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Research Article

Nonlocal Conditions for Lower Semicontinuous Parabolic Inclusions

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We discuss conditions for the existence of at least one solution of a discontinuous parabolic equation with lower semicontinuous right hand side and a nonlocal initial condition of integral type. Our technique is based on fixed point theorems for multivalued maps.

1. Introduction

Let Ω be an open bounded domain in $\mathbb{R}^N$, $N \geq 2$, with a smooth boundary $\partial\Omega$. We denote the norm (usually the Euclidean norm) of $x \in \Omega$ by $\|x\|$. Let T be a positive real number. Set $Q_T = \Omega \times (0,T)$ and $\Gamma_T = \partial\Omega \times [0,T]$. For $u : D \to \mathbb{R}$ we denote its partial derivatives (when they exist) by $u_t = \partial u/\partial t$, $u_{x_i} = \partial u/\partial x_i$, $u_{x_i x_j} = \partial^2 u/\partial x_i \partial x_j$, $i, j = 1, \ldots, N$.

Let $X = C(Q_T)$ denote the Banach space of continuous functions $u : Q_T \to \mathbb{R}$, endowed with the norm

$$|u|_0 = \sup\{|u(x,t)|; (x,t) \in Q_T\}$$
$$u \in C^{2,1}(Q_T) \quad \text{if } u(\cdot,t) \in C^2(\Omega), \quad t \in (0,T), \quad u(x,\cdot) \in C^1(0,T), \quad x \in \Omega. \tag{1.1}$$

For $1 \leq p < +\infty$, we say that $u : Q_T \to \mathbb{R}$ is in $L^p(Q_T)$ if u is measurable and $\int_{Q_T} |u(x,t)|^p dxdt < +\infty$, in which case we define its norm by

$$|u|_{L^p} = \left(\int_{QT} |u(x,t)|^p dxdt\right)^{1/p}. \tag{1.2}$$

Consider the linear nonhomogeneous problem

$$u_t + Lu = f(x,t), \quad (x,t) \in Q_T, \tag{1.3}$$

$$u(x,t) = 0, \quad (x,t) \in \Gamma_T, \tag{1.4}$$

with the following nonlocal initial condition:

$$u(x,0) = \int_0^T k(x,t,u(x,t))dt, \quad x \in \Omega. \tag{1.5}$$

Here, L is an elliptic operator given by

$$Lu = -\sum_{i,j=1}^{N} a_{ij}(x,t)u_{x_i x_j} + c(x,t)u. \tag{1.6}$$

We will assume throughout this paper that the functions $a_{ij}, c : Q_T \to \mathbb{R}$ are Hölder continuous, $a_{ij} = a_{ji}$, and moreover, there exist positive numbers λ_0, λ_1 such that

$$\lambda_0\|\xi\|^2 \leq \sum_{i,j=1}^{N} a_{ij}(x,t)\xi_i\xi_j \leq \lambda_1\|\xi\|^2, \quad \forall \xi \in \mathbb{R}^N, \forall (x,t) \in Q_T. \tag{1.7}$$

Let $u_0 : \Omega \to \mathbb{R}$ be continuous. For the problem (1.3), (1.4) together with initial condition

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{1.8}$$

we have the following classical result.

Lemma 1.1 (see [1–4]). *Assume that the function f is Hölder continuous on Q_T and u_0 is continuous on Ω. Then problem* (1.3), (1.4), (1.8) *has a unique solution $u \in C^{2,1}(Q_T) \cap C(\overline{Q_T})$, which for each $(x,t) \in Q_T$, is given by*

$$u(x,t) = \int_\Omega G(x,t;y,0)u_0(y)dy + \int_0^t \int_\Omega G(x,t;y,s)f(y,s)dyds, \tag{1.9}$$

where $G(x,t;y,s)$, is the Green's function corresponding to the linear homogeneous problem. This function has the following properties (see [1, 4]).

(i) $D_tG + LG = \delta(t-s)\delta(x-y)$, $s < t$, $x, y \in \Omega$.

(ii) $G(x,t;y,s) = 0$, $s > t$, $x, y \in \Omega$.

(iii) $G(x,t;y,s) = 0$, $(x,t), (y,s) \in \Gamma_T$.

(iv) $G(x,t;y,s) > 0$ for $(x,t) \in Q_T$.

(v) G, G_t, G_x, G_{xx} are continuous functions of (x,t), $(y,s) \in Q_T$, $t-s>0$.

In addition to the above, $G(x,t;y,s)$ satisfies the following important estimate.

(vi) $|G(x,t;y,s)| \leq C(t-s)^{-N/2}\exp((-a\|x-y\|^2)/(t-s))$, for some positive constants C, a (see [2]).

Since $u \in C^{2,1}(Q_T) \cap C(\overline{Q_T})$, it is clear that the functions $(x,t) \to \int_\Omega G(x,t;y,0)dy$ and $(x,t) \to \int_0^t \int_\Omega G(x,t;y,s)dyds$ are continuous. Let $d_0 := \max_{(x,t)\in Q_T} \int_\Omega G(x,t;y,0)dy$ and let $\delta := \max_{(x,t)\in\overline{Q_T}} \int_0^t \int_\Omega G(x,t;y,s)dyds$. Also, property (vi) above shows that $G \in L^2(Q_T \times Q_T)$.

In this paper, we consider a nonlocal problem for a class of nonlinear parabolic equations with a lower semicontinuous multivalued right hand side. More specifically, we consider the following problem,

$$\begin{gathered} u_t + Lu \in F(x,t,u), \quad (x,t) \in Q_T, \\ u(x,t) = 0, \quad (x,t) \in \Gamma_T, \\ u(x,0) = \int_0^T k(x,t,u(x,t))dt, \quad x \in \Omega. \end{gathered} \tag{1.10}$$

Parabolic problems with discontinuous nonlinearities arise as simplified models in the description of porous medium combustion [5], chemical reactor theory [6]. Also, best response dynamics arising in game theory can be modeled by a parabolic equation with a discontinuous right hand side [7, 8]. Parabolic problems with discontinuous nonlinearities have been also investigated in the papers [9–13]. On the other hand, parabolic problems with integral boundary conditions appear in the modeling of concrete problems, such as heat conduction [14, 15] and thermoelasticity [16]. Also, the importance of nonlocal conditions and their applications in different field has been discussed in [17, 18]. Several papers have been devoted to the study of parabolic problems with integral conditions [19, 20]. Next, we state some important facts about multivalued functions and results that will be used in the remainder of the paper.

A subset $\Sigma \subset Q_T \times \mathbb{R}$ is $\mathcal{L}\otimes\mathcal{B}$ measurable if Σ belongs to the σ-algebra generated by all sets of the form $\mathfrak{D}\times\mathcal{J}$ where $\mathfrak{D}$ is Lebesgue measurable in Q_T and $\mathcal{J}$ is Borel measurable in $\mathbb{R}$. Let $(X, |\cdot|_X)$ and $(Y, |\cdot|_Y)$ be Banach spaces. $\wp(Y)$ denotes the set of all nonempty subsets of Y. The domain of a multivalued map $\mathfrak{R} : X \to \wp(Y)$ is the set $\mathrm{Dom}(\mathfrak{R}) = \{u \in X; \mathfrak{R}(u) \neq \emptyset\}$. $\mathfrak{R}$ has closed values if $\mathfrak{R}(u)$ is a closed subset of Y for each $u \in X$ and we write $\mathfrak{R}(u) \in \wp_c(Y)$. Also, $\wp_{cc}(Y)$ denotes the set of all nonempty closed and convex subsets of Y. $\mathfrak{R}$ is bounded if $\sup\{|y|;\, y \in \mathfrak{R}(u)\} < +\infty$. $\mathfrak{R}$ is called lower semicontinuous (lsc) on X if $\mathfrak{R}^{-1}(B)$ is open in X whenever B is open in Y, or the set $\{u \in X; \mathfrak{R}(u) \subset B\}$ is closed in X whenever B is closed in Y. For more details on multivalued maps, we refer the interested reader to the books [21–24].

Let β denote the Kuratowski measure of noncompactness. See [25] for definitions and details.

Theorem 1.2 (see [26, Theorem 3.1]). *Let E be a separable Banach space. Assume the following conditions hold. There exists $M > 0$, independent of λ, with $|u|_{L^p} \neq M$ for any solution $u \in L^2([0,T],E)$ to $u \in \lambda Fu$* a.e. *on $[0,T]$ for each $\lambda \in (0,1)$, $F : X = \{u \in L^2([0,T],E); |u|_{L^p} \leq M\} \to \wp_{cc}(L^2([0,T],E))$ is a closed map, $F(X)$ is a bounded subset of $L^2([0,T],E)$, and $\beta(F(V)) \leq \beta(V)$ for all $V \subseteq X$ with strict inequality if $\beta(V) \neq 0$. Then the inclusion $u \in Fu$ has a solution $u \in X$.*

2. Main Result

By a solution of problem (1.10), (7), (8) we mean a function $u \in L^2(Q_T)$ such that there exists a function $f \in L^2(Q_T)$ with $f(x,t) \in F(x,t,u(x,t))$ for each $(x,t) \in Q_T$ and (1.3), (1.4), (1.5) hold.

Theorem 2.1. *Assume that the following conditions are satisfied.*

(HF) $F : Q_T \times \mathbb{R} \to \wp_{cc}(\mathbb{R})$ *is* $\mathcal{L} \otimes \mathcal{B}$ *measurable,* $u \mapsto F(x,t,u)$ *is* lsc *for* a.e.$(x,t) \in Q_T$*, there exist* $a > 0, b > 0$ *such that* $|F(x,t,u)| \leq a + b|u|$ *with* $2\mathrm{Vol}(Q_T)(b|G|_{L^2(Q_T \times Q_T)})^2 < 1$ *and there exists* $\ell_0 \in L^2(Q_T)$ *such that* $\beta(F(x,t,B)) \leq \ell_0(x,t)\beta(B)$ *for any bounded set* $B \subset \mathbb{R}$*,*

(Hk) $k : Q_T \times \mathbb{R} \to \mathbb{R}$ *is continuous, bounded and there exists* $\ell_1 \in C(Q_T)$ *such that* $\beta(k(x,t,B)) \leq \ell_1(x,t)\beta(B)$*.*

Then problem (1.10)*, (7), (8) has a solution provided that* $d_0|\ell_1|_0 + |\ell_0|_{L^2(Q_T)}|G|_{L^2(Q_T \times Q_T)} < 1$.

Proof. We shall follow the ideas developed in [27]. It follows from the integral representation (1.9) that any solution $u \in L^2(Q_T)$ of (1.10), (7), (8) is a solution of the operator inclusion

$$u \in F(u), \tag{2.1}$$

for $\lambda = 1$, where

$$F(u) = \mathbf{k}(u) + GN_F(u), \tag{2.2}$$

where $\mathbf{k}$ is given by

$$\mathbf{k}(u) = \int_0^T \int_\Omega G(x,t;y,0)k(y,s,u(y,s))dyds, \tag{2.3}$$

while $GN_F(u)$ is given by

$$GN_F(u)(x,t) = \int_0^t \int_\Omega G(x,t;y,s)N_F(u(y,s))dyds, \quad (x,t) \in Q_T. \tag{2.4}$$

First, we show that solutions of (2.1) are a priori bounded. We have

$$u(x,t) = \lambda \int_0^T \int_\Omega G(x,t;y,0)k(y,s,u(y,s))dyds + \lambda \int_0^t \int_\Omega G(x,t;y,s)f(y,s)dyds, \tag{2.5}$$

where $f \in N_F(u)$, that is $f(x,t) \in F(x,t,u)$ for each $(x,t) \in Q_T$. Since k is bounded there exists $C_k > 0$ such that $|k(y,s,u(y,s))| \leq C_k$. It follows from the properties of the Green's function and the assumption (HF) that

$$|u(x,t)| \leq TC_kd_0 + \int_0^t \int_\Omega G(x,t;y,s)(a + b|u(y,s)|)dyds. \tag{2.6}$$

Hence

$$|u(x,t)| \leq TC_k d_0 + a\delta + b|G|_{L^2(Q_T\times Q_T)}|u|_{L^2(Q_T)}. \tag{2.7}$$

Equation (2.7) implies that

$$|u(x,t)|^2 \leq 2(TC_k d_0 + a\delta)^2 + 2\left(b|G|_{L^2(Q_T\times Q_T)}|u|_{L^2(Q_T)}\right)^2, \tag{2.8}$$

or

$$|u|^2_{L^2(Q_T)} \leq \frac{2\mathrm{Vol}(Q_T)(TC_k d_0 + a\delta)^2}{1 - 2\mathrm{Vol}(Q_T)\left(b|G|_{L^2(Q_T\times Q_T)}\right)^2}. \tag{2.9}$$

Therefore, there exists $M > 0$, independent of λ, but depending on Q_T, a, b, C_k and the Green's function such that any possible solution of (2.1) satisfies

$$|u|_{L^2(Q_T)} \leq M. \tag{2.10}$$

Let $U = \{u \in L^2(Q_T); |u|_{L^2(Q_T)} \leq M\}$. Then U is nonempty, closed, and bounded subset of $L^2(Q_T)$.

Since the multifunction F has nonempty, closed and convex values, it follows that N_F has nonempty, closed, and convex values. Since $\mathbf{k}$ is a continuous single valued operator, it is clear that F has nonempty, closed, and convex values. Next, we can easily show that $F : U \to \wp_{cc}(L^2(Q_T))$ is a closed map (i.e., has a closed graph) and $F(U)$ is a bounded subset of $L^2(Q_T)$.

Finally, we show that $\beta(F(B)) \leq \beta(B)$ for any bounded subset $B \subset U$. So, let $u \in B$. Then, since $F(B) = \{F(u); u \in B\}$, we have

$$F(B) = \mathbf{k}(B) + \mathrm{GN}_F(B) = \{\mathbf{k}(u) + \mathrm{GN}_F(u);\ u \in B\}. \tag{2.11}$$

Hence

$$\beta(F(B)) = \beta(\{\mathbf{k}(u) + \mathrm{GN}_F(u);\ u \in B\}). \tag{2.12}$$

It follows from the assumption that

$$\begin{aligned}
\beta(F(B)) &\leq \int_0^T\int_\Omega G(x,t;y,0)\ell_1(y,s)\beta(B)\,dy\,ds + \int_0^t\int_\Omega G(x,t;y,s)\ell_0(y,s)\beta(B)\,dy\,ds \\
&\leq \left(\int_0^T\int_\Omega G(x,t;y,0)\ell_1(y,s)\,dy\,ds + \int_0^t\int_\Omega G(x,t;y,s)\ell_0(y,s)\,dy\,ds\right)\beta(B) \\
&\leq \left(d_0|\ell_1|_0 + |\ell_0|_{L^2(Q_T)}|G|_{L^2(Q_T\times Q_T)}\right)\beta(B) \\
&< \beta(B).
\end{aligned} \tag{2.13}$$

This shows that F is a condensing multivalued map.

By Theorem 3.1 in [26], F has a fixed point in U, which is a solution of problem (1.10), (7), (8). This completes the proof of the main result. □

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