

Research Article

Frequent Oscillatory Behavior of Delay Partial Difference Equations with Positive and Negative Coefficients

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This paper is concerned with a class of nonlinear delay partial difference equations with positive and negative coefficients, which also contains forcing terms. By making use of frequency measures, some new oscillatory criteria are established.

1. Introduction

Partial difference equations are difference equations that involve functions with two or more independent integer variables. Such equations arise from considerations of random walk problems, molecular structure problems, and numerical difference approximation problems. Recently, there have been a large number of papers devoted to partial difference equations, and the problem of oscillatory of solutions and frequent oscillatory solutions for partial difference equations is receiving much attention.

In [1], authors considered oscillatory behavior of the partial difference equations with positive and negative coefficients of the form

$$A_{m+1,n} + A_{m,n+1} - A_{m,n} + p(m,n)A_{m-k,n-l} - q(m,n)A_{m-k',n-l'} = 0, \quad (1.1)$$

but they have not discussed frequent oscillations of this equation.

In [2], authors considered oscillatory behavior for nonlinear partial difference equations with positive and negative coefficients of the form

$$\begin{aligned} \omega f(m, n, A_{m-\sigma, n-\tau}, A_{m-u, n-v}) + A_{m-1, n} + A_{m, n-1} - A_{m, n} + pA_{m+k, n+l} - qA_{m+k', n+l'} = 0, \\ \omega_{m, n} f(m, n, A_{m-\sigma, n-\tau}, A_{m-u, n-v}) + A_{m-1, n} + A_{m, n-1} - A_{m, n} + p_{mn}A_{m+k, n+l} - q_{mn}A_{m+k', n+l'} = 0 \end{aligned} \quad (1.2)$$

In [3], authors considered frequent oscillation in the nonlinear partial difference equation

$$\begin{aligned} u_{m, n} = u_{m+1, n} + u_{m, n+1} + p_{m, n} |u_{m-k_1, n-l_1}|^\alpha \operatorname{sgn} u_{m-k_1, n-l_1} \\ + q_{m, n} |u_{m-k_2, n-l_2}|^\beta \operatorname{sgn} u_{m-k_2, n-l_2} = 0. \end{aligned} \quad (1.3)$$

In [4], authors considered oscillations of the partial difference equations with several nonlinear terms of the form,

$$u_{m+1, n} + u_{m, n+1} - u_{m, n} + \sum_{i=1}^h p_i(m, n) |u_{m-k_i, n-l_i}|^{\alpha_i} \operatorname{sgn} u_{m-k_i, n-l_i} = 0, \quad (1.4)$$

and in [5] authors considered frequent oscillations of these equations.

In [6], authors considered unsaturated solutions for partial difference equations with forcing terms

$$\Delta_1 u(i-1, j) + \Delta_2 u(i, j-1) + P_1(i, j)u(i-1, j) + P_2(i, j)u(i, j-1) + P_3(i, j)u(i, j) = f(i, j). \quad (1.5)$$

Let Z be the set of integers, $Z[k, l] = \{i \in Z \mid i = k, k+1, \dots, l\}$, and $Z[k, \infty] = \{i \in Z \mid i = k, k+1, \dots\}$.

In this paper, we will consider the equation of the following form:

$$u_{m+1, n} + u_{m, n+1} - u_{m, n} + \sum_{i=1}^h p_i(m, n) u_{m-k_i, n-l_i} - \sum_{j=1}^g q_j(m, n) u_{m-s_j, n-t_j} = f(m, n), \quad (1.6)$$

where $m, n \in Z[0, \infty)$, $p_i(m, n) \geq 0$ ($i = 1, 2, \dots, h$), $q_j(m, n) \geq 0$ ($j = 1, 2, \dots, g$) and

(H₁) k_i, l_i ($i = 1, 2, \dots, h$); s_j, t_j ($j = 1, 2, \dots, g$) are nonnegative integers;

(H₂) $p_i = \{p_i(m, n)\}_{m, n \in Z[0, \infty)}$ ($i = 1, 2, \dots, h$), $q_j = \{q_j(m, n)\}_{m, n \in Z[0, \infty)}$ ($j = 1, 2, \dots, g$), $f(m, n)$ are real double sequences.

The usual concepts of oscillation or stability of steady state solutions do not catch all their fine details, and it is necessary to use the concept of frequency measures introduced in [7] to provide better descriptions. In this paper, by employing frequency measures, some new oscillatory criteria of (1.6) are established.

Let

$$\begin{aligned} \bar{k} &= \max_{\substack{1 \leq i \leq h \\ 1 \leq j \leq g}} \{k_i, s_j\} > 0, & \bar{l} &= \max_{\substack{1 \leq i \leq h \\ 1 \leq j \leq g}} \{l_i, t_j\} > 0, \\ \rho &= \min \{u_{m-k_i, n-l_i}\}, & \theta &= \max \{u_{m-s_j, n-t_j}\}, \\ \sigma &= \max \{u_{m-k_i, n-l_i}\}, & \tau &= \min \{u_{m-s_j, n-t_j}\}. \end{aligned} \tag{1.7}$$

In addition to (H_1) and (H_2) , we also assume

$$(H_3) \quad \rho \geq \theta, \quad \sigma \leq \tau;$$

$$(H_4) \quad \sum_{i=1}^h p_i(m, n) - \sum_{j=1}^g q_j(m, n) \geq 0, \quad i = 1, 2, \dots, h; \quad j = 1, 2, \dots, g.$$

For the sake of convenience, $Z[-\bar{k}, \infty) \times Z[-\bar{l}, \infty)$ will be denoted by Ω in the sequel. Given a double sequence $\{u_{m,n}\}$, the partial differences $u_{m+1,n} - u_{m,n}$ and $u_{m,n+1} - u_{m,n}$ will be denoted by $\Delta_1 u_{m,n}$ and $\Delta_2 u_{m,n}$, respectively.

To the best of our knowledge, nothing is known regarding the qualitative behaviour of the solutions of (1.6), because these equations contain positive and negative coefficients, and also contain forcing terms.

Our plan is as follows. In the next section, we will recall some of the basic results related to frequency measures. Then we obtain several criteria for all solutions of (1.6) to be frequently oscillatory and unsaturated. In the final section, we give one example to illustrate our results.

2. Preliminary

The union, intersection, and difference of two sets A and B will be denoted by $A + B$, $A \cdot B$, and $A \setminus B$, respectively. The number of elements of a set S will be denoted by $|S|$. Let Φ be a subset of Ω , then

$$\begin{aligned} X^m \Phi &= \{(i + m, j) \in \Omega \mid (i, j) \in \Phi\}, \\ Y^m \Phi &= \{(i, j + m) \in \Omega \mid (i, j) \in \Phi\} \end{aligned} \tag{2.1}$$

are the translations of Φ . Let $\alpha, \beta, \lambda,$ and δ be integers satisfying $\alpha \leq \beta$ and $\lambda \leq \delta$. The union $\sum_{i=\alpha}^{\beta} \sum_{j=\lambda}^{\delta} X^i Y^j \Phi$ will be denoted by $X_{\alpha}^{\beta} Y_{\lambda}^{\delta} \Phi$. Clearly,

$$(i, j) \in \Omega \setminus X_{\alpha}^{\beta} Y_{\lambda}^{\delta} \Phi \iff (i - d, j - e) \in \Omega \setminus \Phi \tag{2.2}$$

for $\alpha \leq d \leq \beta$ and $\lambda \leq e \leq \delta$.

For any $m, n \in Z[0, \infty)$, we set

$$\Phi^{(m,n)} = \{(i, j) \in \Phi \mid -k_1 \leq i \leq m, -l_1 \leq j \leq n\}. \tag{2.3}$$

If

$$\limsup_{m,n \rightarrow \infty} \frac{|\Phi^{(m,n)}|}{mn} \quad (2.4)$$

exists, then the superior limit, denoted by $\mu^*(\Phi)$, will be called the upper frequency measure of Φ . Similarly, if

$$\liminf_{m,n \rightarrow \infty} \frac{|\Phi^{(m,n)}|}{mn} \quad (2.5)$$

exists, then the inferior limit, denoted by $\mu_*(\Phi)$, will be called the lower frequency measure of Φ . If $\mu_*(\Phi) = \mu^*(\Phi)$, then the common limit is denoted by $\mu(\Phi)$ and is called the frequency measure of Φ .

Clearly, $\mu(\emptyset) = 0, \mu(\Omega) = 1$, and $0 \leq \mu_*(\Phi) \leq \mu^*(\Phi) \leq 1$ for any subset Φ of Ω ; furthermore, if Φ is finite, then $\mu(\Phi) = 0$.

The following results are concerned with the frequency measures and their proofs are similar to those in [8].

Lemma 2.1. *Let Φ and Γ be subsets of Ω . Then $\mu^*(\Phi + \Gamma) \leq \mu^*(\Phi) + \mu^*(\Gamma)$. Furthermore, if Φ and Γ are disjoint, then*

$$\mu_*(\Phi) + \mu_*(\Gamma) \leq \mu_*(\Phi + \Gamma) \leq \mu_*(\Phi) + \mu^*(\Gamma) \leq \mu^*(\Phi + \Gamma) \leq \mu^*(\Phi) + \mu^*(\Gamma), \quad (2.6)$$

so that

$$\mu_*(\Phi) + \mu^*(\Omega \setminus \Phi) = 1. \quad (2.7)$$

Lemma 2.2. *Let Φ be a subset of Ω , and let α, β, λ , and δ be integers such that $\alpha \leq \beta$ and $\lambda \leq \delta$. Then*

$$\mu^*(X_\alpha^\beta Y_\lambda^\delta \Phi) \leq (\beta - \alpha + 1)(\delta - \lambda + 1)\mu^*(\Phi), \quad (2.8)$$

$$\mu_*(X_\alpha^\beta Y_\lambda^\delta \Phi) \leq (\beta - \alpha + 1)(\delta - \lambda + 1)\mu_*(\Phi).$$

Lemma 2.3. *Let Φ_1, \dots, Φ_n be subsets of Ω . Then*

$$\begin{aligned} \mu^*\left(\sum_{i=1}^n \Phi_i\right) &\leq \sum_{i=1}^n \mu^*(\Phi_i) - (n-1)\mu_*\left(\prod_{i=1}^n \Phi_i\right), \\ \mu_*\left(\sum_{i=1}^n \Phi_i\right) &\leq \mu_*(\Phi_1) + \mu^*\left(\sum_{i=2}^n \Phi_i\right) - (n-1)\mu_*\left(\prod_{i=1}^n \Phi_i\right). \end{aligned} \tag{2.9}$$

Lemma 2.4. *Let Φ and Γ be subsets of Ω . If $\mu_*(\Phi) + \mu^*(\Gamma) > 1$, then the intersection $\Phi \cdot \Gamma$ is infinite.*

For any real double sequence $\{v_{i,j}\}$ defined on a subset of Ω , the level set $\{(i, j) \in \Omega \mid v_{i,j} > c\}$ is denoted by $(v > c)$. The notations $(v \geq c)$, $(v < c)$, and $(v \leq c)$ are similarly defined. Let $u = \{u_{i,j}\}_{(i,j) \in \Omega}$ be a real double sequence. If $\mu^*(u \leq 0) = 0$, then u is said to be frequently positive, and if $\mu^*(u \geq 0) = 0$, then u is said to be frequently negative.

u is said to be frequently oscillatory if it is neither frequently positive nor frequently negative. If $\mu^*(u > 0) = \omega \in (0, 1)$, then u is said to have unsaturated upper positive part, and if $\mu_*(u > 0) = \omega \in (0, 1)$, then u is said to have unsaturated lower positive part. u is said to have unsaturated positive part if $\mu^*(u > 0) = \mu_*(u > 0) = \omega \in (0, 1)$.

The concepts of frequently oscillatory and unsaturated double sequences were introduced in [5–11]. It was also observed that if a double sequence $u = \{u_{i,j}\}_{(i,j) \in \Omega}$ is frequently oscillatory or has unsaturated positive part, then it is oscillatory; that is, u is not positive for all large m and n , nor negative for all large m and n . Thus if we can show that every solution of (1.6) is frequently oscillatory or has unsaturated positive part, then every solution of (1.6) is oscillatory.

3. Frequently Oscillatory Solutions

Lemma 3.1. *Suppose there exist $m_0 \geq 2\bar{k}$ and $n_0 \geq 2\bar{l}$ such that*

$$p_i(m, n) \geq 0 \quad (i = 1, 2, \dots, h), \quad q_j(m, n) \geq 0 \quad (j = 1, 2, \dots, g) \tag{3.1}$$

for $(m, n) \in Z[m_0 - 2\bar{k}, m_0 + 1] \times Z[n_0 - 2\bar{l}, n_0 + 1]$. Let $\{u_{m,n}\}$ be a solution of (1.6). If $u_{m,n} \geq 0$, $f(m, n) \leq 0$ for $(m, n) \in Z[m_0 - 2\bar{k}, m_0 + 1] \times Z[n_0 - 2\bar{l}, n_0 + 1]$, then

$$\Delta_1 u_{m,n} \leq 0, \quad \Delta_2 u_{m,n} \leq 0 \quad \text{for } (m, n) \in Z[m_0 - \bar{k}, m_0] \times Z[n_0 - \bar{l}, n_0], \tag{3.2}$$

and if $u_{m,n} \leq 0$, $f(m, n) \geq 0$ for $(m, n) \in Z[m_0 - 2\bar{k}, m_0 + 1] \times Z[n_0 - 2\bar{l}, n_0 + 1]$, then

$$\Delta_1 u_{m,n} \geq 0, \quad \Delta_2 u_{m,n} \geq 0 \quad \text{for } (m, n) \in Z[m_0 - \bar{k}, m_0] \times Z[n_0 - \bar{l}, n_0]. \tag{3.3}$$

Proof. If $u_{m,n} \geq 0$, $f(m,n) \leq 0$ for $(m,n) \in Z[m_0 - 2\bar{k}, m_0 + 1] \times Z[n_0 - 2\bar{l}, n_0 + 1]$, it follows from (1.6) and (H_3) that

$$\begin{aligned} u_{m,n} &= u_{m+1,n} + u_{m,n+1} + \sum_{i=1}^h p_i(m,n)u_{m-k_i,n-l_i} - \sum_{j=1}^g q_j(m,n)u_{m-s_j,n-t_j} - f(m,n) \\ &\geq u_{m+1,n} + u_{m,n+1} + \theta \left[\sum_{i=1}^h p_i(m,n) - \sum_{j=1}^g q_j(m,n) \right] - f(m,n) \\ &\geq u_{m+1,n} + u_{m,n+1}. \end{aligned} \quad (3.4)$$

Hence $\Delta_1 u_{m,n} \leq 0$, $\Delta_2 u_{m,n} \leq 0$ for $(m,n) \in Z[m_0 - \bar{k}, m_0] \times Z[n_0 - \bar{l}, n_0]$.

Similarly, we also have $\Delta_1 u_{m,n} \geq 0$, $\Delta_2 u_{m,n} \geq 0$ for $(m,n) \in Z[m_0 - \bar{k}, m_0] \times Z[n_0 - \bar{l}, n_0]$. \square

Theorem 3.2. *Suppose that*

$$\begin{aligned} \mu^*(p_i < 0) &= \omega_i, \quad (i=1, 2, \dots, h), \quad \mu^*(q_j < 0) = \omega_j, \quad (j=1, 2, \dots, g), \quad \mu^*(f > 0) = \omega_f^+, \\ \mu^*(f < 0) &= \omega_f^-, \quad \mu_* \left(\prod_{i=1}^h (p_i < 0) \prod_{j=1}^g (q_j < 0) \cdot (f > 0) \right) = \omega^+, \\ \mu_* \left(\prod_{i=1}^h (p_i < 0) \prod_{j=1}^g (q_j < 0) \cdot (f < 0) \right) &= \omega^-, \\ \mu_* \left(\left(\sum_{i=1}^h p_i - \sum_{j=1}^g q_j \right) > 1 \right) &> 4(\bar{k} + 1)(\bar{l} + 1) \left(\sum_{i=1}^h \omega_i + \sum_{j=1}^g \omega_j + \omega_f - (h + g)\omega \right), \end{aligned} \quad (3.5)$$

where $\omega_f = \max\{\omega_f^+, \omega_f^-\}$ and $\omega = \min\{\omega^+, \omega^-\}$. Then every nontrivial solution of (1.6) is frequently oscillatory.

Proof. Suppose to the contrary that $u = \{u_{m,n}\}$ is a frequently positive solution of (1.6). Then $\mu^*(u \leq 0) = 0$. By Lemmas 2.1–2.3, we have

$$\begin{aligned} 1 &= \mu^* \left\{ \Omega \setminus X_{-1}^{2\bar{k}} Y_{-1}^{2\bar{l}} \left[\sum_{i=1}^h (p_i < 0) + \sum_{j=1}^g (q_j < 0) + (f > 0) + (u \leq 0) \right] \right\} \\ &\quad + \mu_* \left\{ X_{-1}^{2\bar{k}} Y_{-1}^{2\bar{l}} \left[\sum_{i=1}^h (p_i < 0) + \sum_{j=1}^g (q_j < 0) + (f > 0) + (u \leq 0) \right] \right\} \end{aligned}$$

$$\begin{aligned}
 &\leq \mu^* \left\{ \Omega \setminus X_{-1}^{2\bar{k}} Y_{-1}^{2\bar{l}} \left[\sum_{i=1}^h (p_i < 0) + \sum_{j=1}^g (q_j < 0) + (f > 0) + (u \leq 0) \right] \right\} \\
 &\quad + 4(\bar{k} + 1)(\bar{l} + 1) \\
 &\quad \times \left\{ \mu_* \left(\sum_{i=1}^h (p_i < 0) + \sum_{j=1}^g (q_j < 0) + (f > 0) \right) + \mu^*(u \leq 0) \right\} \\
 &\leq \mu^* \left\{ \Omega \setminus X_{-1}^{2\bar{k}} Y_{-1}^{2\bar{l}} \left[\sum_{i=1}^h (p_i < 0) + \sum_{j=1}^g (q_j < 0) + (f > 0) + (u \leq 0) \right] \right\} \\
 &\quad + 4(\bar{k} + 1)(\bar{l} + 1) \left(\sum_{i=1}^h \omega_i + \sum_{j=1}^g \omega_j + \omega_f - (h + g)\omega \right) \\
 &< \mu^* \left\{ \Omega \setminus X_{-1}^{2\bar{k}} Y_{-1}^{2\bar{l}} \left[\sum_{i=1}^h (p_i < 0) + \sum_{j=1}^g (q_j < 0) + (f > 0) + (u \leq 0) \right] \right\} \\
 &\quad + \mu_* \left(\left(\sum_{i=1}^h p_i - \sum_{j=1}^g q_j \right) > 1 \right).
 \end{aligned} \tag{3.6}$$

Therefore, by Lemma 2.4, the intersection

$$\left\{ \Omega \setminus X_{-1}^{2\bar{k}} Y_{-1}^{2\bar{l}} \left[\sum_{i=1}^h (p_i < 0) + \sum_{j=1}^g (q_j < 0) + (f > 0) + (u \leq 0) \right] \right\} \cdot \left(\left(\sum_{i=1}^h p_i - \sum_{j=1}^g q_j \right) > 1 \right) \tag{3.7}$$

is infinite. This implies that there exist $m_0 \geq 2\bar{k}$ and $n_0 \geq 2\bar{l}$ such that

$$\left(\sum_{i=1}^h p_i(m_0, n_0) - \sum_{j=1}^g q_j(m_0, n_0) \right) > 1, \tag{3.8}$$

$$p_i(m, n) \geq 0, \quad (i = 1, 2, \dots, h), \quad q_j(m, n) \geq 0, \quad (j = 1, 2, \dots, g), \tag{3.9}$$

$$f(m, n) \leq 0, \quad u_{m,n} > 0$$

hold for $(m, n) \in Z[m_0 - 2\bar{k}, m_0 + 1] \times Z[n_0 - 2\bar{l}, n_0 + 1]$. In view of (3.9) and Lemma 3.1, we may see that $\Delta_1 u_{m,n} \leq 0$ and $\Delta_2 u_{m,n} \leq 0$ for $(m, n) \in Z[m_0 - \bar{k}, m_0] \times Z[n_0 - \bar{l}, n_0]$, and hence $\theta \geq u_{m_0, n_0}$, so by (3.9) and (H_3) , we have that

$$\begin{aligned}
0 &\geq u_{m_0+1, n_0} + u_{m_0, n_0+1} - u_{m_0, n_0} + \rho \sum_{i=1}^h p_i(m_0, n_0) - \theta \sum_{j=1}^g q_j(m_0, n_0) - f(m_0, n_0) \\
&\geq u_{m_0+1, n_0} + u_{m_0, n_0+1} - u_{m_0, n_0} + \theta \left(\sum_{i=1}^h p_i(m_0, n_0) - \sum_{j=1}^g q_j(m_0, n_0) \right) - f(m_0, n_0) \\
&\geq u_{m_0+1, n_0} + u_{m_0, n_0+1} - u_{m_0, n_0} + u_{m_0, n_0} \left(\sum_{i=1}^h p_i(m_0, n_0) - \sum_{j=1}^g q_j(m_0, n_0) \right) - f(m_0, n_0) \\
&\geq u_{m_0, n_0} \left(\sum_{i=1}^h p_i(m_0, n_0) - \sum_{j=1}^g q_j(m_0, n_0) - 1 \right) > 0,
\end{aligned} \tag{3.10}$$

which is a contradiction.

In a similar manner, if $u = \{u_{m,n}\}$ is a frequently negative solution of (1.6) such that $\mu^*(u \geq 0) = 0$, then we may show that

$$\left\{ \Omega \setminus X_{-1}^{2\bar{k}} Y_{-1}^{2\bar{l}} \left[\sum_{i=1}^h (p_i < 0) + \sum_{j=1}^g (q_j < 0) + (f < 0) + (u \geq 0) \right] \right\} \cdot \left(\left(\sum_{i=1}^h p_i - \sum_{j=1}^g q_j \right) > 1 \right) \tag{3.11}$$

is infinite. Again we may arrive at a contradiction as above. The proof is complete. \square

Theorem 3.3. *Suppose that*

$$\mu^*(p_i < 0) = \omega_i, \quad (i = 1, 2, \dots, h), \quad \mu^*(q_j < 0) = \omega_j, \quad (j = 1, 2, \dots, g), \quad \mu^*(f > 0) = \omega_f^+,$$

$$\mu^*(f < 0) = \omega_f^-, \quad \mu^* \left(\left(\sum_{i=1}^h p_i - \sum_{j=1}^g q_j \right) \leq 1 \right) = \omega',$$

$$\mu_* \left(\prod_{i=1}^h (p_i < 0) \prod_{j=1}^g (q_j < 0) \cdot (f > 0) \cdot \left(\left(\sum_{i=1}^h p_i - \sum_{j=1}^g q_j \right) \leq 1 \right) \right) = \omega^+,$$

$$\mu_* \left(\prod_{i=1}^h (p_i < 0) \prod_{j=1}^g (q_j < 0) \cdot (f < 0) \cdot \left(\left(\sum_{i=1}^h p_i - \sum_{j=1}^g q_j \right) \leq 1 \right) \right) = \omega^-,$$

$$\omega > \frac{\sum_{i=1}^h \omega_i + \sum_{j=1}^g \omega_j + \omega_f + \omega'}{h + g + 1} - \frac{1}{4(h + g + 1)(\bar{k} + 1)(\bar{l} + 1)},$$

(3.12)

where $\omega_f = \max\{\omega_f^+, \omega_f^-\}$, and $\omega = \min\{\omega^+, \omega^-\}$. Then every nontrivial solution of (1.6) is frequently oscillatory.

Proof. Suppose to the contrary that $u = \{u_{m,n}\}$ is frequently positive solution of (1.6). Then $\mu^*(u \leq 0) = 0$. By Lemmas 2.1–2.3, we know

$$\begin{aligned} & \mu^* \left\{ \Omega \setminus X_{-1}^{2\bar{k}} Y_{-1}^{2\bar{l}} \left[\sum_{i=1}^h (p_i < 0) + \sum_{j=1}^g (q_j < 0) + (f > 0) + \left(\left(\sum_{i=1}^h p_i - \sum_{j=1}^g q_j \right) \leq 1 \right) + (u \leq 0) \right] \right\} \\ &= 1 - \mu_* \left\{ X_{-1}^{2\bar{k}} Y_{-1}^{2\bar{l}} \left[\sum_{i=1}^h (p_i < 0) + \sum_{j=1}^g (q_j < 0) + (f > 0) + \left(\left(\sum_{i=1}^h p_i - \sum_{j=1}^g q_j \right) \leq 1 \right) + (u \leq 0) \right] \right\} \\ &\geq 1 - 4(\bar{k} + 1)(\bar{l} + 1) \\ &\quad \times \left\{ \mu_* \left[\sum_{i=1}^h (p_i < 0) + \sum_{j=1}^g (q_j < 0) + (f > 0) + \left(\left(\sum_{i=1}^h p_i - \sum_{j=1}^g q_j \right) \leq 1 \right) \right] + \mu^*(u \leq 0) \right\} \\ &\geq 1 - 4(\bar{k} + 1)(\bar{l} + 1) \left[\sum_{i=1}^h \mu^*(p_i < 0) + \sum_{j=1}^g \mu^*(q_j < 0) \right. \\ &\quad \left. + \mu^*(f > 0) + \mu^* \left(\left(\sum_{i=1}^h p_i - \sum_{j=1}^g q_j \right) \leq 1 \right) - (h + g + 1) \right. \\ &\quad \left. \cdot \mu_* \left(\prod_{i=1}^h (p_i < 0) \prod_{j=1}^g (q_j < 0) \cdot (f > 0) \cdot \left(\left(\sum_{i=1}^h p_i - \sum_{j=1}^g q_j \right) \leq 1 \right) \right) \right] \\ &> 0. \end{aligned} \tag{3.13}$$

Therefore, by Lemma 2.4, we know that

$$\left\{ \Omega \setminus X_{-1}^{2\bar{k}} Y_{-1}^{2\bar{l}} \left[\sum_{i=1}^h (p_i < 0) + \sum_{j=1}^g (q_j < 0) + (f > 0) + \left(\left(\sum_{i=1}^h p_i - \sum_{j=1}^g q_j \right) \leq 1 \right) + (u \leq 0) \right] \right\} \tag{3.14}$$

is infinite. This implies that there exist $m_0 \geq 2\bar{k}$ and $n_0 \geq 2\bar{l}$ such that (3.8) and

$$\begin{aligned} p_i(m, n) &\geq 0, \quad (i = 1, 2, \dots, h), & q_j(m, n) &\geq 0, \quad (j = 1, 2, \dots, g), \\ f(m, n) &\leq 0, & u_{m,n} &> 0 \end{aligned} \tag{3.15}$$

hold for $(m, n) \in Z[m_0 - 2\bar{k}, m_0 + 1] \times Z[n_0 - 2\bar{l}, n_0 + 1]$. By similar discussions as in the proof of Theorem 3.2, we may arrive at a contradiction against (3.8).

In case $u = \{u_{m,n}\}$ is a frequently negative solution of (1.6), then $\mu^*(u \geq 0) = 0$. In an analogous manner, we may see that

$$\left\{ \Omega \setminus X_{-1}^{2\bar{k}} Y_{-1}^{2\bar{l}} \left[\sum_{i=1}^h (p_i < 0) + \sum_{j=1}^g (q_j < 0) + (f < 0) + \left(\left(\sum_{i=1}^h p_i - \sum_{j=1}^g q_j \right) \leq 1 \right) + (u \geq 0) \right] \right\} \quad (3.16)$$

is infinite. This can lead to a contradiction again. The proof is complete. \square

4. Unsaturated Solutions

The methods used in the above proofs can be modified to obtain the following results for unsaturated solutions.

Theorem 4.1. *Suppose there exists constant $\omega_0 \in (0, 1)$ such that*

$$\begin{aligned} \mu^*(p_i < 0) &= \omega_i, \quad (i = 1, 2, \dots, h), & \mu^*(q_j < 0) &= \omega_j, \quad (j = 1, 2, \dots, g), & \mu^*(f > 0) &= \omega_f^+, \\ \mu^*(f < 0) &= \omega_f^-, & \mu_* \left(\prod_{i=1}^h (p_i < 0) \prod_{j=1}^g (q_j < 0) \cdot (f > 0) \right) &= \omega^+, \\ \mu_* \left(\prod_{i=1}^h (p_i < 0) \prod_{j=1}^g (q_j < 0) \cdot (f < 0) \right) &= \omega^-, \\ \mu_* \left(\left(\sum_{i=1}^h p_i - \sum_{j=1}^g q_j \right) > 1 \right) &> 4(\bar{k} + 1)(\bar{l} + 1) \left(\sum_{i=1}^h \omega_i + \sum_{j=1}^g \omega_j + \omega_f + \omega_0 - (h + g)\omega \right), \end{aligned} \quad (4.1)$$

where $\omega_f = \max\{\omega_f^+, \omega_f^-\}$, and $\omega = \max\{\omega^+, \omega^-\}$. Then every nontrivial solution of (1.6) has unsaturated upper positive part.

Proof. Let $u = \{u_{m,n}\}$ be a nontrivial solution of (1.6). We assert that $\mu^*(u > 0) \in (\omega_0, 1)$. Otherwise, then $\mu^*(u > 0) \leq \omega_0$ or $\mu^*(u > 0) = 1$. In the former case, applying arguments similar to the proof of Theorem 3.2, we may then arrive at the fact that

$$\left\{ \Omega \setminus X_{-1}^{2\bar{k}} Y_{-1}^{2\bar{l}} \left[\sum_{i=1}^h (p_i < 0) + \sum_{j=1}^g (q_j < 0) + (f < 0) + (u > 0) \right] \right\} \cdot \left(\left(\sum_{i=1}^h p_i - \sum_{j=1}^g q_j \right) > 1 \right) \quad (4.2)$$

is infinite and a subsequent contradiction.

In the latter case, we have $\mu_*(u \leq 0) = 0$. By Lemmas 2.1–2.3, we have

$$\begin{aligned}
 1 &= \mu^* \left\{ \Omega \setminus X_{-1}^{2\bar{k}} Y_{-1}^{2\bar{l}} \left[\sum_{i=1}^h (p_i < 0) + \sum_{j=1}^g (q_j < 0) + (f > 0) + (u \leq 0) \right] \right\} \\
 &\quad + \mu_* \left\{ X_{-1}^{2\bar{k}} Y_{-1}^{2\bar{l}} \left[\sum_{i=1}^h (p_i < 0) + \sum_{j=1}^g (q_j < 0) + (f > 0) + (u \leq 0) \right] \right\} \\
 &\leq \mu^* \left\{ \Omega \setminus X_{-1}^{2\bar{k}} Y_{-1}^{2\bar{l}} \left[\sum_{i=1}^h (p_i < 0) + \sum_{j=1}^g (q_j < 0) + (f > 0) + (u \leq 0) \right] \right\} \\
 &\quad + 4(\bar{k} + 1)(\bar{l} + 1) \left\{ \mu^* \left(\sum_{i=1}^h (p_i < 0) + \sum_{j=1}^g (q_j < 0) + (f > 0) \right) + \mu_*(u \leq 0) \right\} \\
 &\leq \mu^* \left\{ \Omega \setminus X_{-1}^{2\bar{k}} Y_{-1}^{2\bar{l}} \left[\sum_{i=1}^h (p_i < 0) + \sum_{j=1}^g (q_j < 0) + (f > 0) + (u \leq 0) \right] \right\} \\
 &\quad + 4(\bar{k} + 1)(\bar{l} + 1) \left(\sum_{i=1}^h \omega_i + \sum_{j=1}^g \omega_j + \omega_f + \omega_0 - (h + g)\omega \right) \\
 &< \mu^* \left\{ \Omega \setminus X_{-1}^{2\bar{k}} Y_{-1}^{2\bar{l}} \left[\sum_{i=1}^h (p_i < 0) + \sum_{j=1}^g (q_j < 0) + (f > 0) + (u \leq 0) \right] \right\} \\
 &\quad + \mu_* \left(\left(\sum_{i=1}^h p_i - \sum_{j=1}^g q_j \right) > 1 \right).
 \end{aligned} \tag{4.3}$$

Therefore, by Lemma 2.4, we know that the set

$$\left\{ \Omega \setminus X_{-1}^{2\bar{k}} Y_{-1}^{2\bar{l}} \left[\sum_{i=1}^h (p_i < 0) + \sum_{j=1}^g (q_j < 0) + (f > 0) + (u \leq 0) \right] \right\} \cdot \left(\left(\sum_{i=1}^h p_i - \sum_{j=1}^g q_j \right) > 1 \right) \tag{4.4}$$

is infinite. Then by discussions similar to these in the proof of Theorem 3.2 again, we may arrive at a contradiction. The proof is complete. \square

Combining Theorems 3.3 and 4.1, we have the following Theorem 4.2 and the proof of this theorem is omitted.

Theorem 4.2. Suppose there exists constant $\omega_0 \in (0, 1)$ such that

$$\begin{aligned} \mu^*(p_i < 0) &= \omega_i, \quad (i = 1, 2, \dots, h), & \mu^*(q_j < 0) &= \omega_j, \quad (j = 1, 2, \dots, g), & \mu^*(f > 0) &= \omega_f^+, \\ \mu^*(f < 0) &= \omega_f^-, & \mu^*\left(\left(\sum_{i=1}^h p_i - \sum_{j=1}^g q_j\right) \leq 1\right) &= \omega', \\ \mu_*\left(\prod_{i=1}^h (p_i < 0) \prod_{j=1}^g (q_j < 0) \cdot (f > 0) \cdot \left(\left(\sum_{i=1}^h p_i - \sum_{j=1}^g q_j\right) \leq 1\right)\right) &= \omega^+, \\ \mu_*\left(\prod_{i=1}^h (p_i < 0) \prod_{j=1}^g (q_j < 0) \cdot (f < 0) \cdot \left(\left(\sum_{i=1}^h p_i - \sum_{j=1}^g q_j\right) \leq 1\right)\right) &= \omega^-, \\ \omega &> \frac{\sum_{i=1}^h \omega_i + \sum_{j=1}^g \omega_j + \omega_f + \omega' + \omega_0}{h + g + 1} - \frac{1}{4(h + g + 1)(\bar{k} + 1)(\bar{l} + 1)}, \end{aligned} \tag{4.5}$$

where $\omega_f = \max\{\omega_f^+, \omega_f^-\}$, and $\omega = \min\{\omega^+, \omega^-\}$. Then every nontrivial solution of (1.6) has unsaturated upper positive part.

Theorem 4.3. Suppose there exists constant $\omega_0 \in (0, 1)$ such that

$$\begin{aligned} \mu^*(p_i < 0) &= \omega_i, \quad (i = 1, 2, \dots, h), & \mu^*(q_j < 0) &= \omega_j, \quad (j = 1, 2, \dots, g), & \mu^*(f > 0) &= \omega_f^+, \\ \mu^*(f < 0) &= \omega_f^-, & \mu^*\left(\left(\sum_{i=1}^h p_i - \sum_{j=1}^g q_j\right) \leq 1\right) &= \omega', \\ \mu_*\left(\prod_{i=1}^h (p_i < 0) \prod_{j=1}^g (q_j < 0) \cdot (f > 0) \cdot \left(\left(\sum_{i=1}^h p_i - \sum_{j=1}^g q_j\right) \leq 1\right)\right) &= \omega^+, \\ \mu_*\left(\prod_{i=1}^h (p_i < 0) \prod_{j=1}^g (q_j < 0) \cdot (f < 0) \cdot \left(\left(\sum_{i=1}^h p_i - \sum_{j=1}^g q_j\right) \leq 1\right)\right) &= \omega^-, \\ 4(\bar{k} + 1)(\bar{l} + 1) \left(\sum_{i=1}^h \omega_i + \sum_{j=1}^g \omega_j + \omega_f + \omega' + \omega_0 - (h + g + 1)\omega\right) &< 1, \end{aligned} \tag{4.6}$$

where $\omega_f = \max\{\omega_f^+, \omega_f^-\}$, and $\omega = \min\{\omega^+, \omega^-\}$. Then every nontrivial solution of (1.6) has an unsaturated upper positive part.

Proof. We claim that $\mu^*(u > 0) \in (\omega_0, 1)$. First, we prove that $\mu^*(u > 0) > \omega_0$. Otherwise, if $\mu^*(u > 0) \leq \omega_0$, by Lemmas 2.1–2.3, we have

$$\begin{aligned}
 & \mu_* \left\{ \Omega \setminus X_{-1}^{2\bar{k}} Y_{-1}^{2\bar{l}} \left[\sum_{i=1}^h (p_i < 0) + \sum_{j=1}^g (q_j < 0) + (f < 0) \right. \right. \\
 & \quad \left. \left. + \left(\left(\sum_{i=1}^h p_i - \sum_{j=1}^g q_j \right) \leq 1 \right) \right] \right\} + \mu^* \left\{ \Omega \setminus X_{-1}^{2\bar{k}} Y_{-1}^{2\bar{l}} [(u > 0)] \right\} \\
 & = 2 - \mu_* \left\{ X_{-1}^{2\bar{k}} Y_{-1}^{2\bar{l}} \left[\sum_{i=1}^h (p_i < 0) + \sum_{j=1}^g (q_j < 0) + (f < 0) \right. \right. \\
 & \quad \left. \left. + \left(\left(\sum_{i=1}^h p_i - \sum_{j=1}^g q_j \right) \leq 1 \right) \right] \right\} - \mu_* \left\{ X_{-1}^{2\bar{k}} Y_{-1}^{2\bar{l}} [(u > 0)] \right\} \\
 & \geq 2 - 4(\bar{k} + 1)(\bar{l} + 1) \left\{ \sum_{i=1}^h \mu^*(p_i < 0) + \sum_{j=1}^g \mu^*(q_j < 0) \right. \\
 & \quad \left. + \mu^*(f < 0) + \mu^* \left(\left(\sum_{i=1}^h p_i - \sum_{j=1}^g q_j \right) \leq 1 \right) + \mu^*(u > 0) \right. \\
 & \quad \left. - (h + g + 1) \mu_* \left(\prod_{i=1}^h (p_i < 0) \prod_{j=1}^g (q_j < 0) \right. \right. \\
 & \quad \left. \left. \cdot (f < 0) \cdot \left(\left(\sum_{i=1}^h p_i - \sum_{j=1}^g q_j \right) \leq 1 \right) \right) \right\} > 1.
 \end{aligned} \tag{4.7}$$

Hence, by Lemma 2.4, we see that

$$\begin{aligned}
 & \left\{ \Omega \setminus X_{-1}^{2\bar{k}} Y_{-1}^{2\bar{l}} \left[\sum_{i=1}^h (p_i < 0) + \sum_{j=1}^g (q_j < 0) + (f < 0) + \left(\left(\sum_{i=1}^h p_i - \sum_{j=1}^g q_j \right) \leq 1 \right) \right] \right\} \\
 & \quad \cdot \left\{ \Omega \setminus X_{-1}^{2\bar{k}} Y_{-1}^{2\bar{l}} [(u > 0)] \right\}
 \end{aligned} \tag{4.8}$$

is infinite. Then there exist $m_0 \geq 2\bar{k}$ and $n_0 \geq 2\bar{l}$ such that (3.8) and

$$\begin{aligned}
 p_i(m, n) & \geq 0, \quad (i = 1, 2, \dots, h), & q_j(m, n) & \geq 0, \quad (j = 1, 2, \dots, g), \\
 f(m, n) & \leq 0, & u_{m,n} & \leq 0
 \end{aligned} \tag{4.9}$$

hold for $(m, n) \in Z[m_0 - 2\bar{k}, m_0 + 1] \times Z[n_0 - 2\bar{l}, n_0 + 1]$. Applying similar discussions as in the proof of Theorem 3.2, we can get a contradiction. Next, we prove that $\mu^*(u > 0) < 1$. Otherwise, $\mu_*(u \leq 0) = 0$. Analogously, we see that

$$\left\{ \Omega \setminus X_{-1}^{2\bar{k}} Y_{-1}^{2\bar{l}} \left[\sum_{i=1}^h (p_i < 0) + \sum_{j=1}^g (q_j < 0) + (f > 0) + \left(\left(\sum_{i=1}^h p_i - \sum_{j=1}^g q_j \right) \leq 1 \right) \right] \right\} \quad (4.10)$$

$$\cdot \left\{ \Omega \setminus X_{-1}^{2\bar{k}} Y_{-1}^{2\bar{l}} [(u \leq 0)] \right\}$$

is infinite. Then, we can also lead to a contradiction. The proof is complete. \square

We remark that every nontrivial solution of (1.6) has an unsaturated lower positive part under the same conditions as Theorems 4.1, 4.2, or 4.3. So we can obtain that every nontrivial solution of (1.6) has an unsaturated positive part.

5. Examples

We give one example to illustrate our previous results.

Example 5.1. Consider the partial difference equation

$$u_{m+1,n} + u_{m,n+1} - u_{m,n} + p_1(m,n)u_{m-4,n-3} + p_2(m,n)u_{m-3,n-2} - q_1(m,n)u_{m-1,n-1} = f(m,n), \quad (5.1)$$

where

$$f(m,n) = \begin{cases} 1, & m = 30\eta, \quad n = 10\xi, \quad \eta, \xi \in N \\ 0, & \text{otherwise} \end{cases}, \quad p_1(m,n) = 1, \quad p_2(m,n) = 3, \quad (5.2)$$

and $q_1(m,n) = 2$.

It is clear that $\mu^*(p_1 < 0) = 0$, $\mu^*(p_2 < 0) = 0$, $\mu^*(q_1 < 0) = 0$, $\mu^*(f > 0) = 1/300$, $\mu^*(f < 0) = 0$, and $\mu^*((p_1 + p_2 + q_1) > 1) = 1$.

Moreover,

$$\mu_*((p_1 < 0)(p_2 < 0)(q_1 < 0)(f < 0)) = 0, \quad \mu_*((p_1 < 0)(p_2 < 0)(q_1 < 0)(f > 0)) = 0,$$

$$\mu^*((p_1 + p_2 + q_1) > 1) = 1 > 4 \times (4 + 1) \times (3 + 1) \times \frac{1}{300} = 4(\bar{k} + 1)(\bar{l} + 1)\omega_f. \quad (5.3)$$

Then according to Theorems 3.2 or 3.3, we know that every nontrivial solution of (5.1) is frequently oscillatory. If $\omega_0 \in (0, 11/1200)$, we see that all conditions in Theorems 4.1, 4.2, or 4.3 are satisfied. Thus, every nontrivial solution of (5.1) has an unsaturated upper positive part.

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