*

(F_1) *there exist constants $a_1 > 0$, $a_2 > 0$, and $\beta > \delta + 1$ such that*

$$\int_0^z f(n,s)ds \leq -a_1|z|^{\beta} + a_2, \quad \forall z \in \mathbb{R}; \tag{3.1}$$

(F_2)

$$q_n \leq 0, \quad \forall n \in \mathbb{Z}. \tag{3.2}$$

Then the functional

$$\mathbf{J}(x) = \frac{1}{\delta+1}\sum_{n=1}^{T} p_n(\Delta x_{n-1})^{\delta+1} - \frac{1}{\delta+1}\sum_{n=1}^{T} q_n x_n^{\delta+1} + \sum_{n=1}^{T} F(n,x_n) \tag{3.3}$$

satisfies P-S condition.

Proof. For any sequence $\{x^{(l)}\} \subset E_T$, with $J(x^{(l)})$ being bounded and $J'(x^{(l)}) \to 0$ as $l \to +\infty$, there exists a positive constant M such that $|J(x^{(l)})| \leq M$. Thus, by (F_1),

$$\begin{aligned} -M \leq J(x^{(l)}) &= \frac{1}{\delta+1}\sum_{n=1}^{T}\left[p_n\left(x_n^{(l)} - x_{n-1}^{(l)}\right)^{\delta+1} - q_n\left(x_n^{(l)}\right)^{\delta+1}\right] + \sum_{n=1}^{T} F(n,x_n^{(l)}) \\ &\leq \frac{1}{\delta+1}\sum_{n=1}^{T} p_n 2^{\delta+1}\left(\left(x_n^{(l)}\right)^{\delta+1} + \left(x_{n-1}^{(l)}\right)^{\delta+1}\right) - \frac{1}{\delta+1}\sum_{n=1}^{T} q_n\left(x_n^{(l)}\right)^{\delta+1} + \sum_{n=1}^{T} F(n,x_n^{(l)}) \\ &\leq \frac{2^{\delta+1}}{\delta+1}\sum_{n=1}^{T}(p_n + p_{n+1})\left(x_n^{(l)}\right)^{\delta+1} - \frac{1}{\delta+1}\sum_{n=1}^{T} q_n\left(x_n^{(l)}\right)^{\delta+1} - a_1\sum_{n=1}^{T}\left|x_n^{(l)}\right|^{\beta} + a_2 T \\ &= \frac{1}{\delta+1}\sum_{n=1}^{T}\left[2^{\delta+1}(p_n + p_{n+1}) - q_n\right]\left(x_n^{(l)}\right)^{\delta+1} - a_1\left\|x^{(l)}\right\|_{\beta}^{\beta} + a_2 T. \end{aligned} \tag{3.4}$$

Set

$$A_0 = \max_{n\in\mathbb{Z}(1,T)}\left[2^{\delta+1}(p_n + p_{n+1}) - q_n\right]. \tag{3.5}$$

Then $A_0 > 0$. Also, by the above inequality, we have

$$-M \leq J(x^{(l)}) \leq \frac{A_0}{\delta+1}\left\|x^{(l)}\right\|_{\delta+1}^{\delta+1} - a_1\left\|x^{(l)}\right\|_{\beta}^{\beta} + a_2 T. \tag{3.6}$$

In view of

$$\sum_{n=1}^{T}\left|x_n^{(l)}\right|^{\delta+1} \le T^{(\beta-\delta-1)/\beta}\left(\sum_{n=1}^{T}\left|x_n^{(l)}\right|^{\beta}\right)^{(\delta+1)/\beta}, \tag{3.7}$$

we have

$$\left\|x^{(l)}\right\|_{\beta}^{\beta} \ge T^{(\delta+1-\beta)/(\delta+1)}\left\|x^{(l)}\right\|_{\delta+1}^{\beta}. \tag{3.8}$$

Then we get

$$-M \le J(x^{(l)}) \le \frac{A_0}{\delta+1}\left\|x^{(l)}\right\|_{\delta+1}^{\delta+1} - a_1T^{(\delta+1-\beta)/(\delta+1)}\left\|x^{(l)}\right\|_{\delta+1}^{\beta} + a_2T. \tag{3.9}$$

Therefore, for any $l \in \mathbb{N}$,

$$a_1T^{(\delta+1-\beta)/(\delta+1)}\left\|x^{(l)}\right\|_{\delta+1}^{\beta} - \frac{A_0}{\delta+1}\left\|x^{(l)}\right\|_{\delta+1}^{\delta+1} \le M + a_2T. \tag{3.10}$$

Since $\beta > \delta+1$, the above inequality implies that $\{x^{(l)}\}$ is a bounded sequence in E_T. Thus $\{x^{(l)}\}$ possesses convergent subsequences, and the proof is complete. □

Theorem 3.2. *Suppose that* (F_1) *and following conditions hold:*

(F_3) *for each* $n \in \mathbb{Z}$,

$$\lim_{z\to 0}\frac{f(n,z)}{z^{\delta}} = 0; \tag{3.11}$$

(F_4)

$$q_n < 0, \quad \forall n \in \mathbb{Z}(1,T). \tag{3.12}$$

Then there exist at least two nontrivial T-periodic solutions for (1.1).

Proof. We will use Lemma 1.1 to prove Theorem 3.2. First, by Lemma 3.1, J satisfies P-S condition. Next, we will prove that conditions (I_1) and (I_2) hold. In fact, by (F_3), there exists $\rho > 0$ such that for any $|z| < \rho$ and $n \in \mathbb{Z}(1,T)$,

$$|F(n,z)| \le -\frac{q_{\max}}{2(\delta+1)}z^{\delta+1}, \tag{3.13}$$

where $q_{\max} = \max_{n\in\mathbb{Z}(1,T)} q_n < 0$. Thus for any $x \in E_T$, $\|x\| \le \rho$ for all $n \in \mathbb{Z}(1,T)$, we have

$$\begin{aligned} J(x) &\ge -\frac{q_{\max}}{\delta+1}\sum_{n=1}^{T}x_n^{\delta+1} + \frac{q_{\max}}{2(\delta+1)}\sum_{n=1}^{T}x_n^{\delta+1} \\ &= -\frac{q_{\max}}{2(\delta+1)}\|x\|_{\delta+1}^{\delta+1} \\ &\ge -\frac{q_{\max}}{2(\delta+1)}c_1^{\delta+1}\|x\|_2^{\delta+1}. \end{aligned} \tag{3.14}$$

Taking $a = -c_1^{\delta+1}(q_{\max}/2(\delta+1))\rho^{\delta+1}$, we have

$$J(x)|_{\partial B_\rho} \geq a > 0, \tag{3.15}$$

and the assumption (I_1) is verified. Clearly, $J(0) = 0$. For any given $w \in E_T$ with $\|w\| = 1$ and a constant $\alpha > 0$,

$$\begin{aligned} J(\alpha w) &= \frac{1}{\delta+1}\sum_{n=1}^{T}\left[p_n(\alpha w_n - \alpha w_{n-1})^{\delta+1} - q_n(\alpha w_n)^{\delta+1}\right] + \sum_{n=1}^{T}F(n,\alpha w_n) \\ &\leq \frac{1}{\delta+1}\sum_{n=1}^{T}\left[p_n(2\alpha)^{\delta+1} - q_n\alpha^{\delta+1}\right] - a_1\sum_{n=1}^{T}|\alpha w_n|^{\beta} + a_2 T \\ &\leq \frac{1}{\delta+1}\sum_{n=1}^{T}\left[2^{\delta+1}p_n - q_n\right]\alpha^{\delta+1}\alpha^{\delta+1} - a_1 T^{(2-\beta)/2}\alpha^{\beta} + a_2 T \\ &\longrightarrow -\infty, \quad (\alpha \longrightarrow +\infty). \end{aligned} \tag{3.16}$$

Thus we can easily choose a sufficiently large α such that $\alpha > \rho$ and for $\overline{x} = \alpha w \in E_T$, $J(\overline{x}) < 0$. Therefore, by Lemma 1.1, there exists at least one critical value $c \geq a > 0$. We suppose that $\tilde{x}$ is a critical point corresponding to c, that is, $J(\tilde{x}) = c$, and $J'(\tilde{x}) = 0$. By a similar argument to the proof of Lemma 3.1, for any $x \in E_T$, there exists $\widehat{x} \in E_T$ such that $J'(\widehat{x}) = c_{\max}$. Clearly, $\widehat{x} \neq 0$. If $\tilde{x} \neq \widehat{x}$, and the proof is complete; otherwise, $\tilde{x} = \widehat{x}$ and $c = c_{\max}$. By Lemma 1.1,

$$c = \inf_{h\in\Gamma}\sup_{s\in[0,1]} J(h(s)), \tag{3.17}$$

where $\Gamma = \{h \in C([0,1], E_T) \mid h(0) = 0,\ h(1) = \overline{x}\}$. Then for any $h \in \Gamma$, $c_{\max} = \max_{s\in[0,1]} J(h(s))$. By the continuity of $J(h(s))$ in s, $J(0) \leq 0$ and $J(\overline{x}) < 0$ show that there exists some $s_0 \in (0,1)$ such that $J(h(s_0)) = c_{\max}$. If we choose $h_1, h_2 \in \Gamma$ such that the intersection $\{h_1(s) \mid s \in (0,1)\} \bigcap \{h_2(s) \mid s \in (0,1)\}$ is empty, then there exist $s_1, s_2 \in (0,1)$ such that $J(h_1(s_1)) = J(h_2(s_2)) = c_{\max}$. Thus we obtain two different critical points $x^1 = h_1(s_1)$, $x^2 = h_2(s_2)$ of J in E_T. In this case, in fact, we may obtain at least two nontrivial critical points which correspond to the critical value $c_{\max}$. The proof of Theorem 3.2 is complete. When $f(n, x_n) \equiv h_n$, we have the following results. □

Theorem 3.3. *Assume that the following conditions hold:*

(G_1)

$$qn < 0, \quad \forall n \in \mathbb{Z}(1,T); \tag{3.18}$$

(G_2)

$$\frac{1}{c_1^{\delta+1}}\left(\sum_{n=1}^{T}h_n^2\right)^{(\delta+1)/2}\sum_{n=1}^{T}(-q_n) < \left(p_{\min}\lambda_2^{(\delta+1)/2} - q_{\max}\right)\left(\sum_{n=1}^{T}h_n\right)^{\delta+1}, \tag{3.19}$$

where $p_{\min} = \min_{n\in\mathbb{Z}(1,T)} p_n$, $q_{\max} = \max_{n\in\mathbb{Z}(1,T)} q_n$, c_1 *is a constant in* (2.10), *and* λ_2 *is the minimal positive eigenvalue of the matrix*

$$A = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & -1 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ -1 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix}_{T\times T}. \tag{3.20}$$

Then equation

$$\Delta\left[p_n \Delta x_{n-1}\right]^{\delta} + q_n x_n^{\delta} = h_n, \quad n \in \mathbb{Z}, \tag{3.21}$$

possesses at least one T*-periodic solution.*

First, we proved the following lemma.

Lemma 3.4. *Assume that* (G_1) *holds, then the functional*

$$J(x) = \frac{1}{\delta+1}\sum_{n=1}^{T} p_n\left(\Delta x_{n-1}\right)^{\delta+1} - \frac{1}{\delta+1}\sum_{n=1}^{T} q_n x_n^{\delta+1} + \sum_{n=1}^{T} h_n x_n \tag{3.22}$$

satisfies P-S condition on E_T.

Proof. For any sequence $\{x^{(l)}\} \subset E_T$ with $J(x^{(l)})$ being bounded and $J'(x^{(l)}) \to 0$ as $n \to +\infty$, there exists a positive constant M such that $|J(x^{(l)})| \le M$. In view of (G_3) and

$$\sum_{n=1}^{T} |h_n x_n^{(l)}| \le \left(\sum_{n=1}^{T} h_n^2\right)^{1/2} \left(\sum_{n=1}^{T} \left(x_n^{(l)}\right)^2\right)^{1/2}, \tag{3.23}$$

we have

$$\begin{aligned} M \ge J(x^{(l)}) &= \frac{1}{\delta+1}\sum_{n=1}^{T}\left[p_n\left(\Delta x_{n-1}^{(l)}\right)^{\delta+1}\right] - \frac{1}{\delta+1}\sum_{n=1}^{T} q_n\left(x_n^{(l)}\right)^{\delta+1} + \sum_{n=1}^{T} h_n x_n^{(l)} \\ &\ge -\frac{1}{\delta+1}\sum_{n=1}^{T} q_n\left(x_n^{(l)}\right)^{\delta+1} - \sum_{n=1}^{T}\left|h_n x_n^{(l)}\right| \\ &\ge -\frac{1}{\delta+1} q_{\max}\sum_{n=1}^{T}\left(x_n^{(l)}\right)^{\delta+1} - \left(\sum_{n=1}^{T} h_n^2\right)^{1/2}\left(\sum_{n=1}^{T}\left(x_n^{(l)}\right)^2\right)^{1/2} \\ &= -\frac{q_{\max}}{\delta+1}\|x^{(l)}\|_{\delta+1}^{\delta+1} - \left(\sum_{n=1}^{T} h_n^2\right)^{1/2}\|x^{(l)}\| \\ &\ge -\frac{q_{\max}}{\delta+1} c_1^{\delta+1}\|x^{(l)}\|^{\delta+1} - \left(\sum_{n=1}^{T} h_n^2\right)^{1/2}\|x^{(l)}\|. \end{aligned} \tag{3.24}$$

By $\delta + 1 > 1$, the above inequality implies that $\{x^{(l)}\}$ is a bounded sequence in E_T. Thus $\{x^{(l)}\}$ possesses a convergent subsequence, and the proof of Lemma 3.4 is complete. Now we prove Theorem 3.3 by the saddle point theorem. □

Proof of Theorem 3.3. For any $w = (z, z, \dots, z)^T \in W$, we have

$$J(w) = -\frac{1}{\delta+1}\sum_{n=1}^{T} q_n z^{\delta+1} + \sum_{n=1}^{T} h_n z. \tag{3.25}$$

Take $z = \left(\sum_{n=1}^{T} h_n / \sum_{n=1}^{T} q_n\right)^{1/\delta}$ and $\rho = \|w\| = T^{1/2}\left|\sum_{n=1}^{T} h_n / \sum_{n=1}^{T} q_n\right|^{1/\delta}$, then

$$J(w) = \frac{\delta}{\delta+1}\frac{\left(\sum_{n=1}^{T} h_n\right)^{(\delta+1)/\delta}}{\left|\sum_{n=1}^{T} q_n\right|^{1/\delta}}. \tag{3.26}$$

Set

$$\sigma = \frac{\delta}{\delta+1}\frac{\left(\sum_{n=1}^{T} h_n\right)^{(\delta+1)/\delta}}{\left|\sum_{n=1}^{T} q_n\right|^{1/\delta}}, \tag{3.27}$$

then we have

$$J(w) = \sigma, \quad \forall w \in \partial B_\rho \bigcap Y. \tag{3.28}$$

On the other hand, for any $x \in Y$, we have

$$\begin{aligned} J(x) &= \frac{1}{\delta+1}\sum_{n=1}^{T} p_n (\Delta x_{n-1})^{\delta+1} - \frac{1}{\delta+1}\sum_{n=1}^{T} q_n x_n^{\delta+1} + \sum_{n=1}^{T} h_n x_n \\ &\geq \frac{p_{\min}}{\delta+1}\sum_{n=1}^{T} (\Delta x_{n-1})^{\delta+1} - \frac{q_{\max}}{\delta+1}\sum_{n=1}^{T} x_n^{\delta+1} - \sum_{n=1}^{T} |h_n x_n| \\ &\geq \frac{p_{\min}}{\delta+1} c_1^{\delta+1}\left[\sum_{n=1}^{T} (\Delta x_{n-1})^2\right]^{(\delta+1)/2} - \frac{q_{\max}}{\delta+1}\|x\|_{\delta+1}^{\delta+1} - \left(\sum_{n=1}^{T} h_n^2\right)^{1/2}\|x\| \\ &= \frac{p_{\min}}{\delta+1} c_1^{\delta+1}(x^T A x)^{(\delta+1)/2} - \frac{q_{\max}}{\delta+1}\|x\|_{\delta+1}^{\delta+1} - \sum_{n=1}^{T} |h_n x_n|, \end{aligned} \tag{3.29}$$

where $x^T = (x_1, x_2, \dots, x_T)$.

Clearly, $\lambda_1 = 0$ is an eigenvalue of the matrix A and $\xi = (v, v, \dots, v)^T \in E_T$ is an eigenvector of A corresponding to 0, where $v \neq 0$, $v \in R$. Let $\lambda_2, \lambda_3, \dots, \lambda_T$ be the other eigenvalues of A. By matrix theory, we have $\lambda_j > 0$ for all $j \in \mathbb{Z}(2, T)$. Without loss of generality, we may assume that $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_T$, then for any $x \in Y$,

$$\begin{aligned} J(x) &\geq \frac{p_{\min}}{\delta+1} c_1^{\delta+1}\lambda_2^{(\delta+1)/2}\|x\|^{\delta+1} - \frac{q_{\max}}{\delta+1}\|x\|_{\delta+1}^{\delta+1} - \left(\sum_{n=1}^{T} h_n^2\right)^{1/2}\|x\| \\ &= \left[\frac{p_{\min}}{\delta+1} c_1^{\delta+1}\lambda_2^{(\delta+1)/2} - \frac{q_{\max}}{\delta+1} c_1^{\delta+1}\right]\|x\|^{\delta+1} - \left(\sum_{n=1}^{T} h_n^2\right)^{1/2}\|x\| \\ &\geq -\frac{\delta}{\delta+1}\left(\sum_{n=1}^{T} h_n^2\right)^{1/2}\left(\frac{\left(\sum_{n=1}^{T} h_n^2\right)^{1/2}}{p_{\min} c_1^{\delta+1}\lambda_2^{(\delta+1)/2} - q_{\max} c_1^{\delta+1}}\right)^{1/\delta}, \end{aligned} \tag{3.30}$$

as one finds by minimizing with respect to $\|x\|$. That is

$$J(x) \geq -\frac{\delta}{\delta+1} \frac{\left(\sum_{n=1}^{T} h_n^2\right)^{(\delta+1)/2\delta} (1/c_1)^{(\delta+1)/\delta}}{\left(p_{\min} \lambda_2^{(\delta+1)/2} - q_{\max}\right)^{1/\delta}}. \tag{3.31}$$

Set

$$w_0 = -\frac{\delta}{\delta+1} \frac{\left(\sum_{n=1}^{T} h_n^2\right)^{(\delta+1)/2\delta} (1/c_1)^{(\delta+1)/\delta}}{\left(p_{\min} \lambda_2^{(\delta+1)/2} - q_{\max}\right)^{1/\delta}}, \tag{3.32}$$

then by (G_2), we have

$$J(x) \geq w_0 > \sigma, \quad \forall x \in Y. \tag{3.33}$$

This implies that the assumption of saddle point theorem is satisfied. Thus there exists at least one critical point of J on E_T, and the proof is complete. When $q_n > 0$, we have the following result. □

Theorem 3.5. *Assume that the following conditions are satisfied:*

(G_3) $2^{\delta+1}[p_n + p_{n+1}] < q_n$, $q_n > 0$ *for all* $n \in \mathbb{Z}(1,T)$;

(G_4) $(\sum_{n=1}^{T} h_n^2)^{(\delta+1)/2\delta} (\sum_{n=1}^{T} q_n)^{1/\delta} C_1^{\delta+1} < -A_0 (\sum_{n=1}^{T} h_n)^{(\delta+1)/\delta}$,

where $A_0 = \max_{n\in\mathbb{Z}(1,T)} [2^{\delta+1}(p_n + p_{n+1}) - q_n]$.

Then (3.21) *possesses at least one* T*-periodic solution.*

Before proving Theorem 3.5, first, we prove the following result.

Lemma 3.6. *Assume that* (G_3) *holds, then* $J(x)$ *defined by* (3.22) *satisfies P-S condition.*

Proof. For any sequence $\{x^{(l)}\} \in E_T$ with $J(x^{(l)})$ being bounded and $J'(x^{(l)}) \to 0$ as $n \to +\infty$, there exists a positive constant M such that $|J(x^{(l)})| \leq M$.

Thus

$$\begin{aligned}
-M \leq J(x^{(l)}) &\leq \frac{1}{\delta+1} \sum_{n=1}^{T} p_n (\Delta x_{n-1}^{(l)})^{\delta+1} - \frac{1}{\delta+1} \sum_{n=1}^{T} q_n (x_n^{(l)})^{\delta+1} + \sum_{n=1}^{T} h_n x_n^{(l)} \\
&\leq \frac{2^{\delta+1}}{\delta+1} \sum_{n=1}^{T} (p_n + p_{n+1}) (x_n^{(l)})^{\delta+1} - \frac{1}{\delta+1} \sum_{n=1}^{T} q_n (x_n^{(l)})^{\delta+1} + \sum_{n=1}^{T} |h_n x_n^{(l)}| \\
&\leq \frac{1}{\delta+1} \sum_{n=1}^{T} [2^{\delta+1}(p_n + p_{n+1}) - q_n] (x_n^{(l)})^{\delta+1} + \left(\sum_{n=1}^{T} h_n^2\right)^{1/2} \|x^{(l)}\| \\
&\leq \frac{1}{\delta+1} A_0 \|x^{(l)}\|_{\delta+1}^{\delta+1} + \left(\sum_{n=1}^{T} h_n^2\right)^{1/2} \|x^{(l)}\| \\
&\leq \frac{A_0}{\delta+1} c_2^{\delta+1} \|x^{(l)}\|^{\delta+1} + \left(\sum_{n=1}^{T} h_n^2\right)^{1/2} \|x^{(l)}\|.
\end{aligned} \tag{3.34}$$

That is,

$$-c_2^{\delta+1}\frac{A_0}{\delta+1}\left\|x^{(l)}\right\|^{\delta+1}-\left(\sum_{n=1}^{T}h_n^2\right)^{1/2}\left\|x^{(l)}\right\|\leq M,\quad \forall n\in\mathbb{N}. \tag{3.35}$$

By $\delta+1>1$, the above inequality implies that $\{x^{(l)}\}$ is a bounded sequence in E_T. Thus $\{x^{(l)}\}$ possesses convergent subsequences, and the proof is complete. □

Proof of Theorem 3.5. For any $w=(z,z,\dots,z)^T\in\mathbf{W}$, we have

$$J(\omega)=-\frac{1}{\delta+1}\sum_{n=1}^{T}q_nz^{\delta+1}+\sum_{n=1}^{T}h_nz. \tag{3.36}$$

Take $z=(\sum_{n=1}^{T}h_n/\sum_{n=1}^{T}q_n)$, $\rho=\|w\|=T^{1/2}|\sum_{n=1}^{T}h_n/\sum_{n=1}^{T}|^{1/\delta}$, then

$$J(w)=\frac{\delta}{\delta+1}\frac{\left(\sum_{n=1}^{T}h_n\right)^{(\delta+1)/\delta}}{\left|\sum_{n=1}^{T}q_n\right|^{1/\delta}},\quad \forall w\in\partial B_\rho\bigcap W. \tag{3.37}$$

Set

$$\sigma=\frac{\delta}{\delta+1}\frac{\left(\sum_{n=1}^{T}h_n\right)^{(\delta+1)/\delta}}{|\sum_{n=1}^{T}q_n|^{1/\delta}}, \tag{3.38}$$

then $J(w)=\sigma$ for all $w\in\partial B_\rho\bigcap W$. On the other hand, for any $x\in Y$, we have

$$\begin{aligned}J(x)&\leq\frac{1}{\delta+1}\sum_{n=1}^{T}[2^{\delta+1}(p_n+p_{n+1})-q_n]x_n^{\delta+1}+\left(\sum_{n=1}^{T}h_n^2\right)^{1/2}\|x\|\\&\leq\frac{A_0}{\delta+1}c_2^{\delta+1}\|x\|^{\delta+1}+\left(\sum_{n=1}^{T}h_n^2\right)^{1/2}\|x\|\\&\leq-\frac{\delta}{\delta+1}\left(\frac{1}{A_0}\right)^{1/\delta}\left(\frac{1}{c_2}\right)^{(\delta+1)/\delta}\left(\sum_{n=1}^{T}h_n^2\right)^{(\delta+1)/2\delta}.\end{aligned} \tag{3.39}$$

Set $w_0=-\delta/(\delta+1)(1/A_0)^{1/\delta}(1/c_2)^{(\delta+1)/\delta}(\sum_{n=1}^{T}h_n^2)^{(\delta+1)/2\delta}$, then $J(x)\leq w_0<\sigma$. Thus $-J(x)$ satisfies the assumption of saddle point theorem, that is, there exists at least one critical point of J on E_T. This completes the proof of Theorem 3.5. □

Acknowledgment

This project is supported by specialized research fund for the doctoral program of higher education, Grant no. 20020532014.

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