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Research Article

Fractional-Order Variational Calculus with Generalized Boundary Conditions

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This paper presents the necessary and sufficient optimality conditions for fractional variational problems involving the right and the left fractional integrals and fractional derivatives defined in the sense of Riemman-Liouville with a Lagrangian depending on the free end-points. To illustrate our approach, two examples are discussed in detail.

1. Introduction

Fractional calculus is one of the generalizations of the classical calculus. Several fields of application of fractional differentiation and fractional integration are already well established, some others have just started. Many applications of fractional calculus can be found in turbulence and fluid dynamics, stochastic dynamical system, plasma physics and controlled thermonuclear fusion, nonlinear control theory, image processing, nonlinear biological systems, astrophysics, and so forth (see [1–11] and the references therein).

Real integer variational calculus plays a significant role in many areas of science, engineering, and applied mathematics. In recent years, there has been a growing interest in the area of fractional variational calculus and its applications which include classical and quantum mechanics, field theory, and optimal control (see [10, 12–20]).

In the papers cited above, the problems have been formulated mostly in terms of two types of fractional derivative, namely, Riemann-Liouville (RL) and Caputo derivatives.

The natural boundary conditions for fractional variational problems, in terms of the RL and the Caputo derivatives, are presented in [13, 14].

The necessary optimality conditions for problems of the fractional calculus of variations with a Lagrangian that may also depend on the unspecified end-points *y(a), y(b)* is proven in [19].

In [18] the two authors discussed the fractional variational problems with fractional integral and fractional derivative in the sense of Riemann-Liouville and the Caputo derivatives and give the fractional Euler-Lagrange equations with the natural boundary conditions.

Here we develop the theory of fractional variational calculus further by proving the necessary optimality conditions for more general problems of the fractional calculus of variations with a fractional integral and a Lagrangian that may also depend on the unspecified end-points *y(a)* or *y(b)*. The novelty is the dependence of the integrand L on the free end-points *y(a), y(b)* with replacing the ordinary integral by fractional integral in the functional.

We consider two types of fractional variational calculus

$$J(y) = I_{a+}^{\gamma} L\left(x, y(x), {}^{R}D_{a+}^{\alpha} y, y(a)\right), \tag{1.1}$$

$$J(y) = I_{b-}^{\gamma} L\left(x, y(x), {}^{R}D_{b-}^{\alpha} y, y(b)\right). \tag{1.2}$$

The paper is organized as follows.

In Section 2, we present the principal definitions used in this paper. In Section 3, the necessary optimality conditions are proved for problems (1.1) and (1.2) by giving some special cases which prove the generalization of our problems. Sufficient conditions are shown in Section 4, and two examples are depicted in Section 5.

2. Preliminaries

Here we give the standard definitions of left and right Riemann-Liouville fractional integral, Riemann-Liouville fractional derivatives, and Caputo fractional derivatives (see [1, 2, 4, 21]).

Definition 2.1. If $f(t) \in L_1(a,b)$, the set of all integrable functions, and $\alpha > 0$, then the left and right Riemann-Liouville fractional integrals of order α, denoted, respectively, by I_{a+}^{α} and I_{b-}^{α}, are defined by

$$\begin{aligned} I_{a+}^{\alpha} f(t) &= \frac{1}{\Gamma(\alpha)} \int_{a}^{t} (t-\tau)^{\alpha-1} f(\tau) d\tau, \\ I_{b-}^{\alpha} f(t) &= \frac{1}{\Gamma(\alpha)} \int_{t}^{b} (\tau-t)^{\alpha-1} f(\tau) d\tau. \end{aligned} \tag{2.1}$$

Definition 2.2. For $\alpha > 0$, the left and right Riemann-Liouville fractional derivatives of order α, denoted, respectively, by ${}^{R}D^{\alpha}_{a+}$ and ${}^{R}D^{\alpha}_{b-}$, are defined by

$$
\begin{aligned}
{}^{R}D^{\alpha}_{a+}f(t) &= \frac{1}{\Gamma(n-\alpha)} D^{n} \int_{a}^{t} (t-\tau)^{n-\alpha-1} f(\tau)d\tau, \\
{}^{R}D^{\alpha}_{b-}f(t) &= \frac{1}{\Gamma(n-\alpha)} (-D)^{n} \int_{t}^{b} (\tau-t)^{n-\alpha-1} f(\tau)d\tau,
\end{aligned}
\tag{2.2}
$$

where n is such that $n-1 < \alpha < n$ and $D = d/dt$

If α is an integer, these derivatives are defined in the usual sense

$$
{}^{R}D^{\alpha}_{a+} := D^{\alpha}, \quad {}^{R}D^{\alpha}_{b-} := (-D)^{\alpha}, \quad \alpha = 1,2,3,\ldots. \tag{2.3}
$$

Definition 2.3. For $\alpha > 0$, the left and right Caputo fractional derivatives of order α, denoted, respectively, by ${}^{C}D^{\alpha}_{a+}$ and ${}^{C}D^{\alpha}_{b-}$, are defined by

$$
\begin{aligned}
{}^{C}D^{\alpha}_{a+}f(t) &= \frac{1}{\Gamma(n-\alpha)} \int_{a}^{t} (t-\tau)^{n-\alpha-1} D^{n} f(\tau)d\tau, \\
{}^{C}D^{\alpha}_{b-}f(t) &= \frac{1}{\Gamma(n-\alpha)} \int_{t}^{b} (\tau-t)^{n-\alpha-1} (-D)^{n} f(\tau)d\tau,
\end{aligned}
\tag{2.4}
$$

where n is such that $n-1 < \alpha < n$ and $D = d/d\tau$.

If α is an integer, then these derivatives take the ordinary derivatives

$$
{}^{C}D^{\alpha}_{a+} = D^{\alpha}, \quad {}^{C}D^{\alpha}_{b-} = (-D)^{\alpha}, \quad \alpha = 1,2,3,\ldots. \tag{2.5}
$$

3. Necessary Optimality Conditions

3.1. Necessary Optimality Conditions for Problem (1.1)

To develop the necessary conditions for the extremum for (1.1), assume that $y^{*}(x)$ is the desired function, let $\epsilon \in R$, and define a family of curves $y(x) = y^{*}(x) + \epsilon\eta(x)$ since ${}^{R}D^{\alpha}_{a+}$ is a linear operator; then we get (1.1) in the form

$$
J(\epsilon) = \int_{a}^{x} \frac{(x-t)^{\gamma-1}}{\Gamma(\gamma)} L\Big(t, y(t) + \epsilon\eta(t),\, {}^{R}D^{\alpha}_{a+}y + \epsilon\, {}^{R}D^{\alpha}_{a+}\eta, y(a) + \epsilon\eta(a)\Big) dt \tag{3.1}
$$

and where *J*(ϵ) is extremum at $\epsilon = 0$, we get by differentiating both sides with respect to ϵ and set $dJ/d\epsilon = 0$, for all admissible *η*(*x*),

$$
\int_{a}^{x} \frac{(x-t)^{\gamma-1}}{\Gamma(\gamma)} \left[\frac{\partial L}{\partial y}\eta + \frac{\partial L}{\partial\, {}^{R}D^{\alpha}_{a+}y}\, {}^{R}D^{\alpha}_{a+}\eta + \frac{\partial L}{\partial y(a)}\eta(a) \right] dt = 0. \tag{3.2}
$$

But we have (by integration by parts in classic and fractional calculus)

$$
\begin{aligned}
&\int_a^x \left(\frac{(x-t)^{\gamma-1}}{\Gamma(\gamma)} \frac{\partial L}{\partial\, {}^{R}D_{a+}^{\alpha} y}\, {}^{R}D_{a+}^{\alpha} \eta \right) dt \\
&\quad = \left(\frac{(x-t)^{\gamma-1}}{\Gamma(\gamma)} \frac{\partial L}{\partial\, {}^{R}D_{a+}^{\alpha} y} \right) I_{a+}^{1-\alpha} \eta(t) \Big|_a^x - \int_a^x I_{a+}^{1-\alpha} \eta(t) D \left(\frac{(x-t)^{\gamma-1}}{\Gamma(\gamma)} \frac{\partial L}{\partial\, {}^{C}D_{a+}^{\alpha} y} \right) dt \\
&\quad = \left(\frac{(x-t)^{\gamma-1}}{\Gamma(\gamma)} \frac{\partial L}{\partial\, {}^{R}D_{a+}^{\alpha} y} \right) I_{a+}^{1-\alpha} \eta(t) \Big|_a^x + \int_a^x \eta(t)\, {}^{R}D_{x-}^{\alpha} \left(\frac{(x-t)^{\gamma-1}}{\Gamma(\gamma)} \frac{\partial L}{\partial\, {}^{C}D_{a+}^{\alpha} y} \right) dt.
\end{aligned}
\tag{3.3}
$$

Substituting in (3.2), we get

$$
\begin{aligned}
&\int_a^x \eta(t) \left[\frac{(x-t)^{\gamma-1}}{\Gamma(\gamma)} \frac{\partial L}{\partial y} + {}^{C}D_{x-}^{\alpha} \left(\frac{(x-t)^{\gamma-1}}{\Gamma(\gamma)} \frac{\partial L}{\partial\, {}^{R}D_{a+}^{\alpha} y} \right) \right] dt \\
&\quad + \left(\frac{(x-t)^{\gamma-1}}{\Gamma(\gamma)} \frac{\partial L}{\partial\, {}^{R}D_{a+}^{\alpha} y} \right) I_{a+}^{1-\alpha} \eta(t) \Big|_{t=x} - \left(\frac{(x-t)^{\gamma-1}}{\Gamma(\gamma)} \frac{\partial L}{\partial\, {}^{R}D_{a+}^{\alpha} y} \right) I_{a+}^{1-\alpha} \eta(t) \Big|_{t=a} \\
&\quad + \eta(a) \left[\int_a^x \frac{(x-t)^{\gamma-1}}{\Gamma(\gamma)} \frac{\partial L}{\partial y(a)} dt \right] = 0.
\end{aligned}
\tag{3.4}
$$

Since $\eta(t)$ is arbitrary, we get $I_{a+}^{1-\alpha}\eta(t)|_{t=a} = 0$ and $I_{a+}^{1-\alpha}\eta(t)|_{t=x} \neq 0$ which gives the fractional Euler-Lagrange equation in the form

$$
\frac{(x-t)^{\gamma-1}}{\Gamma(\gamma)} \frac{\partial L}{\partial y} + {}^{C}D_{x-}^{\alpha} \left(\frac{(x-t)^{\gamma-1}}{\Gamma(\gamma)} \frac{\partial L}{\partial\, {}^{R}D_{a+}^{\alpha} y} \right) = 0
\tag{3.5}
$$

with the natural boundary condition (transversality conditions)

$$
\left(\frac{(x-t)^{\gamma-1}}{\Gamma(\gamma)} \frac{\partial L}{\partial\, {}^{R}D_{a+}^{\alpha} y} \right) \Big|_{t=x} = 0.
\tag{3.6}
$$

If $y(a)$ is specified, then we have $\eta(a) = 0$, but if it is not specified, then we get the boundary condition

$$
\int_a^x \frac{(x-t)^{\gamma-1}}{\Gamma(\gamma)} \frac{\partial L}{\partial y(a)} dt = 0.
\tag{3.7}
$$

Remark 3.1. These conditions are only necessary for an extremum. The question of sufficient conditions for the existence of an extremum is considered in the next section.

Special Cases

Case 1. If y is a local extremizer to

$$J(y) = \int_a^b L\left(t, y(t), {}^R D_{a+}^{\alpha} y\right) dt, \tag{3.8}$$

by putting $\gamma = 1$ and $x = b$ in (3.5), (3.6), and (3.7), we get the fractional Euler-Lagrange equation in the form

$$\frac{\partial L}{\partial y} + {}^C D_{b-}^{\alpha}\left(\frac{\partial L}{\partial\, {}^R D_{a+}^{\alpha} y}\right) = 0 \tag{3.9}$$

for all $t \in [a, b]$, with the boundary condition

$$\left.\left(\frac{\partial L}{\partial\, {}^R D_{a+}^{\alpha} y}\right)\right|_{t=x} = 0. \tag{3.10}$$

Case 2. If y is a local extremizer to

$$J(y) = I^{\gamma} L\left(x, y(x), {}^R D_{a+}^{\alpha} y\right), \tag{3.11}$$

we get similar results as in [18].

3.2. Necessary Optimality Conditions for Problem (1.2)

To develop the necessary conditions for the extremum for (1.2), assume that $y^*(x)$ is the desired function, let $\epsilon \in R$, and define a family of curves $y(x) = y^*(x) + \epsilon\eta(x)$ since ${}^R D_{b-}^{\beta}$ is a linear operator; then we get (1.2) in the form

$$J(\epsilon) = \int_x^b \frac{(t-x)^{\gamma-1}}{\Gamma(\gamma)} L\left(t, y(t) + \epsilon\eta(t), {}^R D_{b-}^{\alpha} y + \epsilon\, {}^R D_{b-}^{\alpha}\eta, y(b) + \epsilon\eta(b)\right) dt \tag{3.12}$$

and where $J(\epsilon)$ is extremum at $\epsilon = 0$, we get by differentiating both sides with respect to ϵ and set $dJ/d\epsilon = 0$, for all admissible $\eta(x)$,

$$\int_x^b \frac{(t-x)^{\gamma-1}}{\Gamma(\gamma)} \left[\frac{\partial L}{\partial y}\eta + \frac{\partial L}{\partial\, {}^R D_{b-}^{\beta} y}\, {}^R D_{b-}^{\beta}\eta + \frac{\partial L}{\partial y(b)}\eta(b)\right] dt = 0. \tag{3.13}$$

But we have (by integration by parts) that

$$\int_x^b \left(\frac{(t-x)^{\gamma-1}}{\Gamma(\gamma)} \frac{\partial L}{\partial\, {}^R D_{b-}^{\beta} y}\, {}^R D_{b-}^{\beta} \eta \right) dt$$
$$= -\left(\left(\frac{(t-x)^{\gamma-1}}{\Gamma(\gamma)} \frac{\partial L}{\partial\, {}^R D_{b-}^{\beta} y} \right) I_{b-}^{1-\beta} \eta \right) \Bigg|_x^b + \int_x^b \eta\, {}^C D_{x+}^{\beta} \left(\frac{(t-x)^{\gamma-1}}{\Gamma(\gamma)} \frac{\partial L}{\partial\, {}^R D_{b-}^{\beta} y} \right) dt. \tag{3.14}$$

Substituting in (3.13), we get

$$\int_x^b \eta(t) \left[\frac{(t-x)^{\gamma-1}}{\Gamma(\gamma)} \frac{\partial L}{\partial y} + {}^C D_{x+}^{\beta} \left(\frac{(t-x)^{\gamma-1}}{\Gamma(\gamma)} \frac{\partial L}{\partial\, {}^R D_{b-}^{\beta} y} \right) \right] dt$$
$$- \left(\left(\frac{(t-x)^{\gamma-1}}{\Gamma(\gamma)} \frac{\partial L}{\partial\, {}^R D_{b-}^{\beta} y} \right) I_{b-}^{1-\beta} \eta \right) \Bigg|_x^b + \eta(b) \int_x^b \frac{(t-x)^{\gamma-1}}{\Gamma(\gamma)} \frac{\partial L}{\partial y(b)} dt = 0. \tag{3.15}$$

Since $\eta(t)$ is arbitrary, we get $I_{b-}^{1-\alpha}\eta(t)|_{t=b} = 0$ and $I_{b-}^{1-\alpha}\eta(t)|_{t=x} \neq 0$ which gives the fractional Euler-Lagrange equation in the form

$$\frac{(t-x)^{\gamma-1}}{\Gamma(\gamma)} \frac{\partial L}{\partial y} + {}^C D_{x+}^{\beta} \left(\frac{(t-x)^{\gamma-1}}{\Gamma(\gamma)} \frac{\partial L}{\partial\, {}^R D_{b-}^{\beta} y} \right) = 0 \tag{3.16}$$

with the natural boundary condition (transversality conditions)

$$\left(\left(\frac{(t-x)^{\gamma-1}}{\Gamma(\gamma)} \frac{\partial L}{\partial\, {}^R D_{b-}^{\beta} y} \right) \right) \Bigg|_{t=x} = 0. \tag{3.17}$$

If $y(b)$ is specified, then we have $\eta(b) = 0$, but if it is not specified, then we get the boundary condition

$$\int_x^b \frac{(t-x)^{\gamma-1}}{\Gamma(\gamma)} \frac{\partial L}{\partial y(b)} dt = 0. \tag{3.18}$$

4. Sufficient Conditions

In this section, we prove the sufficient conditions that ensure the existence of a minimum (maximum). Some conditions of convexity (concavity) are in order.

Given a function $L = L(t, y, z, u)$, we say that L is jointly convex (concave) in (y, z, u) if $\partial L/\partial y, \partial L/\partial z, \partial L/\partial u$ exist and are continuous and verify the following condition:

$$L(t, y + y_1, z + z_1, u + u_1) - L(t, y, z, u) \geq (\leq) \frac{\partial L}{\partial y} y_1 + \frac{\partial L}{\partial z} z_1 + \frac{\partial L}{\partial u} u_1 \tag{4.1}$$

for all $(t, y, z, u), (t, y + y_1, z + z_1, u + u_1) \in [a, b] \times R^3$.

Theorem 4.1. *Let L (t, y, z, u) be jointly convex (concave) in (y, z, u). If y_0 satisfies conditions* (3.5) (3.7), *then y_0 is a global minimizer (maximizer) to problem* (1.1).

Proof. We will give the proof for only the convex case (and similarly we can prove it for the concave case). Since L is jointly convex in (y, z, u, v) for any admissible function $y_0 + h$, we have

$$\begin{aligned} J(y_0 + h) - J(y_0) &= \int_a^x \frac{(x-t)^{\gamma-1}}{\Gamma(\gamma)} \Big[L\Big(t, y_0(t) + h(t),\ {}^R D_{a+}^{\alpha}(y_0(t) + h(t)) y_0(a) + h(a)\Big) \\ &\quad - L\Big(t, y_0(t),\ {}^R D_{a+}^{\alpha} y_0(t),\ {}^R D_{b-}^{\beta} y_0(t), y_0(a)\Big)\Big] dt \\ &\geq \int_a^x \frac{(x-t)^{\gamma-1}}{\Gamma(\gamma)} \left[\frac{\partial L}{\partial y_0} h + \frac{\partial L}{\partial\, {}^R D_{a+}^{\alpha} y_0}\, {}^R D_{a+}^{\alpha} h + \frac{\partial L}{\partial y_0(a)} h(a) \right] dt. \end{aligned} \tag{4.2}$$

By using integration by parts (as in proving (3.5)–(3.7)), we get

$$\begin{aligned} J(y_0 + h) - J(y_0) &\geq \int_a^x h(t) \left[\frac{(x-t)^{\gamma-1}}{\Gamma(\gamma)} \frac{\partial L}{\partial y} + {}^C D_{x-}^{\beta} \left(\frac{(x-t)^{\gamma-1}}{\Gamma(\gamma)} \frac{\partial L}{\partial\, {}^R D_{a+}^{\beta} y} \right) \right] dt \\ &\quad - \left(\left(\frac{(x-t)^{\gamma-1}}{\Gamma(\gamma)} \frac{\partial L}{\partial\, {}^R D_{a+}^{\beta} y} \right) I_{a+}^{1-\beta} h(t) \right) \Bigg|_a^x + h(a) \int_x^b \frac{(x-t)^{\gamma-1}}{\Gamma(\gamma)} \frac{\partial L}{\partial y(b)} dt. \end{aligned} \tag{4.3}$$

Since y_0 satisfies conditions (3.5)–(3.7), thus we obtain $J(y_0 + h) - J(y_0) \geq 0$ which completes the proof. □

Similar to proving the previous theorem, we can prove the following theorem.

Theorem 4.2. *Let L (t, y, z, u) be jointly convex (concave) in (y,z,u). If y_0 satisfies conditions* (3.16)–(3.18), *then y_0 is a global minimizer (maximizer) to problem* (1.2).

5. Examples

We will provide in this section two examples in order to illustrate our main results.

Example 5.1. Consider the following problem:

$$\min J(y) = \frac{1}{2} I_{0+}^{\gamma} \left[y^2(t) + \left({}^{R}D_{0+}^{\alpha} y(t) \right)^2 + \delta (y(0))^2 \right], \quad x \in [0,1], \; \delta \geq 0. \tag{5.1}$$

For this problem, we get the generalized fractional Euler-Lagrange equational and the natural boundary conditions, respectively, in the following form:

$$\begin{gathered} \frac{(x-t)^{\gamma-1}}{\Gamma(\gamma)} y(t) + {}^{C}D_{x-}^{\alpha} \left(\frac{(x-t)^{\gamma-1}}{\Gamma(\gamma)} \, {}^{R}D_{0+}^{\alpha} y(t) \right) = 0, \\ \left. \left(\frac{(x-t)^{\gamma-1}}{\Gamma(\gamma)} \, {}^{R}D_{0+}^{\alpha} y \right) \right|_{t=x} = 0, \\ \int_0^x \frac{(x-t)^{\gamma-1}}{\Gamma(\gamma)} \delta y(0) dt = 0. \end{gathered} \tag{5.2}$$

Note that it is difficult to solve the above fractional equations; for $0 < \alpha < 1$, a numerical method should be used, and where $L(y, z, u) = 1/2(y^2 + z^2 + \delta u^2)$ is a jointly convex then the obtained solution is a global minimizer to problem (5.1).

Example 5.2. Consider the following problem:

$$\min J(y) = \frac{1}{2} I_{1-}^{\gamma} \left[y^2(t) + \left({}^{R}D_{1-}^{\beta} y(t) \right)^2 + \lambda (y(1))^2 \right], \quad x \in [0,1], \; \lambda \geq 0. \tag{5.3}$$

For this problem, we get the generalized fractional Euler-Lagrange equational and the natural boundary conditions, respectively, in the following form:

$$\begin{gathered} \frac{(t-x)^{\gamma-1}}{\Gamma(\gamma)} y + {}^{C}D_{x+}^{\beta} \left(\frac{(t-x)^{\gamma-1}}{\Gamma(\gamma)} \, {}^{R}D_{1-}^{\beta} y \right) = 0, \\ \left. \left(\frac{(t-x)^{\gamma-1}}{\Gamma(\gamma)} \, {}^{R}D_{1-}^{\beta} y \right) \right|_{t=x} = 0, \\ \int_x^1 \frac{(t-x)^{\gamma-1}}{\Gamma(\gamma)} \lambda y(1) dt = 0. \end{gathered} \tag{5.4}$$

Using a numerical method, we get the solution which is a global minimizer to problem (5.3) where $L(y, z, u) = 1/2(y^2 + z^2 + \lambda u^2)$ is a jointly convex.

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