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Research Article

Necessary and Sufficient Conditions for the Existence of Positive Solution for Singular Boundary Value Problems on Time Scales

Meiqiang Feng,[1] Xuemei Zhang,[2,3] Xiangui Li,[1] and Weigao Ge[3]

[1] *School of Science, Beijing Information Science & Technology University, Beijing 100192, China*
[2] *Department of Mathematics and Physics, North China Electric Power University, Beijing 102206, China*
[3] *Department of Applied Mathematics, Beijing Institute of Technology, Beijing 100081, China*

Correspondence should be addressed to Xuemei Zhang, zxm74@sina.com

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By constructing available upper and lower solutions and combining the Schauder's fixed point theorem with maximum principle, this paper establishes sufficient and necessary conditions to guarantee the existence of $C_{ld}[0,1]_{\mathbb{T}}$ as well as $C_{ld}^{\Delta}[0,1]_{\mathbb{T}}$ positive solutions for a class of singular boundary value problems on time scales. The results significantly extend and improve many known results for both the continuous case and more general time scales. We illustrate our results by one example.

1. Introduction

Recently, there have been many papers working on the existence of positive solutions to boundary value problems for differential equations on time scales; see, for example, [1–22]. This has been mainly due to its unification of the theory of differential and difference equations. An introduction to this unification is given in [10, 14, 23, 24]. Now, this study is still a new area of fairly theoretical exploration in mathematics. However, it has led to several important applications, for example, in the study of insect population models, neural networks, heat transfer, and epidemic models; see, for example, [9, 10].

Motivated by works mentioned previously, we intend in this paper to establish sufficient and necessary conditions to guarantee the existence of positive solutions for the singular dynamic equation on time scales:

$$x^{\Delta\nabla} + f(t,x) = 0, \quad t \in (0,1)_{\mathbb{T}}, \tag{1.1}$$

subject to one of the following boundary conditions:

$$x(0) = x(1) = 0, \tag{1.2}$$

or

$$x(0) = x^{\Delta}(1) = 0, \tag{1.3}$$

where $\mathbb{T}$ is a time scale, $(0,1)_{\mathbb{T}} = (0,1) \cap \mathbb{T}$, where 0 is right dense and 1 is left dense. and (H) $f : (0,1)_{\mathbb{T}} \times [0,+\infty) \to [0,+\infty)$ is continuous. Suppose further that $f(t,x)$ is nonincreasing with respect to x, and there exists a function $g(k) : [0,1] \to [1,\infty)$ such that

$$f(t,kx) \leq g(k)f(t,x), \quad \forall (t,x) \in (0,1)_{\mathbb{T}} \times [0,+\infty). \tag{1.4}$$

A necessary and sufficient condition for the existence of $C_{ld}[0,1]_{\mathbb{T}}$ as well as $C_{ld}^{\Delta}[0,1]_{\mathbb{T}}$ positive solutions is given by constructing upper and lower solutions and with the maximum principle. The nonlinearity $f(t,x)$ may be singular at $t = 0$ and/or $t = 1$. By singularity we mean that the functions f in (1.1) is allowed to be unbounded at the points $t = 0$ and/or $t = 1$. A function $x(t) \in C_{ld}[0,1]_{\mathbb{T}} \cap C_{ld}^{\Delta\nabla}(0,1)_{\mathbb{T}}$ is called a $C_{ld}[0,1]_{\mathbb{T}}$ (positive) solution of (1.1) if it satisfies (1.1) $(x(t) > 0, \text{for}\, t \in (0,1)_{\mathbb{T}})$; if even $x^{\Delta}(0^+), x^{\Delta}(1^-)$ exist, we call it is a $C_{ld}^{\Delta}[0,1]_{\mathbb{T}}$ solution.

To the best of our knowledge, there is very few literature giving sufficient and necessary conditions to guarantee the existence of positive solutions for singular boundary value problem on time scales. So it is interesting and important to discuss these problems. Many difficulties occur when we deal with them. For example, basic tools from calculus such as Fermat's theorem, Rolle's theorem, and the intermediate value theorem may not necessarily hold. So we need to introduce some new tools and methods to investigate the existence of positive solutions for problem (1.1) with one of the above boundary conditions.

The time scale related notations adopted in this paper can be found, if not explained specifically, in almost all literature related to time scales. The readers who are unfamiliar with this area can consult, for example, [6, 11–13, 25, 26] for details.

The organization of this paper is as follows. In Section 2, we provide some necessary background. In Section 3, the main results of problem (1.1)-(1.2) will be stated and proved. In Section 4, the main results of problem (1.1)–(1.3) will be investigated. Finally, in Section 5, one example is also included to illustrate the main results.

2. Preliminaries

In this section we will introduce several definitions on time scales and give some lemmas which are useful in proving our main results.

Definition 2.1. A time scale $\mathbb{T}$ is a nonempty closed subset of R.

Definition 2.2. Define the forward (backward) jump operator $\sigma(t)$ at t for $t < \sup \mathbb{T}(\rho(t)$ at t for $t > \inf \mathbb{T})$ by

$$\sigma(t) = \inf\{\tau > t : \tau \in \mathbb{T}\}\left(\rho(t) = \sup\{\tau < t : \tau \in \mathbb{T}\}\right) \tag{2.1}$$

for all $t \in \mathbb{T}$. We assume throughout that $\mathbb{T}$ has the topology that it inherits from the standard topology on R and say t is right scattered, left scattered, right dense and left dense if $\sigma(t) > t, \rho(t) < t, \sigma(t) = t$, and $\rho(t) = t$, respectively. Finally, we introduce the sets $\mathbb{T}^k$ and $\mathbb{T}_k$ which are derived from the time scale $\mathbb{T}$ as follows. If $\mathbb{T}$ has a left-scattered maximum t_1, then $\mathbb{T}^k = \mathbb{T} - t_1$, otherwise $\mathbb{T}^k = \mathbb{T}$. If $\mathbb{T}$ has a right-scattered minimum t_2, then $\mathbb{T}_k = \mathbb{T} - t_2$, otherwise $\mathbb{T}_k = \mathbb{T}$.

Definition 2.3. Fix $t \in \mathbb{T}$ and let $y : \mathbb{T} \to R$. Define $y^{\Delta}(t)$ to be the number (if it exists) with the property that given $\varepsilon > 0$ there is a neighborhood U of t with

$$\left|[y(\sigma(t)) - y(s)] - y^{\Delta}(t)[\sigma(t) - s]\right| < \varepsilon|\sigma(t) - s| \tag{2.2}$$

for all $s \in U$, where y^{Δ} denotes the (delta) derivative of y with respect to the first variable, then

$$g(t) := \int_a^t \omega(t,\tau)\Delta\tau \tag{2.3}$$

implies

$$g^{\Delta}(t) = \int_a^t \omega^{\Delta}(t,\tau)\Delta\tau + \omega(\sigma(t),\tau). \tag{2.4}$$

Definition 2.4. Fix $t \in \mathbb{T}$ and let $y : \mathbb{T} \to R$. Define $y^{\nabla}(t)$ to be the number (if it exists) with the property that given $\varepsilon > 0$ there is a neighborhood U of t with

$$\left|[y(\rho(t)) - y(s)] - y^{\nabla}(t)[\rho(t) - s]\right| < \varepsilon\left|\rho(t) - s\right| \tag{2.5}$$

for all $s \in U$. Call $y^{\nabla}(t)$ the (nabla) derivative of $y(t)$ at the point t.

If $\mathbb{T} = \mathbb{R}$ then $f^{\Delta}(t) = f^{\nabla}(t) = f'(t)$. If $\mathbb{T} = \mathbb{Z}$ then $f^{\Delta}(t) = f(t+1) - f(t)$ is the forward difference operator while $f^{\nabla}(t) = f(t) - f(t-1)$ is the backward difference operator.

Definition 2.5. A function $f : \mathbb{T} \to R$ is called rd-continuous provided that it is continuous at all right-dense points of $\mathbb{T}$ and its left-sided limit exists (finite) at left-dense points of $\mathbb{T}$. We let $C^0_{\mathrm{rd}}(\mathbb{T})$ denote the set of rd-continuous functions $f : \mathbb{T} \to R$.

Definition 2.6. A function $f : \mathbb{T} \to R$ is called *ld*-continuous provided that it is continuous at all left-dense points of $\mathbb{T}$ and its right-sided limit exists (finite) at right-dense points of $\mathbb{T}$. We let $C_{ld}(\mathbb{T})$ denote the set of *ld*-continuous functions $f : \mathbb{T} \to R$.

Definition 2.7. A function $F : \mathbb{T}^k \to R$ is called a delta-antiderivative of $f : \mathbb{T}^k \to R$ provided that $F^{\Delta}(t) = f(t)$ holds for all $t \in \mathbb{T}^k$. In this case we define the delta integral of f by

$$\int_a^t f(s)\Delta s = F(t) - F(a), \tag{2.6}$$

for all $a, t \in \mathbb{T}$.

Definition 2.8. A function $\Phi : \mathbb{T}_k \to R$ is called a nabla-antiderivative of $f : \mathbb{T}_k \to R$ provided that $\Phi^{\nabla}(t) = f(t)$ holds for all $t \in \mathbb{T}_k$. In this case we define the delta integral of f by

$$\int_a^t f(s)\nabla s = \Phi(t) - \Phi(a) \tag{2.7}$$

for all $a, t \in \mathbb{T}$.

Throughout this paper, we assume that $\mathbb{T}$ is a closed subset of R with $0, 1 \in \mathbb{T}$.

Let $E = C_{ld}[0,1]_{\mathbb{T}}$, equipped with the norm

$$\|x\| := \sup_{t \in [0,1]_{\mathbb{T}}} |x(t)|. \tag{2.8}$$

It is clear that E is a real Banach space with the norm.

Lemma 2.9 (Maximum Principle). *Let $a, b \in [0,1]_{\mathbb{T}}$ and $a < b$. If $x \in C_{ld}[0,1]_{\mathbb{T}} \cap C_{ld}^{\Delta\nabla}(0,1)_{\mathbb{T}}$, $x(a) \geq 0, x(b) \geq 0$, and $x^{\Delta\nabla}(t) \leq 0, t \in (a,b)_{\mathbb{T}}$. Then $x(t) \geq 0, t \in [a,b]_{\mathbb{T}}$.*

3. Existence of Positive Solution to (1.1)-(1.2)

In this section, by constructing upper and lower solutions and with the maximum principle Lemma 2.9, we impose the growth conditions on f which allow us to establish necessary and sufficient condition for the existence of (1.1)-(1.2).

We know that

$$G(t,s) = \begin{cases} s(1-t), & \text{if } 0 \leq s \leq t \leq 1, \\ t(1-s), & \text{if } 0 \leq t \leq s \leq 1 \end{cases} \tag{3.1}$$

is the Green's function of corresponding homogeneous BVP of (1.1)-(1.2).

We can prove that $G(t,s)$ has the following properties.

Proposition 3.1. *For $(t,s) \in [0,1]_{\mathbb{T}} \times [0,1]_{\mathbb{T}}$, one has*

$$\begin{gathered} G(t,s) \geq 0, \\ e(t)e(s) \leq G(t,s) \leq G(t,t) = t(1-t) = e(t). \end{gathered} \tag{3.2}$$

To obtain positive solutions of problem (1.1)-(1.2), the following results of Lemma 3.2 are fundamental.

Lemma 3.2. *Assume that* (H) *holds. If* $\int_0^{t_0}\nabla s\int_0^s f(s,\overline{u})\Delta t$ *and* $\int_0^{t_0}\Delta t\int_t^{t_0} f(s,\overline{u})\nabla s$ *exist and are finite, then one has*

$$\int_0^{t_0}\nabla s\int_0^s f(s,\overline{u})\Delta t=\int_0^{t_0}\Delta t\int_t^{t_0} f(s,\overline{u})\nabla s. \tag{3.3}$$

Proof. Without loss of generality, we suppose that there is only one right-scattered point $t_1\in[0,1]_{\mathbb{T}}$. Then we have

$$\begin{aligned}
\int_0^{t_0}\nabla s\int_0^s f(s,\overline{u})\Delta t&=\int_0^{t_1}\nabla s\int_0^s f(s,\overline{u})\Delta t+\int_{\sigma(t_1)}^{t_0}\nabla s\int_0^s f(s,\overline{u})\Delta t+\int_{t_1}^{\sigma(t_1)}\nabla s\int_0^s f(s,\overline{u})\Delta t\\
&=\int_0^{t_1}\Delta t\int_t^{t_1} f(s,\overline{u})\nabla s+\int_0^{\sigma(t_1)}\Delta t\int_{\sigma(t_1)}^{t_0} f(s,\overline{u})\nabla s\\
&\quad+\int_{\sigma(t_1)}^{t_0}\Delta t\int_t^{t_0} f(s,\overline{u})\nabla s+\mu(t_1)f(\sigma(t_1),\overline{u})\sigma(t_1)\\
&=\int_0^{t_1}\Delta t\int_t^{t_1} f(s,\overline{u})\nabla s+\int_{\sigma(t_1)}^{t_0}\Delta t\int_t^{t_0} f(s,\overline{u})\nabla s+\sigma(t_1)\int_{\sigma(t_1)}^{t_0} f(s,\overline{u})\nabla s\\
&\quad+\mu(t_1)f(\sigma(t_1),\overline{u})\sigma(t_1),\\
\int_0^{t_0}\Delta t\int_0^s f(s,\overline{u})\nabla s&=\int_0^{t_1}\Delta t\int_t^{t_0} f(s,\overline{u})\nabla s+\int_{t_1}^{\sigma(t_1)}\Delta t\int_t^{t_0} f(s,\overline{u})\nabla s+\int_{\sigma(t_1)}^{t_0}\Delta t\int_t^{t_0} f(s,\overline{u})\nabla s\\
&=\int_0^{t_1}\Delta t\int_t^{t_1} f(s,\overline{u})\nabla s+\int_0^{t_1}\Delta t\int_{t_1}^{t_0} f(s,\overline{u})\nabla s+\mu(t_1)\int_{t_1}^{t_0} f(s,\overline{u})\nabla s\\
&\quad+\int_{\sigma(t_1)}^{t_0}\Delta t\int_t^{t_0} f(s,\overline{u})\nabla s\\
&=\int_0^{t_1}\Delta t\int_t^{t_1} f(s,\overline{u})\nabla s+\left(t_1+\mu(t_1)\right)\int_{t_1}^{t_0} f(s,\overline{u})\nabla s+\int_{\sigma(t_1)}^{t_0}\Delta t\int_t^{t_0} f(s,\overline{u})\nabla s\\
&=\int_0^{t_1}\Delta t\int_t^{t_1} f(s,\overline{u})\nabla s+\int_{\sigma(t_1)}^{t_0}\Delta t\int_t^{t_0} f(s,\overline{u})\nabla s\\
&\quad+\sigma(t_1)\left[\int_{t_1}^{\sigma(t_1)} f(s,\overline{u})\nabla s+\int_{\sigma(t_1)}^{t_0} f(s,\overline{u})\nabla s\right]\\
&=\int_0^{t_1}\Delta t\int_t^{t_1} f(s,\overline{u})\nabla s+\int_{\sigma(t_1)}^{t_0}\Delta t\int_t^{t_0} f(s,\overline{u})\nabla s+\sigma(t_1)\int_{\sigma(t_1)}^{t_0} f(s,\overline{u})\nabla s\\
&\quad+\mu(t_1)f(\sigma(t_1),\overline{u})\sigma(t_1),
\end{aligned} \tag{3.4}$$

that is,

$$\int_0^{t_0} \Delta t \int_0^s f(s,\overline{u}) \nabla s = \int_0^{t_0} \Delta t \int_0^s f(s,\overline{u}) \nabla s. \tag{3.5}$$

Similarly, we can prove

$$\int_{\sigma(t_0)}^1 \nabla s \int_s^1 f(s,\overline{u}) \Delta t = \int_{\sigma(t_0)}^1 \Delta t \int_{\sigma(t_0)}^t f(s,\overline{u}) \nabla s. \tag{3.6}$$

The proof is complete. □

Theorem 3.3. *Suppose that (H) holds. Then problem (1.1)-(1.2) has a $C_{ld}[0,1]_{\mathbb{T}}$ positive solution if and only if the following integral condition holds:*

$$0 < \int_0^1 e(s) f(s,1) \nabla s < +\infty. \tag{3.7}$$

Proof. (1) Necessity

By (H), there exists $g(k) : [0,1] \to [1,\infty)$ such that $f(t,kx) \le g(k) f(t,x)$. Without loss of generality, we assume that $g(k)$ is nonincreasing on $[0,1]$ with $g(1) \ge 1$.

Suppose that u is a positive solution of problem (1.1)-(1.2), then

$$u^{\Delta\nabla}(t) = -f(t,u(t)) \le 0, \tag{3.8}$$

which implies that u is concave on $[0,1]_{\mathbb{T}}$. Combining this with the boundary conditions, we have $u^{\Delta}(0) > 0, u^{\Delta}(1) < 0$. Therefore $u^{\Delta}(0)u^{\Delta}(1) < 0$. So by [10, Theorem 1.115], there exists $t_0 \in (0,1)_{\mathbb{T}}$ satisfying $u^{\Delta}(t_0) = 0$ or $u^{\Delta}(t_0)u^{\Delta}(\sigma(t_0)) \le 0$. And $u^{\Delta}(t) > 0$ for $t \in (0,t_0)$, $u^{\Delta}(t) < 0$, for $t \in (\sigma(t_0),1)$. Denote $\overline{u} = \max\{u(t_0), u(\sigma(t_0))\}$, then $\overline{u} = \max_{t\in[0,1]_{\mathbb{T}}} u(t)$.

First we prove $0 < \int_0^1 e(s) f(s,1) \nabla s$.

By (H), for any fixed $u, v > 0$, we have

$$f(t,u) = f\left(t, \frac{u}{v} v\right) \le g\left(\frac{u}{v}\right) f(t,v), \quad u \le v. \tag{3.9}$$

It follows that

$$f(t,u) \le g\left(\frac{2u}{u+v+|u-v|}\right) f(t,v) \quad \forall u,v \in R^+ = [0,+\infty). \tag{3.10}$$

If $f(t,1) \equiv 0$, then we have by (3.10)

$$0 \le f(t,u) \le g\left(\frac{2u}{u+1+|u-1|}\right) f(t,1) \quad \forall t \in (0,1)_{\mathbb{T}}. \tag{3.11}$$

This means $f(t,u(t)) \equiv 0$, then $u(t) \equiv 0$, which is a contradiction with $u(t)$ being positive solution. Thus $f(t,1) \not\equiv 0$, then $0 < \int_0^1 e(s) f(s,1) \nabla s$.

Second, we prove $\int_0^1 e(s) f(s,1) \nabla s < +\infty$.

If $u^{\Delta}(t_0) = 0$, then

$$\begin{aligned}
&\int_t^{t_0} f(s,u(s)) \nabla s = -\int_t^{t_0} u^{\Delta\nabla}(s) \nabla s = -u^{\Delta}(t_0) + u^{\Delta}(t) = u^{\Delta}(t) \quad \text{for } t \in (0,t_0) \\
&\int_{t_0}^{t} f(s,u(s)) \nabla s = -\int_{t_0}^{t} u^{\Delta\nabla}(s) \nabla s = -u^{\Delta}(t) + u^{\Delta}(t_0) = -u^{\Delta}(t) \quad \text{for } t \in (t_0,1).
\end{aligned} \tag{3.12}$$

If $u^{\Delta}(t_0) u^{\Delta}(\sigma(t_0)) < 0$, then $u^{\Delta}(t_0) > 0$, $u^{\Delta}(\sigma(t_0)) < 0$, and

$$\begin{aligned}
&\int_t^{t_0} f(s,u(s)) \nabla s = -\int_t^{t_0} u^{\Delta\nabla}(s) \nabla s = -u^{\Delta}(t_0) + u^{\Delta}(t) \le u^{\Delta}(t) \quad \text{for } t \in (0,t_0) \\
&\int_{\sigma(t_0)}^{t} f(s,u(s)) \nabla s = -\int_{\sigma(t_0)}^{t} u^{\Delta\nabla}(s) \nabla s = -u^{\Delta}(t) + u^{\Delta}(\sigma(t_0)) \le -u^{\Delta}(t) \quad \text{for } t \in (\sigma(t_0),1).
\end{aligned} \tag{3.13}$$

It follows that

$$\begin{aligned}
&\int_t^{t_0} f(s,\overline{u}) \nabla s \le \int_t^{t_0} f(s,u(s)) \nabla s \le u^{\Delta}(t) \quad \text{for } t \in (0,t_0) \\
&\int_{\sigma(t_0)}^{t} f(s,\overline{u}) \nabla s \le \int_{\sigma(t_0)}^{t} f(s,u(s)) \nabla s \le -u^{\Delta}(t) \quad \text{for } t \in (\sigma(t_0),1).
\end{aligned} \tag{3.14}$$

By (3.14) we have

$$\begin{aligned}\int_0^{t_0} s f(s,\overline{u})\nabla s &= \int_0^{t_0}\nabla s\int_0^{s} f(s,\overline{u})\Delta t\\ &= \int_0^{t_0}\Delta t\int_t^{t_0} f(s,\overline{u})\nabla s\\ &\le \int_0^{t_0} u^{\Delta}(t)\Delta t\\ &= u(t_0)-u(0)\\ &= u(t_0) < +\infty,\end{aligned}\tag{3.15}$$

$$\begin{aligned}\int_{\sigma(t_0)}^{1}(1-s) f(s,\overline{u})\nabla s &= \int_{\sigma(t_0)}^{1}\nabla s\int_s^{1} f(s,\overline{u})\Delta t\\ &= \int_{\sigma(t_0)}^{1}\Delta t\int_{\sigma(t_0)}^{t} f(s,\overline{u})\nabla s\\ &\le -\int_{\sigma(t_0)}^{1} u^{\Delta}(t)\Delta t\\ &= u(\sigma(t_0))-u(1)\\ &= u(\sigma(t_0)) < +\infty.\end{aligned}\tag{3.16}$$

Combining this with (3.10) we obtain

$$\begin{aligned}\int_0^{t_0} s f(s,1)\nabla s &\le \int_0^{t_0} s g\left(\frac{2}{1+\overline{u}+|1-\overline{u}|}\right) f(s,\overline{u})\nabla s\\ &= g\left(\frac{2}{1+\overline{u}+|1-\overline{u}|}\right)\int_0^{t_0} s f(s,\overline{u})\nabla s < +\infty.\end{aligned}\tag{3.17}$$

Similarly

$$\int_{\sigma(t_0)}^{1}(1-s) f(s,1)\nabla s < +\infty.\tag{3.18}$$

Then we can obtain

$$0 < \int_0^{1} e(s) f(s,1)\nabla s < +\infty.\tag{3.19}$$

(2) Sufficiency

Let

$$a(t)=\int_0^1 G(t,s)f(s,1)\nabla s,b(t)=\int_0^1 G(t,s)f(s,e(s))\nabla s. \tag{3.20}$$

Then

$$\begin{gathered}e(t)\int_0^1 e(s)f(s,1)\nabla s\le a(t)\le b(t)\le\int_0^1 e(s)f(s,e(s))\nabla s,\\ a^{\Delta\nabla}(t)=-f(t,1),\qquad b^{\Delta\nabla}(t)=-f(t,e(t)).\end{gathered} \tag{3.21}$$

Let

$$k_1=\int_0^1 e(s)f(s,1)\nabla s,\quad l=\min\left\{1,k_1^{-1}\right\},\quad L=\max\left\{1,k_1^{-1}\right\},\quad k_2=\int_0^1 e(s)f(s,e(s))\nabla s, \tag{3.22}$$

then $l\le 1, L\ge 1$.

Let $H(t)=la(t), Q(t)=Lb(t)$, then

$$la(t)\le l\int_0^1 e(s)f(s,1)\nabla s\le 1,\quad Lk_1e(t)\le Lb(t)\le Lk_2\triangleq\rho. \tag{3.23}$$

So, we have

$$\begin{aligned}H^{\Delta\nabla}(t)+f(t,H(t))&=f(t,la(t))-lf(t,1)\\&\ge f(t,1)-lf(t,1)\ge 0,\\Q^{\Delta\nabla}(t)+f(t,Q(t))&=f(t,Lb(t))-Lf(t,e(t))\\&\le f(t,Lk_1e(t))-Lf(t,e(t))\\&\le f(t,e(t))-Lf(t,e(t))\le 0,\end{aligned} \tag{3.24}$$

and $H(0)=H(1)=Q(0)=Q(1)=0$. Hence $H(t),Q(t)$ are lower and upper solutions of problem (1.1)-(1.2), respectively. Obviously $H(t)>0$ for $t\in(0,1)_{\mathbb{T}}$.

Now we prove that problem (1.1)-(1.2) has a positive solution $x^*\in C_{ld}[0,1]_{\mathbb{T}}$ with $0<H(t)\le x^*\le Q(t)$.

Define a function

$$F(t,x)=\begin{cases}f(t,H(t)), & x<H(t),\\ f(t,x), & H(t)\le x\le Q(t),\\ f(t,Q(t)), & x>Q(t).\end{cases} \tag{3.25}$$

Then $F : (0,1)_{\mathbb{T}} \times R^{+} \to R^{+}$ is continuous. Consider BVP

$$-x^{\Delta\nabla}(t) = F(t,x),$$
$$x(0) = x(1) = 0. \tag{3.26}$$

Define mapping $A : E \to E$ by

$$Ax(t) = \int_0^1 G(t,s)F(s,x(s))\nabla s. \tag{3.27}$$

Then problem (1.1)-(1.2) has a positive solution if and only if A has a fixed point $x^* \in C_{ld}[0,1]_{\mathbb{T}}$ with $0 < H(t) \leq x^* \leq Q(t)$.

Obviously A is continuous. Let $D = \{x \mid \|x\| \leq \rho^*, x \in E, \rho^* \in R^+\}$. By (3.7) and (3.16), for all $x \in D$, we have

$$\begin{aligned}\int_0^1 G(t,s)F(s,x(s))\nabla s &\leq \int_0^1 G(t,s)f(s,H(s))\nabla s\\ &\leq \int_0^1 G(t,s)f(s,0)\nabla s\\ &\leq g(0)\int_0^1 G(t,s)f(t,1)\nabla s\\ &\leq g(0)\int_0^1 e(s)f(t,1)\nabla s < +\infty.\end{aligned} \tag{3.28}$$

Then $A(D)$ is bounded. By the continuity of $G(t,s)$ we can easily found that $\{Au(t) \mid u(t) \in D\}$ are equicontinuous. Thus A is completely continuous. By Schauder fixed point theorem we found that A has at least one fixed point $x^* \in D$.

We prove $0 < H(t) \leq x^* \leq Q(t)$. If there exists $t_* \in (0,1)_{\mathbb{T}}$ such that

$$x^*(t_*) > Q(t_*). \tag{3.29}$$

Let $z(t) = Q(t) - x^*, c = \inf\{t_1 \mid 0 \leq t_1 < t_*, z(t) < 0, \forall t \in (t_1, t_*]\}, d = \sup\{t_2 \mid t_* < t_2 \leq 1, z(t) < 0, \forall t \in (t_*, t_2]\}$ then $Q(t) < x^*$ for $t \in (c,d)_{\mathrm{T}}$. Thus $F(t,x^*) = f(t,Q(t)), t \in (c,d)_{\mathrm{T}}$. By (3.24) we know that $-z^{\Delta\nabla}(t) = Q^{\Delta\nabla}(t) - x^{\Delta\nabla}(t) \leq 0$. And $z(c) = Q(c) - x^*(c) \geq 0, z(d) = Q(d) - x^*(d) \geq 0$. By Lemma 2.9 we have $z(t) \geq 0, t \in [c,d]_{\mathbb{T}}$, which is a contradiction. Then $x^* \leq Q(t)$. Similarly we can prove $H(t) \leq x^*$. The proof is complete. □

Theorem 3.4. *Suppose that (H) holds. Then problem* (1.1)-(1.2) *has a* $C^{\Delta}_{ld}[0,1]_{\mathbb{T}}$ *positive solution if and only if the following integral condition holds:*

$$0 < \int_0^1 f(s,e(s))\nabla s < +\infty. \tag{3.30}$$

Proof. (1) Necessity

Let $u(t) \in C_{ld}^{\Delta}[0,1]_{\mathbb{T}}$ be a positive solution of problem (1.1)-(1.2). Then $u^{\Delta}(t)$ is decreasing on $[0,1]_{\mathbb{T}}$. Hence $u^{\Delta\nabla}(t)$ is integrable and

$$\int_0^1 f(t,u(t))\nabla t = -\int_0^1 u^{\Delta\nabla}(t)\nabla t < +\infty. \tag{3.31}$$

By simple computation and using [10, Theorem 1.119], we obtain $\lim_{t\to 0^+}(u(t)/e(t)) > 0, \lim_{t\to 1^-}(u(t)/e(t)) > 0$. So there exist $M > 1 > m > 0$ such that $me(t) \le u(t) \le Me(t)$. By (H) we obtain

$$\begin{gathered} g\left(M^{-1}\right)^{-1} f(t,e(t)) \le f(t,Me(t)) \le f(t,u(t)), \\ \int_0^1 f(t,e(t))\nabla t \le g\left(M^{-1}\right)\int_0^1 f(t,u(t))\nabla t < \infty. \end{gathered} \tag{3.32}$$

By

$$e(t)f(t,1) \le f(t,e(t)) \le g(e(t))f(t,1), \tag{3.33}$$

we have$0 < \int_0^1 e(t)f(t,1)\nabla t \le \int_0^1 f(t,e(t))\nabla t < \infty$.

(2) Sufficiency

Let $r(t) = \int_0^1 G(t,s)f(s,e(s))\nabla s$, then

$$e(t)\int_0^1 G(s,s)f(s,e(s))\nabla s \le r(t) \le \int_0^1 f(s,e(s))\nabla s. \tag{3.34}$$

Similar to Theorem 3.3, let $l' = \min\{1,k_2^{-1}\}, L' = \max\{1,k_2^{-1}\}, H(t) = l'a(t), Q(t) = L'r(t)$, there exists $\omega^*(t)$ satisfying $H(t) \le \omega^*(t) \le Q(t)$, and

$$f(t,\omega^*(t)) \le f(t,H(t)) \le f\left(t,l'k_2e(t)\right) \le g\left(l'k_2\right)f(t,e(t)), \tag{3.35}$$

then $\omega^{*\Delta\nabla}(t)$ is integral and $\omega^{*\Delta}(1-), \omega^{*\Delta}(0+)$ exist, hence $\omega^*(t)$ is a positive solution in $C_{ld}^{\Delta}[0,1]_{\mathbb{T}}$. The proof is complete. □

4. Existence of Positive Solution to (1.1)–(1.3)

Now we deal with problem (1.1)–(1.3). The method is just similar to what we have done in Section 3, so we omit the proof of main result of this section.

Let

$$G_1(t,s) = \begin{cases} s & \text{if } 0 \le s \le t \le 1, \\ t & \text{if } 0 \le t \le s \le 1. \end{cases} \tag{4.1}$$

be the Green's function of corresponding homogeneous BVP of (1.1)–(1.3).

We can prove that $G_1(t,s)$ has the following properties.

Similar to (3.2), we have

$$\begin{gathered} G_1(t,s) \ge 0, \quad (t,s) \in [0,1]_{\mathbb{T}} \times [0,1]_{\mathbb{T}}, \\ e_1(t)e_1(s) \le G_1(t,s) \le G_1(t,t) = t = e_1(t), \quad (t,s) \in [0,1]_{\mathbb{T}} \times [0,1]_{\mathbb{T}}. \end{gathered} \tag{4.2}$$

Theorem 4.1. *Suppose that* (H) *holds, then problem* (1.1)–(1.3) *has a* $C_{ld}[0,1]_{\mathbb{T}}$ *positive solution if and only if the following integral condition holds:*

$$0 < \int_0^1 e_1(s) f(s,1) \nabla s < +\infty. \tag{4.3}$$

Theorem 4.2. *Suppose that* (H) *holds, then problem* (1.1)–(1.3) *has a* $C_{ld}^{\Delta}[0,1]_{\mathbb{T}}$ *positive solution if and only if the following integral condition holds:*

$$0 < \int_0^1 f(s, e_1(s)) \nabla s < +\infty. \tag{4.4}$$

5. Example

To illustrate how our main results can be used in practice we present an example.

Example 5.1. We have

$$\begin{gathered} -x^{\Delta\nabla}(t) = t^{-1/2} e^{-x}, \quad t \in (0,1)_{\mathbb{T}}, \\ x(0) = x(1) = 0, \end{gathered} \tag{5.1}$$

where $f(t,x) = t^{-(1/2)e^{-x}}, \mathbb{T} = [0,1/2) \cup \{1/2, 2/3, 3/4, \ldots, n/(n+1), \ldots, 1\}$. Select $g(k) = e(2-k), k \in [0,1]$, then we have $f(t,kx) \leq g(k) f(t,x), \forall (t,x) \in (0,1)_{\mathbb{T}} \times [0,+\infty)$. Moreover, we have

$$\begin{aligned} 0 < \int_0^1 s(1-s)s^{-1/2}e^{-1}\nabla s &= e^{-1}\left[\frac{2}{3}\left(\frac{1}{2}\right)^{3/2} + \frac{2}{5}\left(\frac{1}{2}\right)^{5/2} + \sum_{n=1}^{\infty}\frac{1}{(n+1)^{(7/2)}n^{(1/2)}}\right] \\ &\leq e^{-1}\left[\sum_{n=1}^{\infty}\frac{1}{n^4} + \frac{2}{3}\left(\frac{1}{2}\right)^{3/2} + \frac{2}{5}\left(\frac{1}{2}\right)^{5/2}\right] < +\infty. \end{aligned} \tag{5.2}$$

By Theorem 3.3, problem (5.1) has a positive solution in $C_{ld}[0,1]_{\mathbb{T}}$.

Remark 5.2. Example 5.1 implies that there is a large number of functions that satisfy the conditions of Theorem 3.3. In addition, the conditions of Theorem 3.3 are also easy to check.

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