Hindawi Publishing Corporation
Advances in Difference Equations
Volume 2010, Article ID 202731, 26 pages
doi:10.1155/2010/202731

Research Article

Automorphisms of Submanifolds

Veronika Chrastinová and Václav Tryhuk

Department of Mathematics, Faculty of Civil Engineering, Brno University of Technology, Veveří 331/95, 602 00 Brno, Czech Republic

Correspondence should be addressed to Václav Tryhuk, tryhuk.v@fce.vutbr.cz

Received 26 October 2009; Revised 12 February 2010; Accepted 23 April 2010

Academic Editor: Josef Diblík

The paper deals with local symmetries of the infinite-order jet space of C^∞-smooth n-dimensional submanifolds in $\mathbb{R}^{m+n}$. Transformations under consideration are the most general possible. They need not preserve the distinction between dependent, and independent variables, the order of derivatives and the hierarchy of finite-order jet spaces.

1. Introduction

A huge literature is devoted to the symmetries and equivalences of (partial) differential equations and it might seem that the theory is ultimately established. In spite of this we believe that the actual methods are still insufficient since the problems are investigated in finite-order jet spaces (left-hand side of Figure 1) and all transformations which do not preserve such spaces are passed over in full silence (right-hand side of Figure 1).

We deal with the modest task, with symmetries of the empty system of differential equations, that is, with symmetries of the family of all C^∞-smooth n-dimensional subspaces in an $(m+n)$-dimensional space. Thus we paraphrase and improve our previous results devoted to the automorphisms of curves [1]. The exposition is self-contained but inevitably with rather unorthodox manners. We restrict ourselves to the local theory on open subsets of generic points.

1.1. Transformations of Submanifolds

Our reasoning starts in the space $\mathbb{R}^{m+n}$ $(m, n = 1, 2, \ldots)$ with coordinates

$$x^i, w^k \quad (i = 1, \ldots, n;\ k = 1, \ldots, m), \tag{1.1}$$

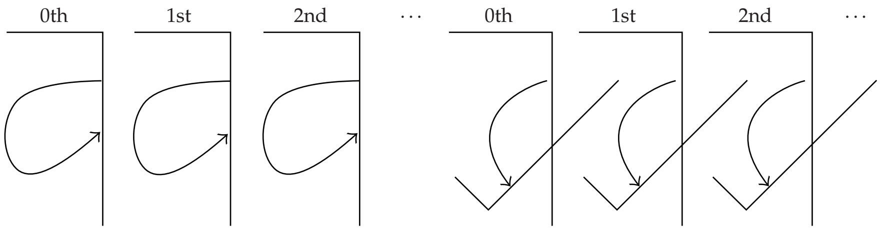


Figure 1

and we are interested in the family of all C^∞-smooth subspaces:

$$w^k = w^k\left(x^1,\dots,x^n\right) \quad (k=1,\dots,m), \tag{1.2}$$

where the definition domains (open subsets of $\mathbb{R}^n$) are not specified. (More precisely, we deal with germs.) Let us recall the multi-index notation

$$w_I^k = \frac{\partial^r w^k}{\partial x^I} = \frac{\partial^r w^k}{\partial x^{i_1}\cdots\partial x^{i_r}} \quad (I = i_1\cdots i_r;\ i_1,\dots,i_r = 1,\dots,n;\ k=1,\dots,m) \tag{1.3}$$

for the derivatives. Our transformations will be (locally) defined by

$$\overline{x}^i = F^i\left(\dots,x^j,w_J^l,\dots\right), \quad \overline{w}^k = G^k\left(\dots,x^j,w_J^l,\dots\right) \quad (i=1,\dots,n;\ k=1,\dots,m), \tag{1.4}$$

where F^i, G^k are C^∞-smooth functions, each depending on a finite number of (a somewhat symbolically indicated) arguments. By virtue of formulae (1.4), a given subspace (1.2) is transformed into a subspace

$$\overline{w}^k = \overline{w}^k\left(\overline{x}^1,\dots,\overline{x}^n\right) \quad (k=1,\dots,m) \tag{1.5}$$

again lying in $\mathbb{R}^{m+n}$ and this is achieved as follows.

A given subspace (1.2) is inserted into $(1.4)_1$ with the result

$$\overline{x}^i = F^i\left(\dots,x^j,\frac{\partial^{|J|}w^l}{\partial x^J}\left(x^1,\dots,x^n\right),\dots\right) = \mathcal{F}^i\left(x^1,\dots,x^n\right). \tag{1.6}$$

Then, assume that

$$\det\left(\frac{\partial \mathcal{F}^{i'}}{\partial x^i}\right) = \det\left(D_i F^{i'}\right) \neq 0 \quad \left(D_i = \frac{\partial}{\partial x^i} + \sum w_{Ii}^k \frac{\partial}{\partial w_I^k}\right). \tag{1.7}$$

Equation (1.6) can be (locally) resolved as

$$x^i = \overline{\mathcal{F}}^i\left(\overline{x}^1,\ldots,\overline{x}^n\right) \quad (i=1,\ldots,n), \tag{1.8}$$

and we obtain the desired subspace

$$\overline{w}^k = G^k\left(\ldots,\overline{\mathcal{F}}^j,\frac{\partial^{|J|}w^l}{\partial x^J}\left(\overline{\mathcal{F}}^1,\ldots,\overline{\mathcal{F}}^n\right),\ldots\right) = \overline{w}^k\left(\overline{x}^1,\ldots,\overline{x}^n\right) \quad (k=1,\ldots,m) \tag{1.9}$$

by using $(1.4)_2$.

The obvious identities

$$\sum \frac{\partial\overline{\mathcal{F}}^{i''}}{\partial\overline{x}^{i'}}\frac{\partial\mathcal{F}^{i'}}{\partial x^i} = \sum \frac{\partial\overline{\mathcal{F}}^{i''}}{\partial\overline{x}^{i'}}D_iF^{i'} = \delta_i^{i''} \quad (i,i''=1,\ldots,n) \tag{1.10}$$

appearing on this occasion express the invertibility of Jacobi matrices

$$\left(\frac{\partial\overline{\mathcal{F}}^i}{\partial\overline{x}^{i'}}\right)^{-1} = \left(\frac{\partial\mathcal{F}^i}{\partial x^{i'}}\right) = \left(D_{i'}F^i\right) \tag{1.11}$$

and will be frequently referred to.

1.2. The Prolongation Procedure

Explicit formulae

$$\overline{w}_I^k = G_I^k\left(\ldots,x^j,w_J^l,\ldots\right) \quad (I=i_1\cdots i_r;\ i_1,\ldots,i_r=1,\ldots,n;\ k=1,\ldots,m) \tag{1.12}$$

for the transformed derivatives

$$\overline{w}_I^k = \frac{\partial^r\overline{w}^k}{\partial\overline{x}^I} = \frac{\partial^r\overline{w}^k}{\partial\overline{x}^{i_1}\cdots\partial\overline{x}^{i_r}} \tag{1.13}$$

can be obtained as follows. Assume that they are known for a certain I. (In particular $G^k = G_\phi^k$ if $I=\phi$ is empty.) Then

$$\begin{aligned}\overline{w}_{Ii}^k &= \frac{\partial}{\partial\overline{x}^i}\overline{w}_I^k = \frac{\partial}{\partial\overline{x}^i}G_I^k\left(\ldots,\overline{\mathcal{F}}^j,\frac{\partial^{|J|}w^l}{\partial\overline{x}^J}\left(\overline{\mathcal{F}}^1,\ldots,\overline{\mathcal{F}}^n\right),\ldots\right)\\ &= \sum D_{i'}G_I^k\left(\ldots,x^j,w_J^l,\ldots\right)\cdot\frac{\partial\overline{\mathcal{F}}^{i'}}{\partial\overline{x}^i} \quad (i=1,\ldots,n).\end{aligned} \tag{1.14}$$

By virtue of (1.10), this is equivalent to the (implicit) recurrence

$$\left(\sum \overline{w}^k_{Ii} D_{i'} F^i =\right) \sum G^k_{Ii} D_{i'} F^i = D_{i'} G^k_I \quad (i' = 1, \ldots, n) \tag{1.15}$$

for the sought functions $G^k_I = \overline{w}^k_I$.

Altogether taken, we have the infinite system

$$\overline{x}^i = F^i\left(\ldots, x^j, w^l_J, \ldots\right), \qquad \overline{w}^k_I = G^k_I\left(\ldots, x^j, w^l_J, \ldots\right) \tag{1.16}$$

subjected to the recurrence (1.15). At this place, functions F^i $(i = 1, \ldots, n)$ satisfying (1.7) and functions G^k $(k = 1, \ldots, m)$ may be quite arbitrary.

1.3. Invertible Transformations

We are interested in transformations (1.15) and (1.16) which can be (locally) inverted by appropriate C^∞-smooth formulae

$$x^i = \overline{F}^i\left(\ldots, \overline{x}^j, \overline{w}^l_J, \ldots\right), \qquad w^k_I = \overline{G}^k_I\left(\ldots, \overline{x}^j, \overline{w}^l_J, \ldots\right) \tag{1.17}$$

analogous to (1.16). If this is possible, we will prove later that the recurrence

$$\sum \overline{G}^k_{Ii} \overline{D}_{i'} \overline{F}^i = \overline{D}_{i'} \overline{G}^k_I \quad \left(i' = 1, \ldots, n;\ \overline{D}_i = \frac{\partial}{\partial \overline{x}^i} + \sum \overline{w}^k_{Ii} \frac{\partial}{\partial \overline{w}^k_I}\right) \tag{1.18}$$

corresponding to (1.15) is satisfied (see Lemma 2.1).

Definition 1.1. One speaks of a *morphism* (1.16) if the recurrence (1.15) holds true and of an *automorphism* (1.16) if moreover the inverse (1.17) exists.

An algorithm for calculation of all automorphisms will be proposed as the final achievement of this paper.

1.4. Elementary Examples

We need not discuss the well-known *point transformations:*

$$\overline{x}^i = F^i\left(\ldots, x^j, w^l, \ldots\right), \quad \overline{w}^k = G^k\left(\ldots, x^j, w^l, \ldots\right) \quad (i = 1, \ldots, n;\ k = 1, \ldots, m) \tag{1.19}$$

which are (locally) invertible if the Jacobi determinant is nonvanishing.

Theorem 1.2. *Let*

$$f^r\left(\overline{x}^1, \ldots, \overline{x}^n, \overline{w}^1, \ldots, \overline{w}^m, x^1, \ldots, x^n, w^1, \ldots, w^m\right) \quad (r = 1, \ldots, R) \tag{1.20}$$

be functions of $2(m+n)$ *variables where* $m+n=(n+1)R$*. Assume that the system of* $(n+1)R$ *equations*

$$f^r = 0, \quad D_i f^r = 0 \quad (i = 1,\dots,n;\ r = 1,\dots,R) \tag{1.21}$$

admits a certain solution:

$$\overline{x}^i = F^i(\cdots), \quad \overline{w}^k = G^k(\cdots) \quad \left(i = 1,\dots,n;\ k = 1,\dots,m,\ \det\left(D_i F^{i'}\right) \neq 0\right) \tag{1.22}$$

where $(\cdots) = (\dots, x^j, w^l, w^l_j, \dots)$*, and moreover the system*

$$f^r = 0, \quad \overline{D}_i f^r = 0 \quad (i = 1,\dots,n;\ r = 1,\dots,R) \tag{1.23}$$

admits a certain solution:

$$x^i = \overline{F}^i(\cdots), \quad w^k = \overline{G}^k(\cdots) \quad \left(i = 1,\dots,n;\ k = 1,\dots,m,\ \det\left(\overline{D}_i \overline{F}^{i'}\right) \neq 0\right), \tag{1.24}$$

where $(\cdots) = (\dots, \overline{x}^j, \overline{w}^l, \overline{w}^l_j, \dots)$ *by applying the implicit function theorem. If* (1.22) *and* (1.24) *are regarded as transformations of submanifolds in* $\mathbb{R}^{m+n}$*, they are inverse one to each other. Alternatively saying, prolongations of* (1.22) *and* (1.24) *are mutually inverse automorphisms.*

Proof. Consider a subspace (1.2). Transformed subspace (1.5) is defined by (1.22) which is equivalent to (1.21). We will see that (1.21) implies (1.23) and hence (1.24). Analogously (1.24) implies (1.22) and we have inverse mappings.

Passing to the proof proper, we suppose that (1.22) and then $(1.21)_1$ read

$$f^r\left(\dots, \overline{x}^j, \overline{w}^l\left(\overline{x}^1,\dots,\overline{x}^n\right), x^j, w^l\left(x^1,\dots,x^n\right),\dots\right) = 0 \quad \left(\overline{x}^j = \mathcal{F}^j\left(x^1,\dots,x^n\right)\right) \tag{1.25}$$

identically. It follows that

$$\frac{d}{dx^i} f^r = \sum \frac{\partial \mathcal{F}^{i'}}{\partial x^i} \overline{D}_{i'} f^r + D_i f^r = 0 \quad (i = 1,\dots,n;\ r = 1,\dots,R). \tag{1.26}$$

Therefore $(1.21)_2$ together with (1.7) implies $\overline{D}_{i'} f^r = 0$ and hence (1.23). The proof is done. □

Remark 1.3. For the particular case $m = R = 1$, our Theorem 1.2 provides the classical Lie's contact transformations. (Indeed, assuming $m = R = 1$ and abbreviating $f = f^1$, $w = w^1$ for a moment, then $(1.21)_1$ implies

$$0 = \mathrm{d}f = \sum D_i f\, \mathrm{d}x^i + \frac{\partial f}{\partial w}\left(\mathrm{d}w - \sum w_i\, \mathrm{d}x^i\right) + \sum \overline{D}_i f\, \mathrm{d}\overline{x}^i + \frac{\partial f}{\partial \overline{w}}\left(\mathrm{d}\overline{w} - \sum \overline{w}_i\, \mathrm{d}\overline{x}^i\right) \tag{1.27}$$

and whence the identity

$$d\overline{w} - \sum \overline{w}_i \, d\overline{x}^i = \lambda \Big(dw - \sum w_i \, dx^i \Big) \quad \left(\lambda = -\frac{\partial f / \partial w}{\partial f / \partial \overline{w}} \right) \tag{1.28}$$

follows by using $(1.21)_2$ and $(1.23)_2$. This is just the classical Lie's definition.) The wave mechanisms of Lie's transformations are rather important and well known. In general (for arbitrary m and R) a geometrical interpretation of our result in terms of waves is possible, as well. Equations $f^r = 0$ with $\overline{x}^j, \overline{w}^k$ kept fixed represent an $(m - R)$-dimensional wave and $D_i f^r = 0$ the intersection with the "close wave" in the space of variables x^i, w^k. The "reverse wave" with parameters and spatial variables interchanged provides the inverse mapping. Alas, the Huyghens principle in general fails.

Remark 1.4. Far-going generalizations can be stated; let us however only very briefly indicate three possibilities without proofs [2] and without any aim for the most possible generality. The particular case $m = R = 1$ of point $(\iota\iota)$ again reduces to the classical Lie's concept.

(ι) *The Multiple Waves*. Let $m + n = (n+1)R + (n+1)n/2$. Assume that equations

$$f^r = 0, \quad D_i f^r = 0, \quad D_i D_{i'} f^1 = 0 \quad (i, i' = 1, \dots, n; \; r = 1, \dots, R) \tag{1.29}$$

admit certain solution (1.22) where $(\cdots) = (\dots, \overline{x}^j, \overline{w}^l, \overline{w}^l_j, \overline{w}^l_{j'}, \dots)$ and moreover the system

$$f^r = 0, \quad \overline{D}_i f^r = 0, \quad \overline{D}_i \overline{D}_{i'} f^1 = 0 \quad (i, i' = 1, \dots, n; \; r = 1, \dots, R) \tag{1.30}$$

admits a solution (1.24) where $(\cdots) = (\dots, x^j, w^l, w^l_j, w^l_{j'}, \dots)$. Then (1.22) and (1.24) are mutually inverse mappings.

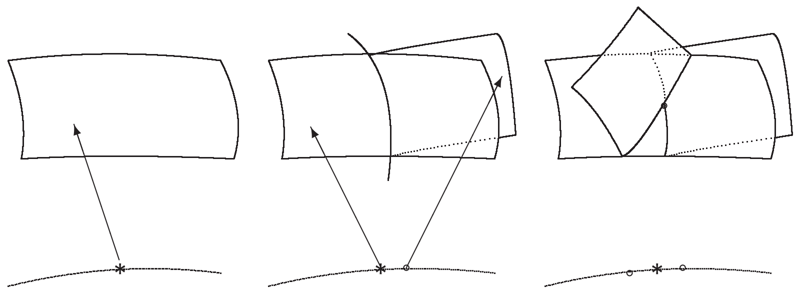
$(\iota\iota)$ *The Degenerate Waves*. Let $m+n = (n+1)R + (n-3)n/2, \; R \geq 2$. Assume that equations

$$\begin{gathered} f^r = 0, \quad D_i f^1 D_{i'} f^2 = D_i f^2 D_{i'} f^1, \quad D_i f^s = 0 \\ (i, i' = 1, \dots, n; \; r = 1, \dots, R; \; s = 3, \dots, R) \end{gathered} \tag{1.31}$$

admit certain solution (1.22) where $(\cdots) = (\dots, x^j, w^l, w^l_j, \dots)$ and moreover the system

$$\begin{gathered} f^r = 0, \quad \overline{D}_i f^1 \overline{D}_{i'} f^2 = \overline{D}_i f^2 \overline{D}_{i'} f^1, \quad \overline{D}_i f^s = 0 \\ (i, i' = 1, \dots, n; \; r = 1, \dots, R; \; s = 3, \dots, R) \end{gathered} \tag{1.32}$$

admits a solution (1.24) where $(\cdots) = (\dots, \overline{x}^j, \overline{w}^l, \overline{w}^l_j, \dots)$. Then (1.22) and (1.24) are mutually inverse mappings.



(a) wave $f = 0$

(b) two infinitesimaly close waves $f = Df = 0$

(c) three waves $f = Df = D^2 f = 0$ the resulting focus

Figure 2

($\iota\iota$) *The Constrained Waves*. Operators D_i, $\overline{D}_i$ need not be applied to all functions f^r under consideration. For instance, assuming $m + n = nR' + R$ $(1 \le R' < R)$ in Theorem 1.2, requirements $(1.21)_2$ and $(1.23)_2$ may be replaced with the weaker conditions $D_i f^r = \overline{D}_i f^r = 0$ $(r = 1, \ldots, R')$.

Remark 1.5. Let us discuss the simplest possible and very particular example of the above automorphisms in more detail for better clarity. We choose $m = 2$, $n = 1$, $R = 1$ and the spherical wave

$$\left(f^1 =\right) f = (\overline{x} - x)^2 + (\overline{y} - y)^2 + (\overline{z} - z)^2 - r^2 \tag{1.33}$$

($r > 0$, abbreviation $x = x^1$, $y = w^1$, $z = w^2$) in the point (ι). Then the final result can be geometrically described as follows. Let $\mathbf{P} = (x, y, z), \overline{\mathbf{P}} = (\overline{x}, \overline{y}, \overline{z})$ denote the original and the transformed curves. The system $f = Df = D^2 f = 0$ (abbreviation $D = D_1$) in point (ι) reads

$$\left(\overline{\mathbf{P}} - \mathbf{P}\right)^2 - r^2 = \left(\overline{\mathbf{P}} - \mathbf{P}\right)\frac{\mathrm{d}\mathbf{P}}{\mathrm{d}s} = \left(\overline{\mathbf{P}} - \mathbf{P}\right)\frac{\mathrm{d}^2\mathbf{P}}{\mathrm{d}s^2} + \left(\frac{\mathrm{d}\mathbf{P}}{\mathrm{d}s}\right)^2 = 0 \tag{1.34}$$

in terms of scalar products and the arclength parametrization $\mathbf{P} = (x(s), y(s), z(s))$ of the original curve $\mathbf{P} = \mathbf{P}(s)$. Then the solution

$$\overline{\mathbf{P}}_{\pm} = \mathbf{P} + \frac{1}{\kappa}\mathbf{N} \pm \sqrt{r^2 - \frac{1}{\kappa^2}}\mathbf{B} \tag{1.35}$$

(where κ, $\mathbf{N}$, $\mathbf{B}$ are curvature, normal, and binormal vectors) easily follows by using the Frenet formulae [3]. We have obtained two notable "parallel at the distance r" curves to the original curve $\mathbf{P}$. They consist of the "foci at the distance r" which is worth a schematical picture (only one of the two resulting foci $\mathbf{P}_+$, $\mathbf{P}_-$ is noted here). We have an involutory transformation of curves (if the $\pm$ branches are appropriately combined) but *not* the classical Lie's transformation (see Figure 2).

2. General Theory

2.1. Geometrical Approach

The technical tools must be made more precise. So we introduce the infinite-dimensional jet space $\mathbf{M}$ (abbreviation for $\mathbf{M}(m,n)$ in [4]) equipped with jet coordinates

$$x^i, w^k_I \quad (I = i_1 \cdots i_r;\ i, i_1, \dots, i_r = 1, \dots, n;\ r = 0, 1, \dots;\ k = 1, \dots, m) \tag{2.1}$$

(where the order of terms in I is irrelevant) and moreover the module Ω (abbreviation for $\Omega(m,n)$) of contact forms

$$\omega = \sum a^k_I \omega^k_I \quad \left(\text{finite sum, } \omega^k_I = \mathrm{d}w^k_I - \sum w^k_{Ii}\mathrm{d}x^i\right) \tag{2.2}$$

with arbitrary C^∞-smooth coefficients, each depending on a finite number of coordinates. The obvious identities

$$\begin{gathered} \omega(D_i) = D_i \rfloor \omega = 0 \quad (\omega \in \Omega), \quad \mathcal{L}_i \omega^k_I = \omega^k_{Ii}, \quad \mathrm{d}\omega^k_I = \sum \mathrm{d}x^i \wedge \omega^k_{Ii}, \\ \mathrm{d}f = \sum D_i f \mathrm{d}x^i + \sum \frac{\partial f}{\partial w^k_I} \omega^k_I, \end{gathered} \tag{2.3}$$

where $f = f(\cdots, x^i, w^k_I, \cdots)$ is a C^∞-smooth function on $\mathbf{M}$ and

$$\mathcal{L}_i = D_i \rfloor \mathrm{d} + \mathrm{d}D_i \rfloor \quad \left(D_i = \frac{\partial}{\partial x^i} + \sum w^k_{Ii} \frac{\partial}{\partial w^k_I},\ i = 1, \dots, n\right) \tag{2.4}$$

are the Lie derivatives will frequently occur.

We study C^∞-smooth mappings $\mathbf{m}$ (locally) given by certain formulae:

$$\mathbf{m}^* x^i = F^i\left(\dots, x^j, w^l_J, \dots\right), \qquad \mathbf{m}^* w^k_I = G^k_I\left(\dots, x^j, w^l_J, \dots\right). \tag{2.5}$$

They are a mere transcription of (1.16). We are interested in mappings $\mathbf{m}$ that admit the inverse $\mathbf{m}^{-1}$ given by analogous formulae:

$$\left(\mathbf{m}^{-1}\right)^* x^i = \overline{F}^i\left(\dots, x^j, w^l_J, \dots\right), \qquad \left(\mathbf{m}^{-1}\right)^* w^k_I = \overline{G}^k_I\left(\dots, x^j, w^l_J, \dots\right). \tag{2.6}$$

If recurrence (1.15) holds, we have a *morphism* $\mathbf{m}$, and if moreover the inverse (2.6) exists, we have an *automorphism* $\mathbf{m}$.

2.2. On the Recurrences

The recurrences can be expressed in geometrical terms. For this aim, let us begin with the obvious congruence

$$\mathbf{m}^*\omega_I^k = \mathrm{d}G_I^k - \sum G_{Ii}^k \mathrm{d}F^i \cong \sum \left(D_{i'}G_I^k - \sum G_{Ii}^k D_{i'}F^i\right)\mathrm{d}x^{i'} \quad (\mathrm{mod}\,\Omega) \tag{2.7}$$

valid for *every* mapping (2.5). It follows that conditions

$$D_{i'}G_I^k = \sum G_{Ii}^k D_{i'}F^i, \quad \mathbf{m}^*\omega_I^k \in \Omega \tag{2.8}$$

are equivalent.

Consequence 1. A mapping (2.5) is morphism if and only if $\mathbf{m}^*\Omega \subset \Omega$.

Lemma 2.1. *The inverse* $\mathbf{m}^{-1}$ *of a morphism* $\mathbf{m}$ *again is a morphism.*

Proof. We have to verify $(\mathbf{m}^{-1})^*\Omega \subset \Omega$. Let $\omega \in \Omega$ and assume $(\mathbf{m}^{-1})^*\omega \cong \sum f^i \mathrm{d}x^i \ (\mathrm{mod}\,\Omega)$. Then

$$\omega = \mathbf{m}^*\left(\mathbf{m}^{-1}\right)^*\omega \cong \mathbf{m}^* \sum f^i \mathrm{d}x^i \quad (\mathrm{mod}\,\mathbf{m}^*\Omega) \text{ hence } (\mathrm{mod}\,\Omega) \tag{2.9}$$

since $\mathbf{m}$ is a morphism. Consequently

$$\omega \cong \mathbf{m}^* \sum f^i \mathrm{d}x^i = \sum \mathbf{m}^* f^i D_{i'}F^i \mathrm{d}x^{i'} \quad (\mathrm{mod}\,\Omega), \tag{2.10}$$

and hence $\sum D_{i'}F^i \mathbf{m}^* f^i = 0$ $(i' = 1, \dots, n)$. Therefore $\mathbf{m}^* f^i = 0$ by using (1.7) and as a result $\mathbf{m}^*\omega \in \Omega$. □

Consequence 2. If $\mathbf{m}$ is automorphism, then $\mathbf{m}^*\Omega = \Omega$.

Proof. Clearly $(\mathbf{m}^{-1})^*\Omega \subset \Omega$ and whence $\Omega = \mathbf{m}^*(\mathbf{m}^{-1})^*\Omega \subset \mathbf{m}^*\Omega \subset \Omega$. □

Continuing with the recurrences, we have

$$\mathbf{m}^*\mathrm{d}\omega_I^k = \mathbf{m}^* \sum \mathrm{d}x^i \wedge \omega_{Ii}^k \cong \sum D_{i'}F^i \mathrm{d}x^{i'} \wedge \mathbf{m}^*\omega_{Ii}^k \quad (\mathrm{mod}\,\Omega \wedge \Omega). \tag{2.11}$$

On the other hand, assuming $\mathbf{m}^*\omega_I^k = \sum a_{IJ}^{kl}\omega_J^l$, we obtain

$$\mathrm{d}\mathbf{m}^*\omega_I^k \cong \sum D_i a_{IJ}^{kl} \mathrm{d}x^i \wedge \omega_J^l + \sum a_{IJ}^{kl} \mathrm{d}x^i \wedge \omega_{Ji}^l = \sum \mathrm{d}x^i \wedge \mathcal{L}_i \mathbf{m}^*\omega_I^k \quad (\mathrm{mod}\,\Omega \wedge \Omega) \tag{2.12}$$

and the important identities

$$\sum D_{i'} F^i \mathbf{m}^* \omega_{Ii}^k = \mathcal{L}_{i'} \mathbf{m}^* \omega_I^k \quad (i' = 1, \ldots, n) \tag{2.13}$$

for the contact forms immediately follow.

The following result is obvious.

Consequence 3. If $\mathbf{m}$ is a morphism, then $\mathcal{L}_i \overline{\Omega} \subset \overline{\Omega}$ $(i = 1, \ldots, n)$.

2.3. Preparatory Constructions

Roughly saying, our next aim is to convert Consequence 2. In fact a seemingly stronger result more adapted for the practice of calculations will be established by a general method [4]. We will deal with various modules of differential forms on $\mathbf{M}$ over the ring of C^∞-smooth functions. We always suppose that they (locally) have a *basis*, that is, generators linearly independent at every point. The cardinality of a basis is a (finite or infinite) *dimension* of the module.

Passing to the topic proper, let $\Omega_s \subset \Omega$, $\overline{\Omega} \subset \Omega$, $\overline{\Omega}_s \subset \overline{\Omega}$ $(s = 0, 1, \ldots)$ be the submodules of all forms:

$$\sum a_I^k \omega_I^k \quad (\text{only } |I| \le s), \quad \sum a_I^k \mathbf{m}^* \omega_I^k, \quad \sum a_I^k \mathbf{m}^* \omega_I^k \quad (\text{only } |I| \le s), \tag{2.14}$$

respectively. Then

$$\Omega_* : \Omega_0 \subset \Omega_1 \subset \cdots \subset \Omega = \bigcup \Omega_s, \qquad \overline{\Omega}_* : \overline{\Omega}_0 \subset \overline{\Omega}_1 \subset \cdots \subset \overline{\Omega} = \bigcup \overline{\Omega}_s \tag{2.15}$$

are filtrations, and we introduce the corresponding gradations:

$$\mathcal{M} = \bigoplus \mathcal{M}_s, \quad \overline{\mathcal{M}} = \bigoplus \overline{\mathcal{M}}_s \quad \left(\mathcal{M}_s = \frac{\Omega_s}{\Omega_{s-1}}, \ \overline{\mathcal{M}}_s = \frac{\overline{\Omega}_s}{\overline{\Omega}_{s-1}} \right), \tag{2.16}$$

where (formally) $\Omega_{-1} = \overline{\Omega}_{-1} = 0$. Forms ω_I^k $(k = 1, \ldots, m;\ |I| \le s)$ provide a basis of Ω_s and the forms

$$\omega_I^k \quad (k = 1, \ldots, m;\ |I| = s) \tag{2.17}$$

(better the classes of these forms) determine a basis of $\mathcal{M}_s$. Analogously the forms $\mathbf{m}^* \omega_I^k$ $(k = 1, \ldots, m;\ |I| \le s)$ generate $\overline{\Omega}_s$ and the forms

$$\mathbf{m}^* \omega_I^k \quad (k = 1, \ldots, m;\ |I| = s) \tag{2.18}$$

(better: the classes) generate $\overline{\mathcal{M}}_s$. Recall that generators *need not* be linearly independent.

Occasionally alternative generators

$$\ell_I^k = \mathcal{L}_I \mathbf{m}^* \omega^k = \mathcal{L}_{i_1} \cdots \mathcal{L}_{i_r} \mathbf{m}^* \omega^k \quad (r = 0, 1, \ldots;\ k = 1, \ldots, m) \tag{2.19}$$

of modules $\overline{\Omega}$ and $\overline{\Omega}_s$ will be employed. The forms ℓ_I^k indeed are generators as follows from the identities

$$\begin{gathered}\ell^k = \mathbf{m}^* \omega^k, \qquad \ell_i^k = \mathcal{L}_i \mathbf{m}^* \omega^k = \sum D_i F^{i'} \mathbf{m}^* \omega_{i'}^k, \ldots, \\ \ell_{Ii}^k = \mathcal{L}_i \ell_I^k = \cdots + \sum D_i F^{i'} \mathbf{m}^* \omega_{Ii'}^k,\end{gathered} \tag{2.20}$$

where recurrence (2.13) and inequality (1.7) are taken into account.

The number of forms (2.17) with given k and $|I| = s$ is $\binom{s+n-1}{n-1}$ by a well-known combinatorial argument; therefore we have the following dimension:

$$\dim \mathcal{M}_s = m \cdot \binom{s+n-1}{n-1} \quad (s \geq 0). \tag{2.21}$$

Clearly $\dim \overline{\mathcal{M}}_s \leq \dim \mathcal{M}_s$ with strong inequality if classes (2.18) are linearly dependent. We need a slightly stronger assertion.

Assertion 1. Assume that there is a nontrivial linear relation in $\overline{\mathcal{M}}_{s_0}$. Then

$$\dim \overline{\mathcal{M}}_s \leq (m-1) \cdot \binom{s+n-1}{n-1} + H(s) \quad (s \geq s_0), \tag{2.22}$$

where $H(s)$ is a polynomial of degree less than $n-1$, $H(s) = 0$ if $n = 1$.

Hint for Proof

Alternative generators (2.19) are useful. If a class $\ell_{I_0}^{k_0} \in \overline{\mathcal{M}}_{s_0}$ ($|I_0| = s_0$) is linearly dependent on remaining classes lying in $\overline{\mathcal{M}}_{s_0}$, also all classes

$$\mathcal{L}_I \ell_{I_0}^{k_0} = \ell_{II_0}^{k_0} \in \overline{\mathcal{M}}_{s+s_0} \quad (|I| = s) \tag{2.23}$$

are dependent on other generators in $\overline{\mathcal{M}}_{s+s_0}$. Therefore the total number of independent classes is estimated by the lower-order polynomial. Please look at the quite transparent particular cases $n = 1, 2, 3$ for better clarity. The assertion is of elementary nature but a complete formal proof would be rather clumsy.

2.4. The Invertibility Main Theorem

The following result provides the most important technical tool for the study of general automorphisms.

Theorem 2.2. *A morphism* $\mathbf{m}$ *is automorphism if* $\omega^k \in \overline{\Omega}$ $(k = 1, \ldots, m)$.

Proof. We will see in (ι) that the assumption implies even $\Omega \subset \overline{\Omega}$ hence and $\Omega = \overline{\Omega}$. Dimensions (2.21) and (2.22) are related in $(\iota\iota)$ and this ensures the injectivity of $\mathbf{m}^*$ in $(\iota\iota\iota)$. Finally $(\iota\nu)$ provides explicit formulae and (ν) clarifies the definition domain of the inverse $\mathbf{m}^{-1}$.

(ι) *A Simple Reasoning.* We assume $\omega^k = \sum a_J^{kl}\mathbf{m}^*\omega_J^l$. However, if

$$\omega_I^k = \sum a_{IJ}^{kl}\mathbf{m}^*\omega_J^l \quad (\text{fixed } k \text{ and } I) \tag{2.24}$$

then

$$\omega_{Ii}^k = \mathcal{L}_i\omega_I^k = \sum D_i a_{IJ}^{kl}\mathbf{m}^*\omega_J^l + \sum a_{IJ}^{kl}\left(\sum D_{i'}F^i\mathbf{m}^*\omega_{Ji}^l\right) \in \overline{\Omega} \tag{2.25}$$

by using (2.13). It follows that the primary assumption implies $\Omega \subset \overline{\Omega}$.

$(\iota\iota)$ *On the Dimensions.* Since $\overline{\Omega}_*$ is a filtration of $\overline{\Omega}$, we have $\Omega_0 \subset \overline{\Omega}_S$ for appropriate (fixed) S large enough. Therefore

$$\Omega_s \subset \overline{\Omega}_{s+S} \quad (s = 0, 1, \ldots) \tag{2.26}$$

by applying $\mathcal{L}_i$ as in (ι). It follows that

$$\dim\Omega_s = \dim\mathcal{M}_0 + \cdots + \dim\mathcal{M}_s \le \dim\overline{\Omega}_{s+S} = \dim\overline{\mathcal{M}}_0 + \cdots + \dim\overline{\mathcal{M}}_{s+S}, \tag{2.27}$$

and therefore

$$\dim\mathcal{M}_{s_0} + \cdots + \dim\mathcal{M}_s \le \dim\overline{\mathcal{M}}_{s_0} + \cdots + \dim\overline{\mathcal{M}}_{s+S} \tag{2.28}$$

for every s_0 and $s \ge s_0$ (we suppose $\dim\mathcal{M}_s = \dim\overline{\mathcal{M}}_s$ if $s < s_0$ here).

Assuming inequality (2.22) for a moment, then (2.28) gives the inequality

$$\binom{s_0+n-1}{n-1} + \cdots + \binom{s+n-1}{n-1} \le H(s_0) + \cdots + H(s) + \dim\overline{\mathcal{M}}_{s+1} + \dim\overline{\mathcal{M}}_{s+S}, \tag{2.29}$$

symbolically $A(s) \le B(s) + C(s)$, and this inequality is contradictory since $A(s)$ is a polynomial of degree n while $B(s)$ and $C(s)$ are polynomials of degrees $n-1$ at most. It follows that inequality (2.22) cannot be satisfied for any s_0. Therefore (2.18) *are linearly independent classes in* $\overline{\mathcal{M}}_s$ *for every* $s \ge 0$.

$(\iota\iota\iota)$ *On the Injectivity.* We have naturally induced *surjection*

$$\mathbf{m}^* : \mathcal{M}_s \longrightarrow \overline{\mathcal{M}}_s \quad (s \ge 0) \tag{2.30}$$

(class $\omega_I^k \in \mathcal{M}_s$ into class $\mathbf{m}^*\omega_I^k \in \overline{\mathcal{M}}_s$), and we have proved the *injectivity* in (ι). By virtue of well-known principle of algebra, bijectivity between gradations $\mathcal{M}, \overline{\mathcal{M}}$ implies bijectivity between filtrations Ω_*, $\overline{\Omega}_*$. Altogether $\mathbf{m}^* : \Omega \to \overline{\Omega} = \Omega$ *is a bijective mapping*.

The obvious congruence

$$\mathrm{d}\mathbf{m}^* x^i = \mathrm{d}F^i \cong \sum D_{i'} F^i \mathrm{d}x^i \quad (\mathrm{mod}\ \Omega) \tag{2.31}$$

and inequality (1.7) together imply that $\mathbf{m}^*$ *is bijective even on the module of all* 1*-forms*. In other words: $\mathbf{m}^*$ *preserves the linear independence of forms*.

($\iota\nu$) *The Inverse Transformation.* Inclusion (2.26) is expressed by

$$\mathrm{d}w_I^k - \sum w_{Ii}^k \mathrm{d}x^i = \sum a_{IJ}^{kl} \left(\mathrm{d}G_J^l - \sum G_{Jj}^l \mathrm{d}F^j \right). \tag{2.32}$$

Moreover congruence (2.31) alternatively reads $\mathrm{d}x^i \cong \sum b_j^i \mathrm{d}F^j$ (mod Ω) by using (1.7). In more detail

$$\mathrm{d}x^i = \sum b_j^i \mathrm{d}F^j + \sum c_J^{il} \left(\mathrm{d}G_J^l - \sum G_{Jj}^l \mathrm{d}F^j \right), \tag{2.33}$$

where (mod Ω) is replaced with equivalent (mod $\overline{\Omega}$). The latter equations imply that we (locally) deal with certain composed functions:

$$x^i = \overline{F}^i \left(\dots, F^j, G_J^l, \dots \right), \qquad w_I^k = \overline{G}_I^k \left(\dots, F^j, G_J^l, \dots \right) \tag{2.34}$$

since differentials $\mathrm{d}F^j = \mathrm{d}\mathbf{m}^* x^j$, $\mathrm{d}G_J^l = \mathrm{d}\mathbf{m}^* w_I^l$, are linearly independent. *We have obtained formulae* (1.24) *for the inverse morphism* $\mathbf{m}^{-1}$.

(ν) *Definition Domains.* In fact only the existence of functions $\overline{F}^i$, $\overline{G}^k$ ($i = 1, \dots, n$; $k = 1, \dots, m$) causes the main difficulties. The remaining functions $\overline{G}_I^k$ with I nonempty follow from the recurrence (1.18). In particular, *there exists a common definition domain for all functions* $\overline{F}^i$, $\overline{G}_I^k$. □

2.5. Towards the Algorithm

Let us recall our task and briefly approach the strategy to follow. We deal with mappings $\mathbf{m} : \mathbf{M} \to \mathbf{M}$ given by formulae

$$\mathbf{m}^* x^i = F^i \left(\dots, x^j, w_J^l, \dots \right), \quad \mathbf{m}^* w_I^k = G_I^k \left(\dots, x^j, w_J^l, \dots \right) \quad \left(\det \left(D_{i'} F^i \right) \neq 0 \right) \tag{2.35}$$

that are *morphisms*; that is, they satisfy the inclusion $\overline{\Omega} \subset \Omega$. This is expressed by certain equations

$$\ell_I^k = \sum a_{IJ}^{kl} \omega_J^l \tag{2.36}$$

in terms of alternative generators ℓ_I^k of module $\overline{\Omega}$ defined by recurrence:

$$\ell^k = \mathbf{m}^*\omega^k, \qquad \ell_{Ii}^k = \mathcal{L}_i \ell_I^k. \tag{2.37}$$

We are however interested in *automorphisms* $\mathbf{m}$. The invertibility of $\mathbf{m}$ is ensured if (2.36) implies certain identities

$$\omega^k = \sum b_J^{kl} \ell_J^l \quad (k = 1, \dots, m) \tag{2.38}$$

in fact equivalent to the inclusion $\Omega \subset \overline{\Omega}$. The algorithm for determining the automorphisms consists of two parts.

The *algebraical part* concerns the requirements on (2.36) ensuring the existence of identities (2.38). Due to the recurrence (2.37), the requirements can be expressed only in terms of the initial forms ℓ^k and hence the initial coefficients a_J^{kl}. Assuming developments

$$\ell^k = \mathbf{m}^*\omega^k = \sum a_J^{kl}\omega_J^l \quad (k = 1, \dots, m;\ |J| \le S) \tag{2.39}$$

of a given order S, the procedure will be of a finite length.

Then, in the *analytic part* of the algorithm, the explicit transcription

$$\mathrm{d}G^k - \sum G_i^k \mathrm{d}F^i = \sum a_J^{kl}\left(\mathrm{d}w_J^l - \sum w_{Jj}^l \mathrm{d}x^j\right) \quad (k = 1, \dots, m) \tag{2.40}$$

of developments (2.39) determines partial differential equations (more correctly: a Pfaffian system) for the functions F^i and G^k involving moreover the prolongations G_i^k. We assume that coefficients a_J^{kl} satisfy the invertibility requirements here. This is the most toilsome part of the algorithm and moreover the solutions $F^i,\ G^k$ need not exist. The remaining functions G_I^k appearing in transformation formulae (2.35) already follow by a routine prolongation procedure (1.15).

Let us leave the general theory and more precise exposition of the algorithm to other place. For the convenience of reader, we will discuss a few particular examples in the meantime.

3. A Few Simple Examples

We start with the zeroth-order case $S = 0$ of developments (2.39). This will provide a proof of the fundamental Lie-Bäcklund theorem as a by-product. Also the case $m = 1$ of one independent variable is quite easy. In both cases $S = 0$ or $m = 1$ there do not exist any generalized automorphisms. On the contrary, there exists an unimaginable amount of such automorphisms if $S > 0$ and $m > 1$. Even the first-order case $S = 1$ with $m = 2$ cannot be completely analysed on our limited space.

Passing to the topic proper, we recall the simplified notation

$$\overline{x}^i = \mathbf{m}^* x^i, \quad \overline{w}_I^k = \mathbf{m}^* w_I^k, \quad \overline{\omega}_I^k = \mathbf{m}^*\omega_I^k = \mathrm{d}G_I^k - \sum G_{Ii}^k \mathrm{d}F^i \tag{3.1}$$

already employed in Introduction. It is quite sufficient in particular examples to follow.

3.1. The Zeroth-Order Case

Assume $S = 0$. Then the assumption (2.39) reads $\overline{\omega}^k = \sum a^{kl}\omega^l$ and $\det(a^{kl}) \neq 0$ is the necessary and sufficient invertibility condition. Identity (2.38) is expressed by $\omega^k = \sum b^{kl}\overline{\omega}^l$ with the inverse matrix $(b^{kl}) = (a^{kl})^{-1}$. This concludes the *algebraic part* of the algorithm.

Let us turn to the *analysis*. We employ (2.38) which read

$$\mathrm{d}w^k - \sum w_i^k \mathrm{d}x^i = \sum b^{kl}\left(\mathrm{d}\overline{w}^l - \sum \overline{w}_j^l \mathrm{d}\overline{x}^j\right) \quad (k = 1,\ldots,m). \tag{3.2}$$

It follows that

$$w^k = g^k\left(x^1,\ldots,x^n,\overline{w}^1,\ldots,\overline{w}^m,\overline{x}^1,\ldots,\overline{x}^n\right) \quad (k = 1,\ldots,m) \tag{3.3}$$

for appropriate functions g^k. Then the identities

$$\mathrm{d}w^k = \sum \frac{\partial g^k}{\partial x^i}\mathrm{d}x^i + \sum \frac{\partial g^k}{\partial \overline{w}^l}\overline{w}^l + \sum \overline{D}_i g^k \cdot \mathrm{d}\overline{x}^i \tag{3.4}$$

immediately follow.

(ι) *The Free Subcase.* First of all assume that differentials $\mathrm{d}x^1,\ldots,\mathrm{d}x^n,\mathrm{d}\overline{w}^1,\ldots,\mathrm{d}\overline{w}^m,\mathrm{d}\overline{x}^1,\ldots,\mathrm{d}\overline{x}^n$ are linearly independent. Then

$$w_i^k = \frac{\partial g^k}{\partial x^i}, \quad b^{kl} = \frac{\partial g^k}{\partial \overline{w}^l}, \quad \overline{D}_i g^k = 0 \quad (k,l = 1,\ldots,m;\ i = 1,\ldots,n) \tag{3.5}$$

by comparison of (3.4) and (3.2). Let us introduce the Jacobians

$$\Delta^K = \det\begin{pmatrix} \dfrac{\partial^2 g^K}{\partial x^i \partial \overline{x}^j} & \dfrac{\partial^2 g^K}{\partial x^i \partial \overline{w}^l} \\ \dfrac{\partial g^k}{\partial \overline{x}^j} & \dfrac{\partial g^k}{\partial \overline{w}^l} \end{pmatrix} \quad (\text{fixed } K) \tag{3.6}$$

(depending on parameters $x^1,\ldots,x^n$) of functions

$$\frac{\partial g^K}{\partial x^1},\ldots,\frac{\partial g^K}{\partial x^n},g^1,\ldots,g^m \tag{3.7}$$

with respect to the variables $\overline{x}^1,\ldots,\overline{x}^n,\overline{w}^1,\ldots,\overline{w}^m$. Assuming $\Delta^K = 0$ identically for certain K, there would be a nontrivial identity:

$$G\left(\frac{\partial g^K}{\partial x^1},\ldots,\frac{\partial g^K}{\partial x^n},g^1,\ldots,g^m,x^1,\ldots,x^n\right) = 0. \tag{3.8}$$

However, this is impossible due to (3.3) and $(3.5)_1$. Therefore $\Delta^K \neq 0$ for every K. Choosing K fixed, the implicit system

$$\frac{\partial g^K}{\partial x^i} = w_i^K, \quad g^k = w^k \quad (i = 1, \ldots, n;\ k = 1, \ldots, m) \tag{3.9}$$

admits a solution of the kind

$$\overline{x}^i = A^i\left(\ldots, x^j, w^l, w_j^K, \ldots\right), \quad \overline{w}^k = B^k\left(\ldots, x^j, w^l, w_j^K, \ldots\right) \quad (i = 1, \ldots, n;\ k = 1, \ldots, m) \tag{3.10}$$

effectively depending on w_j^K for every $K = 1, \ldots, m$ which may be kept fixed but arbitrary. This is again a contradiction if $m > 1$. So we conclude that $m = 1$ and therefore **m** *is the classical Lie's contact transformation*.

($\iota\iota$) *The Rigid Subcase.* Assume that each differential $\mathrm{d}x^i$ $(i = 1, \ldots, n)$ linearly dependent on $\mathrm{d}\overline{w}^1, \ldots, \mathrm{d}\overline{w}^m, \mathrm{d}\overline{x}^1, \ldots, \mathrm{d}\overline{x}^n$. Then there exist certain identities $x^i = \overline{F}^i(\overline{x}^1, \ldots, \overline{x}^n, \overline{w}^1, \ldots, \overline{w}^m)$ and therefore

$$w^k = g^k\left(F^1, \ldots, F^n, \overline{x}^1, \ldots, \overline{x}^n, \overline{w}^1, \ldots, \overline{w}^m\right) = \overline{G}^k\left(\overline{x}^1, \ldots, \overline{x}^n, \overline{w}^1, \ldots, \overline{w}^m\right). \tag{3.11}$$

We deal with the point transformation **m.**

($\iota\iota\iota$) *The Intermediate Case.* Assume that certain identities

$$x^r = f^r\left(\ldots, x^s, \overline{x}^j, \overline{w}^l, \ldots\right), \quad w^k = g^k\left(\ldots, x^s, \overline{x}^j, \overline{w}^l, \ldots\right) \quad (r = 1, \ldots, R;\ k = 1, \ldots, m) \tag{3.12}$$

where $s = R + 1, \ldots, n$ $(1 \leq R < n)$ and differentials

$$\mathrm{d}x^{R+1}, \ldots, \mathrm{d}x^n, \mathrm{d}\overline{x}^1, \ldots, \mathrm{d}\overline{x}^n, \mathrm{d}\overline{w}^1, \ldots, \mathrm{d}\overline{w}^m \tag{3.13}$$

are linearly independent. The reasoning (ι) can be accepted if the Jacobian of functions

$$f^1, \ldots, f^R, \frac{\partial g^K}{\partial x^{R+1}}, \ldots, \frac{\partial g^K}{\partial x^n}, g^1, \ldots, g^m \quad (\text{fixed } K) \tag{3.14}$$

with respect to variables $\overline{x}^1, \ldots, \overline{x}^n, \overline{w}^1, \ldots, \overline{w}^m$ undertakes the role of Δ^K.*The final result is exactly the same as in* (ι).

Summary 1. Let **m** be automorphism such that $\mathbf{m}^*\Omega_0 \subset \Omega_0$. Then **m** is either a point transformation or **m** is the classical Lie's contact transformation.

The next important folklore result easily follows.

Theorem 3.1 (Lie-Bäcklund [5]). *Let* **m** *be automorphism that preserves the space of jet variables*

$$x^i, w_I^k \quad (i = 1, \ldots, n;\ k = 1, \ldots, m;\ |I| \le S) \tag{3.15}$$

of a certain order S*. Then either* **m** *is a point transformation or* $m = 1$ *and* **m** *is the classical Lie's contact transformation.*

Proof. Clearly $\mathbf{m}^*\Omega_S \subset \Omega_S$. Assuming $S > 0$, then $\Omega_{S-1} \subset \Omega_S$ is the *uniquely determined* submodule of all $\omega \in \Omega_S$ such that $d\omega \cong 0 \pmod{\Omega_S}$ and it follows that necessarily $\mathbf{m}^*\Omega_{S-1} \subset \Omega_{S-1}$. Continuing, we conclude that $\mathbf{m}^*\Omega_0 \subset \Omega_0$ and the Summary 1 applies. □

3.2. *The Case of One Dependent Variable*

Let us state the only remaining simple case of automorphisms.

Theorem 3.2. *In the case* $m = 1$ *of one dependent variable, the classical point and Lie's contact transformations are the only possible automorphisms* **m**.

Proof. Assuming $m = 1$, we abbreviate $\omega_I = \omega_I^1$ and these forms constitute a basis of Ω if only the multiindices I with a different lexicographic order are employed. Let

$$\omega = \sum a_I \omega_I = \cdots + \sum a_{I_0} \omega_{I_0} \in \Omega \quad (a_{I_0} \neq 0) \tag{3.16}$$

be a given form with the top order terms. Then the forms

$$\mathcal{L}_J \omega = \mathcal{L}_J \sum a_i \omega_I = \cdots + \sum a_{I_0} \omega_{I_0 J} \quad (\text{various multi-indices } J) \tag{3.17}$$

are linearly independent. They generate Ω_0 if and only if $I_0 = \phi$ is empty, and therefore if $\omega \in \Omega_0$. A given automorphism **m** preserves this property which implies $\mathbf{m}^*\Omega_0 \subset \Omega_0$ and the Summary 1 applies. □

3.3. *On the First-Order Case*

We have $S = 1$ and let us moreover suppose $m = 2$. Then (2.39) reads

$$\overline{\omega}^k = \sum a^{kl} \omega^l + \sum a_j^{kl} \omega_j^l \in \Omega_1 \quad (k = 1, 2). \tag{3.18}$$

Several requirements will be stated in order to ensure the inclusions $\omega^1, \omega^2 \in \overline{\Omega}$. We start with the proportionality requirement

$$a^{2l}{}_j = C a^{1l}{}_j \quad (j = 1, \ldots, m;\ l = 1, 2) \tag{3.19}$$

for the top order summands in (3.18). Then

$$\ell = \overline{\omega}^2 - C\overline{\omega}^1 = \sum c^l \omega^l \in \Omega_0 \quad \left(c^l = a^{2l} - C a^{1l}\right) \tag{3.20}$$

is a zeroth-order form. We obtain the first-order forms:

$$\ell_i = \mathcal{L}_i \ell = \sum D_i c^l \omega^l + \sum c^l \omega^l_j \in \Omega_1 \quad (i = 1, \dots, n), \tag{3.21}$$

and let us introduce the requirement

$$a^{1l}_j = c^l C_j \quad (j = 1, \dots, m;\ l = 1, 2). \tag{3.22}$$

Then we obtain the zeroth-order form:

$$\overline{\omega}^1 - \sum C_j \ell_j = \sum \left(a^{1l} - \sum C_j D_j c^l \right) \omega^l \in \Omega_0, \tag{3.23}$$

and the invertibility is ensured if

$$\det \begin{pmatrix} c^1 & a^{11} - \sum C_j D_j c^1 \\ c^2 & a^{12} - \sum C_j D_j c^2 \end{pmatrix} \neq 0. \tag{3.24}$$

This concludes the *algebra* and let us turn to the *analysis*. In contrast to Section 3.1, only two rather particular but quite instructive prospects here without any ambitions on thorough theory can be discussed here.

(ι) *A Point-Like Subcase.* Let us insert the assumption

$$\overline{x}^i = F^i\left(\dots, x^j, w^l, \dots\right), \qquad \overline{w}^2 = \mathcal{G}\left(\overline{w}^1, \dots, x^j, w^l, \dots\right) \tag{3.25}$$

into identity (3.20). We obtain the equation

$$\sum \left(\sum \left(\mathcal{G}_{\overline{w}^1} \overline{w}^1_j - \overline{w}^2_j \right) D_i F^j + D_i \mathcal{G} \right) \mathrm{d}x^i + \left(\mathcal{G}_{\overline{w}^1} - C \right) \overline{\omega}^1 + \sum \left(\mathcal{G}_{w^l} - c^l \right) \omega^l = 0 \tag{3.26}$$

which is satisfied if

$$\sum \overline{w}^2_j D_i F^j = \sum \mathcal{G}_{\overline{w}^1} \overline{w}^1_j D_i F^j + D_i \mathcal{G}, \quad \mathcal{G}_{\overline{w}^1} = C, \quad \mathcal{G}_{w^l} = c^l \tag{3.27}$$

for $i = 1, \dots, n$. Let us moreover insert

$$\overline{w}^1 = G\left(\dots, x^j, w^l, w^l_j, \dots\right), \qquad \overline{x}^i = F^i\left(\dots, x^j, w^l_j, \dots\right) \tag{3.28}$$

into identity (3.18) with $k = 1$. We obtain the equation

$$\sum \left(D_i G - \sum \overline{w}^1_j D_i F^j \right) \mathrm{d}x^i + \sum \left(G_{w^l} - \sum \overline{w}^1_j F^j_{w^l} - a^{1l} \right) \omega^l + \sum \left(G_{w^l_j} - a^{1l}_j \right) \omega^l_j = 0 \tag{3.29}$$

which is satisfied if

$$\sum \overline{w}_j^1 D_j F^i = D_i G, \qquad G_{w^l} = \sum \overline{w}_i^1 F^i_{w^l} + a^{1l}, \qquad G_{w^l_j} = a^{1l}{}_j. \tag{3.30}$$

Since $(3.27)_1$ and $(3.30)_2$ are a mere prolongation formulae, we have obtained only the differential equation

$$G_{w^l_j} = a^{1l}_j = c^l C_j = \mathcal{G}_{w^l} C_j \quad (j = 1, \ldots, n;\ l = 1, 2) \tag{3.31}$$

with uncertain coefficients C_j. The equation is satisfied if

$$\mathcal{G} = a\left(\overline{w}^1, \ldots, x^j, \ldots\right) + b\left(\ldots, x^j, w^l, \ldots\right), \qquad G = c\left(\ldots, x^j, w^l, \sum b_{w^l} w^l_j, \ldots\right), \tag{3.32}$$

where $a,\ b,\ c$ may be quite arbitrary functions (direct verification).

Summary 2. Let us choose functions $F^i,\ G,\ \mathcal{G}$ as above. Then, in the "not too special" case (3.24), the formulae

$$x^i = F^i \quad (i = 1, \ldots, n), \qquad \overline{w}^1 = G, \qquad \overline{w}^2 = \mathcal{G}\left(G^1, \cdots, x^j, w^l, \cdots\right) \tag{3.33}$$

determine an automorphism $\mathbf{m}$ of the jet space $\mathbf{M}$.

Remark 3.3. The inequality (3.24) is "in general" satisfied. For instance, we obtain the unit matrix for the simplest choice $F^i = x^i,\ a = 0,\ b = w^2,\ c = w^1 + \sum \lambda_j w_j^2$ (constants $\lambda_j \in \mathbb{R}$) which provide the "quite trivial" Abelian Lie group of automorphisms:

$$\overline{x}^i = x^i, \quad \overline{w}^1 = w^1 + \sum \lambda_i w_i^2, \quad \overline{w}^2 = w^2 \quad (\lambda_i \in \mathbb{R};\ i = 1, \ldots, n). \tag{3.34}$$

Even this group is lying beyond the common symmetry theories since it does not preserve the finite-order jet spaces.

($\iota\iota$) *A Contact-Like Subcase.* Let us assume

$$\overline{x}^r = x^r \quad (r = 2, \ldots, n), \qquad \overline{w}^2 = \mathcal{A}\left(\overline{x}^1, \overline{w}^1, \cdots, x^j, w^l, \cdots\right). \tag{3.35}$$

We obtain the requirement

$$\left(\overline{D}_1\mathcal{A} - \overline{w}_1^2\right)d\overline{x}^1 + D_1\mathcal{A}dx^1 + \sum\left(D_r\mathcal{A} - \overline{w}_r^1\right)dx^r + \left(\mathcal{A}_{\overline{w}^1} - C\right)\overline{w}^1 + \sum\left(\mathcal{A}_{w^l} - c^l\right)\omega^l = 0 \tag{3.36}$$

from identity (3.20) and this is satisfied if

$$\overline{w}_1^2 = \overline{D}_1\mathcal{A} = \mathcal{A}_{\overline{x}^1} + \overline{w}_1^1\mathcal{A}_{\overline{w}^1}, \quad \overline{w}_r^1 = D_r\mathcal{A}, \quad D_1\mathcal{A} = 0, \quad \mathcal{A}_{\overline{w}^1} = C, \quad \mathcal{A}_{w^l} = c^l, \tag{3.37}$$

where $r = 2, \ldots, n$ and $l = 1, 2$. Let us suppose that the equation

$$D_1 \mathcal{H} = \mathcal{H}_{x^1} + \sum w_1^l \mathcal{H}_{w^l} = 0 \tag{3.38}$$

admits a solution $\overline{w}^1 = \mathcal{G}(\overline{x}^1, \ldots, x^j, w^l, w_1^l, \ldots)$. Then identity (3.18) with $k = 1$ provides the requirement

$$\left(\mathcal{G}_{\overline{x}^1} - \overline{w}_1^1\right) \mathrm{d}\overline{x}^1 + D_1 \mathcal{G}\, \mathrm{d}x^1 + \sum \left(D_r \mathcal{G} - \overline{w}_r^1\right) \mathrm{d}x^r + \sum \left(\mathcal{G}_{w^l} - a^{1l}\right) \omega^l + \sum \left(\mathcal{G}_{w_j^l} - a_j^{1l}\right) \omega_j^l = 0. \tag{3.39}$$

This is satisfied if

$$\overline{w}_1^1 = \mathcal{G}_{\overline{x}^1}, \quad \overline{w}_r^1 = D_r \mathcal{G}, \quad D_1 \mathcal{G} = 0, \quad \mathcal{G}_{w^l} = a^{1l}, \quad \mathcal{G}_{w_j^l} = a^{1l}{}_j. \tag{3.40}$$

Altogether taken, besides the prolongation formulae, we have obtained only two equations:

$$D_1 \mathcal{G} = 0, \quad \mathcal{G}_{w_1^l} = a_1^{1l} = c^l C_1 = \mathcal{H}_{w^l} C_1 \quad (l = 1, 2) \tag{3.41}$$

where c^1, c^2 are uncertain coefficients. The second equation is identically satisfied. This follows from the identities:

$$\begin{gathered} D_1 \mathcal{H}\left(\overline{x}^1, \mathcal{G}, \ldots, x^j, w^l, w_1^l, \ldots\right) = 0, \\ \frac{d}{dw_1^k} D_1 \mathcal{H}(\cdots) = \left(\mathcal{H}_{x^1 \overline{w}^1} + \sum w_1^l \mathcal{H}_{w^l \overline{w}^1}\right) \mathcal{G}_{w_1^k} + \mathcal{H}_{w^k} = 0 \quad (k = 1, 2). \end{gathered} \tag{3.42}$$

As the first equation

$$D_1 \mathcal{G} = \mathcal{G}_{x^1} + \sum w_1^l \mathcal{G}_{w^l} + \sum w_{11}^l \mathcal{G}_{w_1^l} = 0 \tag{3.43}$$

is concerned, we suppose that it admits a solution $\overline{x}^1 = F^1(\ldots, x^j, w^l, w_1^l, w_{11}^l, \ldots)$ by applying the implicit function theorem.

Summary 3. Let $\mathcal{H}(\overline{x}^1, \overline{w}^1, \ldots, x^j, w^l, \ldots)$ be a "not too special" function such that the implicit equation $D_1 \mathcal{H} = 0$ admits a solution $\overline{w}^1 = \mathcal{G}(\overline{x}^1, \ldots, x^j, w^l, w_1^l, \ldots)$ and the equation $D_1 \mathcal{G} = 0$ admits a solution $\overline{x}^1 = F^1(\ldots, x^j, w^l, w_1^l, w_{11}^l, \ldots)$. Then the formulae

$$\begin{gathered} \overline{x}^1 = F^1\left(\ldots, x^j, w^l, w_1^l, w_{11}^l \ldots\right), \qquad \overline{x}^r = x^r \quad (r = 2, \ldots, n), \\ \overline{w}^1 = \mathcal{G}\left(\overline{x}^1, \ldots, x^j, w^l, w_1^l, \ldots\right), \qquad \overline{w}^2 = \mathcal{H}\left(F^1, \mathcal{G}, \ldots, x^j, w^l, \ldots\right) \end{gathered} \tag{3.44}$$

determine an automorphism $\mathbf{m}$ of the jet space $\mathbf{M}$.

One can check by direct calculation that condition (3.24) is satisfied if

$$\operatorname{rank}\begin{pmatrix} \mathcal{H}_{w^1} & \mathcal{H}_{w^1w^1} & \mathcal{H}_{w^1w^2} \\ \mathcal{H}_{w^2} & \mathcal{H}_{w^2w^1} & \mathcal{H}_{w^2w^2} \end{pmatrix} = 2. \tag{3.45}$$

Remark 3.4. Assuming $n = 1$, the result may be identified with the particular case of the multiple waves (ι) in Section 1.4, see also [1]. In fact, we have applied this lower dimensional result $n = 1$ along every fibre $x^r = \text{const.}$ ($r = 2, \ldots, n$). Quite analogous construction is possible for any given automorphism in lower dimensions in order to obtain automorphisms with large values of n and m. Then the composition with (e.g.) the point transformations provides automorphisms where the variables x^r ($r = 2, \ldots, n$) in formula $(3.35)_1$ need not be preserved.

4. Concluding Results

In the above examples, the algebraic part of the algorithm looks easier than the subsequent analysis. Nevertheless in full generality, the algebraic reasoning is dominant and moreover strongly affects all fundamental aspects of the theory, for example, the nature of the composition rules of automorphisms together with the structural results. Yet, the analysis remains a toilsome task, though it consists of the well-known compatibility mechanisms for the existence of solutions of differential or Pfaffian equations which does not append much novelties.

4.1. More on the Algebra

We recall filtration $\Omega_* : \Omega_0 \subset \Omega_1 \subset \cdots \subset \Omega = \bigcup \Omega_s$ of module Ω. Here $\Omega_s \subset \Omega$ are submodules with the basis ω^k_I ($k = 1, \ldots, m$; $|I| \le s$) satisfying the recurrence $\mathcal{L}_i \omega^k_I = \omega^k_{Ii}$. A morphism $\mathbf{m}$ is characterized by the inclusion $\overline{\Omega} \subset \Omega$ where $\overline{\Omega}$ is module generated by all forms $\mathbf{m}^* \omega^k_I$. We recall filtration $\overline{\Omega}_* : \overline{\Omega}_0 \subset \overline{\Omega}_1 \subset \cdots \subset \overline{\Omega} = \bigcup \overline{\Omega}_s$. Here $\overline{\Omega}_s \subset \overline{\Omega}$ are submodules generated by all forms $\mathbf{m}^* \omega^k_I (k = 1, \ldots, m;\ |I| \le s)$; however, the alternative generators $\ell^k_I = \mathcal{L}_I \mathbf{m}^* \omega^k$ ($k = 1, \ldots, m$; $|I| \le s$) satisfying the recurrence $\mathcal{L}_i \ell^k_I = \ell^k_{Ii}$ are more appropriate in some respects. It follows that module $\overline{\Omega}_{s+1}$ is generated from $\overline{\Omega}_s$ by using operator $\mathcal{L}_i$. In more detail

$$\overline{\Omega}_{s+1} = \overline{\Omega}_s + \sum \mathcal{L}_i \overline{\Omega}_s \quad (s = 0, 1, \ldots) \tag{4.1}$$

in a brief record where $\mathcal{L}_i \overline{\Omega}_s$ ($i = 1, \ldots, m$) denotes the module with generators $\mathcal{L}_i \omega$ ($\omega \in \overline{\Omega}_s$). In full generality

$$\overline{\Omega}_{r+s} = \sum \mathcal{L}_I \overline{\Omega}_s \quad (\text{sum over } I \text{ with } |I| \le r). \tag{4.2}$$

We are interested in invertible morphisms $\mathbf{m}$, the *automorphisms*. Our task is in fact twofold: *the existence* (the particular examples of automorphisms) and the *criterion* (whether a given morphism is invertible or not).

Due to Theorem 1.2, automorphisms are characterized by the inclusion $\Omega_0 \subset \overline{\Omega}$. The infinite-dimensional module $\overline{\Omega}$ causes some difficulties. It may be replaced by a finite-dimensional one as follows. Let $\mathbf{m}$ be a *morphism of the order S*; that is, we suppose either of the equivalent conditions

$$\overline{\Omega}_0 \subset \Omega_S, \quad \ell^k = \mathbf{m}^* \omega^k \in \Omega_S \quad (k = 1, \ldots, m) \tag{4.3}$$

satisfied. Our reasoning rest on the following simple remark.

Assertion 2. A morphism $\mathbf{m}$ of order S is invertible if and only if $\Omega_0 \subset \Omega_S \cap \overline{\Omega}$.

In order to verify the last inclusion for a given morphism $\mathbf{m}$, we will try to determine the module $\Omega_S \cap \overline{\Omega}$ only in terms of forms lying in Ω_S. This may be regarded as a "finite-dimensional" approach. Let us turn to more details.

Definition 4.1. Let $\mathbf{m}$ be a morphism of the order S. A *saturation* $\mathfrak{I}$ (of submodule $\overline{\Omega}_0 \subset \Omega_S$) is the least submodule $\overline{\Omega}_0 \subset \mathfrak{I} \subset \Omega_S$ such that

$$\sigma(I) \in \mathfrak{I}, \quad \omega = \sum \mathcal{L}_I \sigma(I) \in \Omega_S \quad \text{implies } \omega \in \mathfrak{I}. \tag{4.4}$$

Symbolically

$$\Omega_S \cap \sum \mathcal{L}_I \mathfrak{I} \subset \mathfrak{I} \tag{4.5}$$

for any finite sum over multi-indices I. In particular, if $\sigma \in \Omega_{S-1} \cap \mathfrak{I}$, then $\mathcal{L}_i \sigma \in \Omega_S$ and hence $\mathcal{L}_i \sigma \in \mathfrak{I}$.

Remark 4.2. The rules

$$\begin{gathered} a\mathcal{L}_i \sigma = \mathcal{L}_i (a\sigma) - D_i a \cdot \sigma, \\ a\mathcal{L}_{ii'} \sigma = \mathcal{L}_{ii'} (a\sigma) - \mathcal{L}_{i'} (D_i a \cdot \sigma) - D_{ii'} a \cdot \sigma, \ldots \end{gathered} \tag{4.6}$$

ensure that (4.4) may be replaced by seemingly stronger requirement:

$$\sigma(I') \in \mathfrak{I}, \quad \omega = \sum a(I, I') \mathcal{L}_I \sigma(I') \in \Omega_S \quad \text{implies } \omega \in \mathfrak{I}, \tag{4.7}$$

where $a(I, I')$ are arbitrary coefficients. Moreover the sum $\sum \mathcal{L}_I \mathfrak{I}$ in (4.5) may be regarded as a module.

The following results are self-evident.

Lemma 4.3. *One has* $\mathfrak{I} \subset \Omega_S \cap \overline{\Omega}$.

Proof. Forms $\sigma \in \mathfrak{I}$ arise by applying $\mathcal{L}_i$ to $\overline{\Omega}_0$ and hence $\mathfrak{I} \subset \overline{\Omega}$. □

Consequence 4. If $\Omega_0 \subset \mathfrak{I}$, then $\mathbf{m}$ is invertible morphism.

The latter result is quite sufficient if we search for particular *examples* of automorphism. In this sense, it was latently applied in Section 3.3. Alas, the *criterion* problem is more difficult.

Theorem 4.4. *One has* $\Omega_S \cap \overline{\Omega} = \mathfrak{I}$.

Proof. Due to Lemma 4.3, we have to verify the inclusion $\Omega_S \cap \overline{\Omega} \subset \mathfrak{I}$ or, equivalently, all inclusions $\Omega_S \cap \overline{\Omega}_r \subset \mathfrak{I}$ $(r = 0, 1, \ldots)$ since $\overline{\Omega} = \bigcup \overline{\Omega}_r$. However

$$\Omega_S \cap \overline{\Omega}_r = \Omega_S \cap \sum_{|I| \le r} \mathcal{L}_I \overline{\Omega}_0 \subset \Omega_S \cap \sum \mathcal{L}_I \mathfrak{I} \subset \mathfrak{I} \tag{4.8}$$

by applying (4.2) with $s = 0$, inclusion $\overline{\Omega}_0 \subset \mathfrak{I}$, and (4.5). □

Consequence 5. A morphism $\mathbf{m}$ is invertible if and only if $\Omega_0 = \mathfrak{I}$.

Remark 4.5. Since Ω_S is a finite-dimensional module, we expect that only indices I with a certain limited length $|I| \le s(\mathbf{m})$ are effectively appearing in formulae (4.4) and (4.5). This is the most delicate difficulty of our approach: *to estimate the length* $s(\mathbf{m})$ *which is enough for the calculations with a given morphism* $\mathbf{m}$. In a certain sense, the situation resembles the criterion of involutivity of exterior systems: though the general theory is not easy, the particular examples can be resolved at a limited place.

4.2. One Independent Variable

Assuming $n = 1$ through Section 4.2, we abbreviate $x = x^1$, $\mathcal{L} = \mathcal{L}_1$, $w^k_{(s)} = w^k_I$, $\omega^k_{(s)} = \omega^k_I$ $(I = 1 \cdots 1$ with s terms) and then $\mathcal{L}^r \omega^k_{(s)} = \omega^k_{(s+r)}$. The saturations are very simplified in this case.

Theorem 4.6. *Let* $n = 1$ *and* $\mathbf{m}$ *be a morphism of the order* S. *Each of the following three requirements is equivalent to the condition* (4.4).

(ı) *If* $\sigma(r) \in \mathfrak{I}$ *and* $\omega = \sum \mathcal{L}^r \sigma(r) \in \Omega_S$, *then* $\omega \in \mathfrak{I}$.

(ıı) *If* $\sigma \in \mathfrak{I}$ *and* $\mathcal{L}\sigma \in \Omega_S$ *then* $\mathcal{L}\sigma \in \mathfrak{I}$.

(ııı) *If* $\sigma \in \Omega_{S-1} \cap \mathfrak{I}$, *then* $\mathcal{L}\sigma \in \mathfrak{I}$.

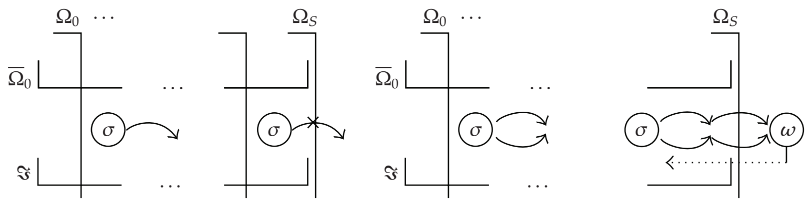
Proof. (ı) is a mere reformulation of (4.4) for the case $n = 1$. Moreover (ı) trivially implies (ıı) and (ıı) is equivalent to (ııı). Let us assume (ıı) in order to prove (ı).

We denote

$$\sigma(r) = \sum a(r)^l_s \omega^l_{(s)} \quad (r = 1, \ldots, R;\ \text{sum over } l = 1, \ldots, m,\ s = 0, \ldots, S), \tag{4.9}$$

where $R > 0$ is supposed. Then $\omega = \cdots + \sum a(R)^l_S \omega^l_{(S+R)}$ and therefore $a(R)^l_S = 0$ $(l = 1, \ldots, m)$. It follows that $\sigma(R) \in \Omega_{S-1}$ and hence $\mathcal{L}\sigma(R) \in \mathfrak{I}$ by using (ııı). Then

$$\omega = \sum^{R-2} \mathcal{L}^r \sigma(r) + \mathcal{L}^{R-1} \sigma'(R) \quad \text{where } \sigma'(R) = \sigma(R-1) + \mathcal{L}\sigma(R) \in \mathfrak{I}. \tag{4.10}$$



$n = 1$: $\mathcal{L}$ acts only inside Ω_S

(a)

$n > 1$: $\mathcal{L}_i$ act even beyond Ω_S

(b)

Figure 3

The form ω was expressed by means of $R-1$ summands. Continuing in this way, we obtain even $\omega = \mathcal{L}\sigma''(1) \in \Omega_S$ with $\sigma''(1) \in \mathfrak{I}$ and whence $\omega \in \mathfrak{I}$. □

Roughly saying, operator $\mathcal{L}$ repeatedly applied to Ω_0 *inside the module* Ω_S leads to the sought saturation $\mathfrak{I}$ and this is achieved after $\dim \Omega_S - \dim \overline{\Omega}_0$ steps at most (Figure 3(a) above). In spite of this lucky reality, a thorough discussion of particular examples need not be easy if the value of the order S is large [1].

4.3. Example

Assuming $n \geq 2$, calculations inside module Ω_S may be quite sufficient (see Section 3.3) or not (Figure 3(b)) and we mention a simplest possible example of this kind.

Suppose $n = m = 2$, $S = 1$ and

$$\begin{aligned} \overline{\omega}^1 &= a^1\omega^1 + a^2\omega^2 + a\omega^1_1 + \omega^2_1, \\ \overline{\omega}^2 &= b^1\omega^1 + b^2\omega^2 + a\omega^1_2 + \omega^2_2. \end{aligned} \tag{4.11}$$

The action of $\mathcal{L}_1$, $\mathcal{L}_2$ inside $\Omega_S = \Omega_1$ is useless at this stage; however,

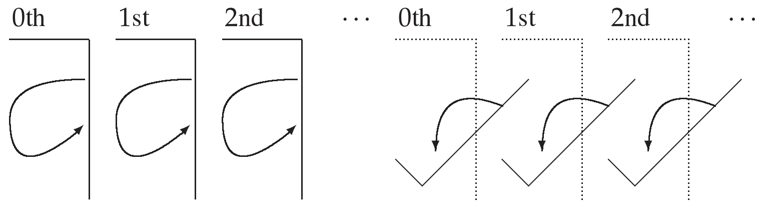
$$\begin{aligned} \omega = \mathcal{L}_2\overline{\omega}^1 - \mathcal{L}_1\overline{\omega}^2 &= \left(D_2a^1 - D_1b^1\right)\omega^1 + \left(D_2a^2 - D_1b^2\right)\omega^2 + \left(-b^1 + D_2a\right)\omega^1_1 \\ &\quad + \left(a^1 - D_1a\right)\omega^1_2 - b^2\omega^2_1 + a^2\omega^2_2 \in \mathfrak{I} \end{aligned} \tag{4.12}$$

in accordance with Definition 4.1 where $\sigma(2) = \overline{\omega}^1$, $\sigma(1) = -\overline{\omega}^2$. Let us assume the identities

$$-b^1 + D_2a = -b^2a, \qquad a^1 - D_1a = a^2a. \tag{4.13}$$

Then

$$\ell = \omega + b^2\overline{\omega}^2 - a^2\overline{\omega}^2 = A\omega^1 + B\omega^2 \in \mathfrak{I}, \tag{4.14}$$



(a) a classical group of symmetries

(b) a group of generalized automorphism

Figure 4

where $A = D_2a^1 - D_1b^1 + b^2a^1 - a^2b^1,\ B = D_2a^2 - D_1b^2$. Moreover $A = Ba$ identically (direct verification using (4.13)) and whence

$$\mathcal{L}_1\ell - B\overline{\omega}^1 = \left(D_1A - Ba^1\right)\omega^1 + \left(D_1B - Ba^2\right)\omega^2 \in \mathfrak{I} \tag{4.15}$$

and also $\mathcal{L}_2\ell - B\overline{\omega}^2 \in \mathfrak{I}$. The inclusion $\Omega_0 \subset \mathfrak{I}$ is guaranteed if

$$\begin{aligned}\operatorname{rank}\begin{pmatrix} A & D_1A - Ba^1 \\ B & D_1B - Ba^2 \end{pmatrix} &= \operatorname{rank}\begin{pmatrix} Ba & D_1(Ba) - B\left(D_1a + a^2a\right) \\ B & D_1B - Ba^2 \end{pmatrix} \\ &= \operatorname{rank} B^2\begin{pmatrix} a & D_1a - a^1 \\ 1 & a^2 \end{pmatrix} = \operatorname{rank} B^2\begin{pmatrix} a & -a^2a \\ 1 & a^2 \end{pmatrix} = 2\end{aligned} \tag{4.16}$$

which is satisfied if $a, a^2, B \neq 0$.

Summary 4. The choice $a \neq 0,\ a^2 \neq 0,\ B \neq 0$ and $a^1 = D_1a + a^2a$ together with the identity $b^1 = D_2a + b^2a$ between remaining coefficients $b^1,\ b^2$ ensure the equality $\Omega_0 = \mathfrak{I}$.

Appendix

The common approach to jets in actual literature rests on the C. Ehresmann mechanisms of smooth sections of fibered manifolds $\pi : Y \to X$ and the jets prolongations $\pi^r : J^rY \to X$ are equipped with a huge family of purely technical concepts. Differential equations with prolongations then appear as a nontrivial achievement due to the difficult compatibility problems. A somewhat paradoxically, this ingenious approach does not rigorously include even the classical Lie's theory of first-order partial differential equations with his generalized solutions and the reductions with respect to the Cauchy characteristics. Also generalized automorphisms and the relevant generalized group symmetries of differential equations are in fact beyond the scope of this theory. We have intentionally used the notation $\mathbf{M} = \mathbf{M}(m, n)$ for the jet spaces instead of the common $J^r(Y, X)$ or $J^\infty(Y, X)$ since we do not regard the jet projections π^r as a reasonable intrinsical concept for this paper. Analogously the notation $\Omega = \Omega(m, n)$ for the contact forms corresponds to the fact that only the number of dependent and independent variables are important, not their actual choice.

Passing to quite general differential equations, they can be introduced without any use of coordinates [4]. In more detail, an infinitely prolonged system of differential equations may be identified with a finite-codimensional submodule $\Omega \subset \Phi(\mathbf{M})$ of the module $\Phi(\mathbf{M})$ of all 1-forms on a manifold $\mathbf{M}$ which satisfies a certain Noetherian property. (The property is as follows. Let $H(\Omega)$ be the module of all vector fields Z such that $\omega(Z) = 0$ $(\omega \in \Omega)$. Then $\mathcal{L}_Z\Omega \subset \Omega$ $(Z \in H(\Omega))$ and module Ω is generated by applying $\mathcal{L}_Z$ to an appropriate finite-dimensional submodule $\Omega_0 \subset \Omega$.) Using this abstract approach, we believe that *generalized automorphisms of systems of differential equations* are available.

Let us finally mention the groups of automorphisms together with the relevant infinitesimal transformations. The classical infinitesimal symmetries preserve the finite-order jet spaces and always generate the group of transformations (Figure 4(a)). They were thoroughly investigated since the times of Lie. On the contrary generalized (Lie-Bäcklund, higher order) infinitesimal symmetries need not generate any group and are only regarded as formal series [5–9]. They generate a true group of (generalized) automorphisms if and only if certain finite-dimensional subspaces depending on the group under consideration are preserved (Figure 4(b)). (In full detail, the criterion is as follows. *A vector field Z on* $\mathbf{M}$ *locally generates a one-parameter group if and only if for every function f on* $\mathbf{M}$, *the infinite family* $Z^n f$ $(n = 0, 1, \ldots)$ *involves only a finite number of functionally independent terms* [4]. For the case of automorphisms of jets, the inspection of functions $f = x$ and $f = w^k$ $(k = 1, \ldots, m)$ is enough.) The subspaces should be determined *together* with the group. This is a serious difficulty and a problem which has not been solved yet.

Acknowledgment

This research has been conducted at the Department of Mathematics as part of the research project CEZ: Progressive reliable and durable structures, MSM 0021630519.

References

[1] V. Tryhuk and V. Chrastinová, "Automorphisms of curves," *Journal of Nonlinear Mathematical Physics*, vol. 16, no. 3, pp. 259–281, 2009.

[2] V. Chrastinová and V. Tryhuk, "Generalized contact transformations," *Journal of Applied Mathematics, Statistics and Informatics*, vol. 3, no. 1, pp. 47–62, 2007.

[3] V. Chrastinová, "Parallel curves in three–dimensional space," in *Proceedings of the 5th Conference on Mathematics and Physics at Technical Colleges*, pp. 120–124, University of Defence, Brno, Czech Republic, 2007.

[4] J. Chrastina, *The Formal Theory of Differential Equations*, vol. 6 of *Folia Facultatis Scientiarium Naturalium Universitatis Masarykianae Brunensis. Mathematica*, Masaryk University, Brno, Czech Republic, 1998.

[5] R. L. Anderson and N. H. Ibragimov, *Lie-Bäcklund Transformations in Applications*, vol. 1 of *SIAM Studies in Applied Mathematics*, SIAM, Philadelphia, Pa, USA, 1979.

[6] P. J. Olver, *Applications of Lie Groups to Differential Equations*, vol. 107 of *Graduate Texts in Mathematics*, Springer, New York, NY, USA, 1986.

[7] P. J. Olver, *Invariants and Symmetry*, vol. 107 of *Graduate Texts in Mathematics*, Cambridge University Press, New York, NY, USA, 1995.

[8] I. S. Krasil'shchik, V. V. Lychagin, and A. M. Vinogradov, *Geometry of Jet Spaces and Nonlinear Partial Differential Equations*, vol. 1 of *Advanced Studies in Contemporary Mathematics*, Gordon and Breach Science Publishers, New York, NY, USA, 1986.

[9] N. Kamran, *Selected Topics in the Geometrical Study of Differential Equations*, vol. 96 of *Regional Conference Series in Mathematics*, AMS, Providence, RI, USA, 2002.