

Research Article

Generalization of Okamoto's Equation to Arbitrary 2×2 Schlesinger System

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The 2×2 Schlesinger system for the case of four regular singularities is equivalent to the Painlevé VI equation. The Painlevé VI equation can in turn be rewritten in the symmetric form of Okamoto's equation; the dependent variable in Okamoto's form of the PVI equation is the (slightly transformed) logarithmic derivative of the Jimbo-Miwa tau-function of the Schlesinger system. The goal of this note is twofold. First, we find a universal formulation of an arbitrary Schlesinger system with regular singularities in terms of appropriately defined Virasoro generators. Second, we find analogues of Okamoto's equation for the case of the 2×2 Schlesinger system with an arbitrary number of poles. A new set of scalar equations for the logarithmic derivatives of the Jimbo-Miwa tau-function is derived in terms of generators of the Virasoro algebra; these generators are expressed in terms of derivatives with respect to singularities of the Schlesinger system.

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1. Introduction

The Schlesinger system is the following nonautonomous system of differential equations for N unknown matrices $A_j \in \mathfrak{sl}(M)$ depending on N variables $\{\lambda_j\}$:

$$\frac{\partial A_j}{\partial \lambda_i} = \frac{[A_j, A_i]}{\lambda_j - \lambda_i}, \quad i \neq j, \quad \frac{\partial A_j}{\partial \lambda_j} = -\sum_{i \neq j} \frac{[A_j, A_i]}{\lambda_j - \lambda_i}. \quad (1.1)$$

System (1.1) determines isomonodromic deformations of a solution of matrix ODE with meromorphic coefficients

$$\frac{\partial \Psi}{\partial \lambda} = A(\lambda) \Psi \equiv \sum_{j=1}^N \frac{A_j}{\lambda - \lambda_j} \Psi, \quad (1.2)$$

if one assumes that $\sum_{j=1}^N A_j = 0$, that is, $\lambda = \infty$ is a regular point of the function Ψ , and the function is normalized by the condition $\Psi(\infty) = I$. The function Ψ solves a matrix Riemann-Hilbert problem with some monodromy matrices around the singularities λ_j .

The Schlesinger equations were discovered almost 100 years ago [1]; however, they continue to play a key role in many areas of mathematical physics such as the theory of random matrices, integrable systems and theory of Frobenius manifolds. System (1.1) is a nonautonomous Hamiltonian system with respect to the Poisson bracket

$$\{A_j^a, A_k^b\} = \delta_{jk} f_c^{ab} A_j^c, \quad (1.3)$$

where f_c^{ab} are structure constants of $\mathfrak{sl}(M)$; $A_j = A_j^a t^a$ with the $\mathfrak{sl}(M)$ generators t^a ; δ_{jk} is the Kronecker symbol (one assumes summation over repeating indices). Obviously, the traces $\text{tr } A_j^n$ are integrals of the Schlesinger system for any value of n . The commuting Hamiltonians defining evolution with respect to the times λ_j are given by

$$H_j = \frac{1}{4\pi i} \oint_{\lambda_i} \text{tr } A^2(\lambda) d\lambda \equiv \frac{1}{2} \sum_{k \neq j} \frac{\text{tr } A_j A_k}{\lambda_k - \lambda_j}. \quad (1.4)$$

The generating function $\tau_M(\{\lambda_j\})$ of the Hamiltonians H_j , defined by

$$\frac{\partial}{\partial \lambda_j} \log \tau_M = H_j, \quad (1.5)$$

was introduced by Jimbo et al. [2, 3]; it is called the τ -function of the Schlesinger system. The τ -function plays a key role in the theory of the Schlesinger equations; in particular, the divisor of zeros of the τ -function coincides with the divisor of singularities of the solution of the Schlesinger system.

In the simplest nontrivial case when the matrix dimension equals $M = 2$ and the number of singularities equals $N = 4$, the Schlesinger system can equivalently be rewritten as a single scalar differential equation of order two—the Painlevé VI equation:

$$\begin{aligned} \frac{d^2 y}{dt^2} = & \frac{1}{2} \left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t} \right) \left(\frac{dy}{dt} \right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t} \right) \frac{dy}{dt} \\ & + \frac{y(y-1)(y-t)}{t^2(t-1)^2} \left(\alpha + \beta \frac{t}{y^2} + \gamma \frac{t-1}{(y-1)^2} + \delta \frac{t(t-1)}{(y-t)^2} \right), \end{aligned} \quad (1.6)$$

where t is the cross-ratio of the four singularities $\lambda_1, \dots, \lambda_4$. Solution y is defined as follows. One maps the set of poles $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ to $(0, t, 1, \infty)$ by a Möbius transformation and then multiplies Ψ from the left by a constant matrix such that A_4 becomes diagonal. Then y coincides with the position of the (unique) zero of the upper right corner element of the transformed matrix $A(\lambda)$.

Let us denote the eigenvalues of the matrices A_j by $\alpha_j/2$ and $-\alpha_j/2$. Then the constants α, β, γ , and δ from the Painlevé VI equation (1.6) are related to the constants α_j as follows:

$$\alpha = \frac{(\alpha_1 - 1)^2}{2}, \quad \beta = -\frac{\alpha_2^2}{2}, \quad \gamma = \frac{\alpha_3^2}{2}, \quad \delta = \frac{1}{2} - \frac{\alpha_4^2}{2}. \quad (1.7)$$

It was further observed by Okamoto [4, 5] that the Painlevé VI equation (and, therefore, the original 2×2 Schlesinger system with four singularities) can be rewritten alternatively in a simple form in terms of the so-called auxiliary Hamiltonian function $h(t)$. To define this function we need to introduce first four constants b_j , which are expressed in terms of the eigenvalues of the matrices A_j as follows:

$$\begin{aligned} b_1 &= \frac{1}{2}(\alpha_2 + \alpha_3), & b_2 &= \frac{1}{2}(\alpha_2 - \alpha_3), \\ b_3 &= \frac{1}{2}(\alpha_4 + \alpha_1), & b_4 &= \frac{1}{2}(\alpha_4 - \alpha_1). \end{aligned} \quad (1.8)$$

The auxiliary Hamiltonian function $h(t)$ is defined in terms of solution y of (1.6) and the constants b_j as follows:

$$\begin{aligned} h &= y(y-1)(y-t) \left(\frac{dy}{dt} \right)^2 \\ &- \{ (b_1 + b_2)(y-1)(y-t) + (b_1 - b_2)y(y-t) + (b_3 + b_4)y(y-1) \} \frac{dy}{dt} \\ &+ \left\{ \frac{1}{4}(2b_1 + b_3 + b_4)^2 - \frac{1}{4}(b_3 - b_4)^2 \right\} (y-t) + \sigma_2'[b]t - \frac{1}{2}\sigma_2[b], \end{aligned} \quad (1.9)$$

where

$$\sigma_2'[b] := b_1b_3 + b_1b_4 + b_3b_4, \quad \sigma_2[b] := \sum_{j,k=1}^4 b_jb_k. \quad (1.10)$$

In terms of the function h , the Painlevé equation (1.6) can be represented in a remarkably symmetric form as follows:

$$\frac{dh}{dt} \left[t(1-t) \frac{d^2h}{dt^2} \right]^2 + \left[\frac{dh}{dt} \left\{ 2h - (2t-1) \frac{dh}{dt} \right\} + b_1b_2b_3b_4 \right]^2 - \prod_{k=1}^4 \left(\frac{dh}{dt} + b_k^2 \right) = 0. \quad (1.11)$$

Okamoto's form (1.11) of the Painlevé VI equation turned out to be extremely fruitful for establishing the hidden symmetries of the equation (the so-called Okamoto symmetries). These symmetries look very simple in terms of the auxiliary Hamiltonian function h but are highly nontrivial on the level of the solution y of the Painlevé VI equation, the corresponding monodromy group, and the solution of the associated Fuchsian system [6, 7].

The goal of this paper is twofold. First, we show how to rewrite the Schlesinger system in an arbitrary matrix dimension in a symmetric universal form. Second, we use this symmetric form to find natural analogues of the Okamoto equation (1.11) for 2×2 Schlesinger systems with an arbitrary number of simple poles. Our approach is similar to the approach used by Harnad to derive analogues of the Okamoto equation for Schlesinger systems corresponding to higher-order poles (nonfuchsian systems) [8].

Namely, introducing the following differential operators (which satisfy the commutation relations of the Virasoro algebra):

$$L_m := \sum_{j=1}^N \lambda_j^{m+1} \frac{\partial}{\partial \lambda_j}, \quad m = -1, 0, 1, \dots, \quad (1.12)$$

and the following dependent variables:

$$\mathcal{B}_n := \sum_j \lambda_j^n A_j \equiv \text{res}_{|\lambda=\infty} \{ \lambda^n A(\lambda) \}, \quad n = 0, 1, \dots, \quad (1.13)$$

one can show that the Schlesinger system (1.1) implies

$$L_m \mathcal{B}_n = \sum_{k=1}^{n-1} [\mathcal{B}_k, \mathcal{B}_{m+n-k}] + n \mathcal{B}_{m+n}, \quad (1.14)$$

for all $n \geq 0$ and $m \geq -1$. The infinite set of (1.14) is of course dependent for any given N . To derive the original Schlesinger system (1.1) from (1.14) it is sufficient to take the set of (1.14) for $n \leq N$ and $m \leq N$. The advantage of system (1.14) is in its universality. Its form is independent of the number of the poles; the positions of the poles enter only the definition of the differential operators L_m .

Consider now the case of 2×2 matrices. To formulate the analog of the Okamoto equation for the case of an arbitrary number of poles we introduce the following ‘‘Hamiltonians’’:

$$\widehat{\mathcal{H}}_m := -\frac{1}{4} \sum_{k=0}^m \text{tr} \mathcal{B}_k \mathcal{B}_{m-k}, \quad (1.15)$$

which can be viewed as symmetrised analogues of the Hamiltonians (1.4); they coincide with $L_m \log \tau_M$ up to an elementary transformation. The simplest equation satisfied by $\widehat{\mathcal{H}}_m$ in the case of 2×2 system is given by

$$\begin{aligned} \frac{1}{8} \left(L_2 L_2 \widehat{\mathcal{H}}_2 + 2L_3 \widehat{\mathcal{H}}_3 - 5L_4 \widehat{\mathcal{H}}_2 - 2\widehat{\mathcal{H}}_6 \right)^2 &= \left(L_3 \widehat{\mathcal{H}}_3 - L_4 \widehat{\mathcal{H}}_2 - \widehat{\mathcal{H}}_6 \right) \left(\left(\widehat{\mathcal{H}}_3 \right)^2 + 4\widehat{\mathcal{H}}_2 \left(L_2 \widehat{\mathcal{H}}_2 - \widehat{\mathcal{H}}_4 \right) \right) \\ &\quad - \left(L_3 \widehat{\mathcal{H}}_2 - \widehat{\mathcal{H}}_5 \right) \left(\widehat{\mathcal{H}}_2 L_3 \widehat{\mathcal{H}}_2 + \widehat{\mathcal{H}}_3 L_2 \widehat{\mathcal{H}}_2 - \widehat{\mathcal{H}}_2 \widehat{\mathcal{H}}_5 \right) \\ &\quad + \left(L_2 \widehat{\mathcal{H}}_2 - \widehat{\mathcal{H}}_4 \right) \left(L_2 \widehat{\mathcal{H}}_2 \right)^2, \end{aligned} \quad (1.16)$$

as we will show in the following.

Since the $\widehat{\mathcal{H}}_m$ themselves are combinations of the first-order derivatives of the tau-function, this equation is of the third order; it also has cubic nonlinearity. In the case $N = 4$, (1.16) boils down to the standard Okamoto equation (1.11).

The paper is organized as follows. In Section 2 we derive the symmetrised form of the Schlesinger system. In Section 3 we derive the generalized Okamoto equations. In Section 4 we show how the usual Okamoto equation is obtained from the generalized equation (1.16) in the case $N = 4$. In Section 5 we discuss some open problems.

2. Universal Form of Schlesinger System in Terms of Virasoro Generators

Consider the differential operators L_m (1.12); these operators satisfy the commutation relations of the Virasoro algebra:

$$[L_m, L_n] = (n - m)L_{m+n}. \quad (2.1)$$

To represent the Schlesinger equations in a universal form we also introduce the symmetric dependent variables given by (1.13). The new variable

$$\mathcal{B}_0 = \sum_j A_j \quad (2.2)$$

plays a distinguished role; it vanishes on-shell (i.e., on solutions of the Schlesinger system); however, off-shell plays the role of a generator (with respect to the Poisson bracket (1.3)) of constant gauge transformations (i.e., constant simultaneous similarity transformations of all matrices A_j).

To describe the dynamics under the action of the differential operators L_m we introduce the symmetrised Hamiltonians \mathcal{H}_m :

$$\mathcal{H}_m := -\frac{1}{2} \operatorname{res}_{|\lambda=\infty} \lambda^{m+1} \operatorname{tr} A^2(\lambda) \equiv \sum_j \lambda_j^{m+1} H_j. \quad (2.3)$$

These Hamiltonians can be expressed in terms of the variables \mathcal{B}_k as follows:

$$\mathcal{H}_m = \widehat{\mathcal{H}}_m + \frac{1}{4}(m+1) \sum_j \lambda_j^m C_j, \quad (2.4)$$

where the modified Hamiltonians $\widehat{\mathcal{H}}$ are given by

$$\widehat{\mathcal{H}}_m := -\frac{1}{4} \sum_{k=0}^m \operatorname{tr} \mathcal{B}_k \mathcal{B}_{m-k}, \quad (2.5)$$

and $C_j := \text{tr } A_j^2 = (1/2)\alpha_j^2$. In particular, $\widehat{\mathcal{H}}_{-1} = \widehat{\mathcal{H}}_0 = \widehat{\mathcal{H}}_1 = 0$ (taking into account that $\mathcal{B}_0 = 0$), such that the first three symmetrised Hamiltonians take the form

$$\mathcal{H}_{-1} = 0, \quad \mathcal{H}_0 = \frac{1}{4} \sum_j C_j, \quad \mathcal{H}_1 = \frac{1}{2} \sum_j \lambda_j C_j. \quad (2.6)$$

In terms of the Virasoro generators L_m , (1.5) for the Jimbo-Miwa τ -function τ_{JM} looks as follows:

$$L_m (\log \tau_{\text{JM}}) = \mathcal{H}_m. \quad (2.7)$$

It is convenient to introduce also a modified τ -function, invariant under Möbius transformations.

Lemma 2.1. *The modified τ -function $\tilde{\tau}$ defined by*

$$\tilde{\tau} \equiv \tau_{\text{JM}} \prod_{i < j} (\lambda_j - \lambda_i)^{-(1/2(N-2))(C_i + C_j) + (2/(N-1)(N-2))\mathcal{H}_0} \quad (2.8)$$

is annihilated by the first three Virasoro generators:

$$L_{-1}\tilde{\tau} = L_0\tilde{\tau} = L_1\tilde{\tau} = 0. \quad (2.9)$$

Proof. By a straightforward computation. \square

In terms of the new variables (1.13), the Schlesinger system (1.1) takes a very compact form.

Theorem 2.2. *The differential operators L_m act on the symmetrised variables \mathcal{B}_n as follows:*

$$L_m \mathcal{B}_n = \sum_{k=1}^{n-1} [\mathcal{B}_k, \mathcal{B}_{m+n-k}] + n \mathcal{B}_{m+n}, \quad (2.10)$$

for $m = -1, 0, 1, 2, \dots$, $n = 1, 2, \dots$.

Proof. Using the Schlesinger equations (1.1), we have

$$L_m \mathcal{B}_n \equiv \sum_{i=1}^N \lambda_i^{m+1} \frac{\partial}{\partial \lambda_i} \left\{ \sum_{j=1}^N A_j \lambda_j^n \right\} = \sum_{i \neq j} \lambda_i^{m+1} \frac{\lambda_i^n - \lambda_j^n}{\lambda_i - \lambda_j} [A_j, A_i] + n \sum_{i=1}^N \lambda_i^{m+n} A_i. \quad (2.11)$$

Expanding

$$\frac{\lambda_i^n - \lambda_j^n}{\lambda_i - \lambda_j} = \lambda_j^{n-1} + \lambda_j^{n-2} \lambda_i + \dots + \lambda_j \lambda_i^{n-2} + \lambda_i^{n-1}, \quad (2.12)$$

we further rewrite this expression for $L_m \mathcal{B}_n$ as

$$[\mathcal{B}_{n-1}, \mathcal{B}_{m+1}] + [\mathcal{B}_{n-2}, \mathcal{B}_{m+2}] + \cdots + [\mathcal{B}_1, \mathcal{B}_{m+n-1}] + n\mathcal{B}_{m+n}, \quad (2.13)$$

which coincides with the right-hand side of (2.10). \square

Remark 2.3. System (2.10) can be equivalently rewritten as follows:

$$L_m \mathcal{B}_n = \sum_{k=1}^m [\mathcal{B}_k, \mathcal{B}_{m+n-k}] + n\mathcal{B}_{m+n}. \quad (2.14)$$

That is, the right-hand side of (2.10) does not change if the upper limit $n-1$ is substituted by m .

The system of (2.10), or (2.14), can be viewed as universal form of the Schlesinger system; formally the number of poles does not enter the system any more. Each Schlesinger system written in the standard form (1.1) can be obtained from (2.10) if one makes the ansatz (1.13) for variables \mathcal{B}_k . Using (2.14) we can express the commutators $[\mathcal{B}_m, \mathcal{B}_n]$ as follows:

$$[\mathcal{B}_m, \mathcal{B}_n] = L_m \mathcal{B}_n - L_{m-1} \mathcal{B}_{n+1} + \mathcal{B}_{m+n}. \quad (2.15)$$

Acting on the modified Hamiltonians $\widehat{\mathcal{H}}_n$ by the operators L_m , we get the following equation:

$$L_m \widehat{\mathcal{H}}_n = -\frac{1}{2} \sum_{k=1}^{n-1} k \operatorname{tr} \mathcal{B}_{m+k} \mathcal{B}_{n-k}. \quad (2.16)$$

In particular, we have

$$L_m \widehat{\mathcal{H}}_n - L_n \widehat{\mathcal{H}}_m = (n-m) \widehat{\mathcal{H}}_{m+n}. \quad (2.17)$$

The same equation holds for the Hamiltonians \mathcal{H}_n as a corollary of the integrability of (2.7).

The Poisson bracket (1.3) induces the following Poisson bracket between variables \mathcal{B}_n , $n = 0, 1, 2, \dots$:

$$\left\{ \mathcal{B}_n^a, \mathcal{B}_m^b \right\} = f_c^{ab} \mathcal{B}_{m+n}^c. \quad (2.18)$$

Then (2.10) can then be written in the following form:

$$L_m \mathcal{B}_n = \{ \mathcal{H}_m, \mathcal{B}_n \} + n\mathcal{B}_{m+n}. \quad (2.19)$$

We note that formally the second term can be absorbed into the symplectic action $\{ \mathcal{H}_m, \mathcal{B}_n \}$ upon extending the affine Poisson structure (2.18) by the standard central extension.

Remark 2.4. Let us briefly discuss the geometric origin of the Virasoro generators L_m . The vectors $\partial/\partial\lambda_j$ span the tangent space to the space of N -punctured spheres with punctures at $\{\lambda_j\}$. On the other hand, there exist several universal ways to vary the moduli of a given Riemann surface. For example, one can vary the moduli by vector fields on a chosen closed contour l (see [9]). In the case of our punctured sphere the contour l can be chosen to be a circle containing all singularities λ_j ; then variation of moduli by the vector field $\lambda^{m+1}\partial/\partial\lambda$ on the circle coincides with variation by the generator L_m . The commutation relations between L_m are then inherited from the commutation relations of vector fields on the circle.

3. Generalized Okamoto Equations

Here we will use the symmetric form (2.10) of the Schlesinger equations to derive an analog of Okamoto's equation (1.11) for an arbitrary 2×2 Schlesinger system. In fact, one can write down a whole family of scalar differential equations for the tau-function in terms of the Virasoro generators L_m . In the next theorem we prove two equations of this kind.

Theorem 3.1. *The τ -function τ_{JM} (1.5) of an arbitrary 2×2 Schlesinger system satisfies the following two differential equations.*

(i) *The third-order equation with cubic nonlinearity:*

$$\begin{aligned} \frac{1}{8} \left(L_2 L_2 \widehat{\mathcal{H}}_2 + 2L_3 \widehat{\mathcal{H}}_3 - 5L_4 \widehat{\mathcal{H}}_2 - 2\widehat{\mathcal{H}}_6 \right)^2 &= \left(L_3 \widehat{\mathcal{H}}_3 - L_4 \widehat{\mathcal{H}}_2 - \widehat{\mathcal{H}}_6 \right) \left(\left(\widehat{\mathcal{H}}_3 \right)^2 + 4\widehat{\mathcal{H}}_2 \left(L_2 \widehat{\mathcal{H}}_2 - \widehat{\mathcal{H}}_4 \right) \right) \\ &\quad - \left(L_3 \widehat{\mathcal{H}}_2 - \widehat{\mathcal{H}}_5 \right) \left(\widehat{\mathcal{H}}_2 L_3 \widehat{\mathcal{H}}_2 + \widehat{\mathcal{H}}_3 L_2 \widehat{\mathcal{H}}_2 - \widehat{\mathcal{H}}_2 \widehat{\mathcal{H}}_5 \right) \\ &\quad + \left(L_2 \widehat{\mathcal{H}}_2 - \widehat{\mathcal{H}}_4 \right) \left(L_2 \widehat{\mathcal{H}}_2 \right)^2. \end{aligned} \tag{3.1}$$

(ii) *The fourth-order equation with quadratic nonlinearity:*

$$\begin{aligned} L_2 L_2 L_2 \widehat{\mathcal{H}}_2 &= 8\widehat{\mathcal{H}}_8 - 9L_4 \widehat{\mathcal{H}}_4 + 10L_5 \widehat{\mathcal{H}}_3 - 4L_3 L_3 \widehat{\mathcal{H}}_2 + 10L_4 L_2 \widehat{\mathcal{H}}_2 \\ &\quad - 4 \left(L_2 \widehat{\mathcal{H}}_2 \left(2\widehat{\mathcal{H}}_4 - 3L_2 \widehat{\mathcal{H}}_2 \right) - \widehat{\mathcal{H}}_3 \left(\widehat{\mathcal{H}}_5 - 2L_3 \widehat{\mathcal{H}}_2 \right) \right) \\ &\quad - 8\widehat{\mathcal{H}}_2 \left(2\widehat{\mathcal{H}}_6 - 3L_3 \widehat{\mathcal{H}}_3 + 4L_4 \widehat{\mathcal{H}}_2 \right), \end{aligned} \tag{3.2}$$

where according to (2.4), (2.7),

$$\widehat{\mathcal{H}}_m = L_m \log \tau_{JM} - \frac{m+1}{4} \sum_{j=1}^N \lambda_j^m C_j. \tag{3.3}$$

Proof. Inverting the system of (2.16), we can express $\text{tr } \mathcal{B}_m \mathcal{B}_n$ in terms of the Hamiltonians $\widehat{\mathcal{H}}_n$ as follows:

$$\text{tr } \mathcal{B}_m \mathcal{B}_n = 4L_m \widehat{\mathcal{H}}_n - 2 \left(L_{m-1} \widehat{\mathcal{H}}_{n+1} + L_{m+1} \widehat{\mathcal{H}}_{n-1} \right). \quad (3.4)$$

From the Schlesinger system (2.10), we furthermore get

$$\begin{aligned} L_k \text{tr}(\mathcal{B}_m \mathcal{B}_n) &= \sum_{j=1}^k (\text{tr}(\mathcal{B}_m [\mathcal{B}_j, \mathcal{B}_{n+k-j}]) + \text{tr}(\mathcal{B}_n [\mathcal{B}_j, \mathcal{B}_{m+k-j}])) \\ &+ n \text{tr } \mathcal{B}_m \mathcal{B}_{k+n} + m \text{tr } \mathcal{B}_n \mathcal{B}_{k+m}. \end{aligned} \quad (3.5)$$

Inverting this relation, we obtain for $k < m < n$

$$\begin{aligned} \text{tr}(\mathcal{B}_k [\mathcal{B}_m, \mathcal{B}_n]) &= \sum_{j=m}^{n-1} \left(L_k \text{tr } \mathcal{B}_j \mathcal{B}_{m+n-j} - \frac{1}{2} L_{k-1} \text{tr } \mathcal{B}_{j+1} \mathcal{B}_{m+n-j} - \frac{1}{2} L_{k+1} \text{tr } \mathcal{B}_j \mathcal{B}_{m+n-1-j} \right) \\ &+ \text{tr } \mathcal{B}_n \mathcal{B}_{k+m} - \text{tr } \mathcal{B}_m \mathcal{B}_{k+n}. \end{aligned} \quad (3.6)$$

Combining this equation with (3.4) we can thus express also $\text{tr}(\mathcal{B}_k [\mathcal{B}_m, \mathcal{B}_n])$ entirely in terms of the action of the operators L_m on the Hamiltonians $\widehat{\mathcal{H}}_n$, which can further be simplified upon using the commutation relations (2.1) and (2.17). This leads to the closed expression

$$\begin{aligned} \text{tr}(\mathcal{B}_k [\mathcal{B}_m, \mathcal{B}_n]) &= 2(L_{n-1} L_{m+1} - L_{n+1} L_{m-1}) \widehat{\mathcal{H}}_k \\ &+ 2(L_{n+1} L_m - L_n L_{m+1}) \widehat{\mathcal{H}}_{k-1} \\ &+ 2(L_n L_{m-1} - L_{n-1} L_m) \widehat{\mathcal{H}}_{k+1} \\ &- 4L_{m+n} \widehat{\mathcal{H}}_k + 2L_{m+n+1} \widehat{\mathcal{H}}_{k-1} - 2L_{m+n-1} \widehat{\mathcal{H}}_{k+1} \\ &- 4L_{k+m} \widehat{\mathcal{H}}_n + 2L_{k+m-1} \widehat{\mathcal{H}}_{n+1} + 2L_{k+m+1} \widehat{\mathcal{H}}_{n-1} \\ &+ 4L_{k+n} \widehat{\mathcal{H}}_m - 2L_{k+n-1} \widehat{\mathcal{H}}_{m+1} - 2L_{k+n+1} \widehat{\mathcal{H}}_{m-1}. \end{aligned} \quad (3.7)$$

In particular, for the lowest values of k, m, n we obtain

$$\text{tr}(\mathcal{B}_1 [\mathcal{B}_2, \mathcal{B}_3]) = -2L_2 L_2 \widehat{\mathcal{H}}_2 - 4L_3 \widehat{\mathcal{H}}_3 + 10L_4 \widehat{\mathcal{H}}_2 + 4\widehat{\mathcal{H}}_6. \quad (3.8)$$

To derive from these relations the desired result, we make use of the following algebraic identity:

$$\text{tr}(M_1 [M_2, M_3]) \text{tr}(M_4 [M_5, M_6]) = -2(\text{tr}(M_1 M_4) \text{tr}(M_2 M_5) \text{tr}(M_3 M_6) + \dots), \quad (3.9)$$

that is valid for an arbitrary set of six matrices $M_j \in \mathfrak{sl}(2)$, where the dots on the right-hand side denote complete antisymmetrisation of the expression with respect to the indices 1, 2, 3. In terms of the structure constants of $\mathfrak{sl}(2)$, this identity reads

$$f_{abc}f^{def} = 6 \delta_{[a}^{[d} \delta_b^e \delta_{c]}^f], \quad (3.10)$$

where adjoint indices a, b, \dots are raised and lowered with the Cartan-Killing form. Setting in (3.9) $M_1 = M_4 = \mathcal{B}_1$, $M_2 = M_5 = \mathcal{B}_2$, and $M_3 = M_6 = \mathcal{B}_3$, and using (3.4), (3.8), we arrive (after some calculation) at (3.1).

Equation (3.2) descends from another algebraic identity

$$\text{tr}([M_1, M_2][M_3, M_4]) = -2 (\text{tr}(M_1 M_3)\text{tr}(M_2 M_4) - \text{tr}(M_2 M_3)\text{tr}(M_1 M_4)), \quad (3.11)$$

that is valid for any four $\mathfrak{sl}(2)$ -valued matrices M_j . In terms of the structure constants of $\mathfrak{sl}(2)$, this identity reads

$$f_{abf}f^{cdf} = 2\delta_{[a}^{[c} \delta_b^{d]} \quad (3.12)$$

and is obtained by contraction from (3.10). We consider the action of L_2 on (3.8) which yields

$$\begin{aligned} 3\text{tr}(\mathcal{B}_1[\mathcal{B}_2, \mathcal{B}_5]) - 2\text{tr}(\mathcal{B}_1[\mathcal{B}_3, \mathcal{B}_4]) &= \text{tr}[\mathcal{B}_1, \mathcal{B}_3][\mathcal{B}_1, \mathcal{B}_3] - \text{tr}[\mathcal{B}_1, \mathcal{B}_2][\mathcal{B}_1, \mathcal{B}_4] - \text{tr}[\mathcal{B}_1, \mathcal{B}_2][\mathcal{B}_2, \mathcal{B}_3] \\ &\quad - 2L_2 L_2 L_2 \widehat{\mathcal{H}}_2 - 4L_2 L_3 \widehat{\mathcal{H}}_3 + 10L_2 L_4 \widehat{\mathcal{H}}_2 + 4L_2 \widehat{\mathcal{H}}_6. \end{aligned} \quad (3.13)$$

The l.h.s. of this equation can be reduced by (3.6) while the first terms on the r.h.s. are reduced by means of the algebraic relations (3.11) together with (3.4). As a result we obtain (3.2). \square

4. Four Simple Poles: Reproducing the Okamoto Equation

As remarked before, the explicit form of the differential equations (3.1), (3.2) for the τ -function τ_{JM} is obtained upon expressing the modified Hamiltonians $\widehat{\mathcal{H}}_m$ in terms of τ_{JM} by virtue of (2.4), (2.7). As an illustration, we will work out these equations for the Schlesinger system with four singularities and show that they reproduce precisely Okamoto's equation (1.11). For $N = 4$, the modified τ -function $\tilde{\tau}$ from (2.8) depends only on the cross-ratio

$$t = \frac{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_4)}{(\lambda_1 - \lambda_4)(\lambda_2 - \lambda_3)}. \quad (4.1)$$

We furthermore define the auxiliary function

$$S(t) = 2t(1+t) \frac{d}{dt} \log \tilde{\tau}(t). \quad (4.2)$$

Then (3.1) in terms of S after lengthy but straightforward calculation gives rise to the second-order differential equation

$$\begin{aligned}
3(t(1-t)S'')^2 &= 3(C_2 - C_3)(C_1 - C_4) (S - tS') - 3(C_1 - C_3)(C_2 - C_4)S' \\
&+ 12 (S - tS')S'^2 + 12(S - tS')^2S' \\
&+ 2\sigma_1[C] \left((S - tS')^2 + (S - tS')S' + S'^2 \right) \\
&- \frac{1}{18}\sigma_1[C]^3 + \frac{1}{2}\sigma_1[C]\sigma_2[C] - 3\sigma_3[C],
\end{aligned} \tag{4.3}$$

where $\sigma_i[C]$ are the elementary symmetric polynomials of the C_i 's

$$\sigma_1[C] = C_1 + C_2 + C_3 + C_4, \quad \sigma_2[C] = \sum_{j < k} C_j C_k, \quad \sigma_3[C] = \sum_{j < k < l} C_j C_k C_l. \tag{4.4}$$

Finally, it is straightforward to verify that with $h(t) \equiv S(t) - (1/12)(1 - 2t)\sigma_1[C]$, (4.3) is equivalent to Okamoto's equation (1.11).

In turn, (3.2) leads to the following quadratic third-order differential equation in the function S :

$$\begin{aligned}
6t(1-t) \left((1-2t)S'' + t(1-t)S^{(3)} \right) &= 3 \left((C_1 - C_3)(C_2 - C_4) - (C_2 - C_3)(C_1 - C_4) \right) t \\
&- 12S^2 + 2(1-2t)S(\sigma_1[C] - 12S') \\
&+ 4\sigma_1[C] \left(1 - t + t^2 \right) S' + 36t(1-t)S'^2.
\end{aligned} \tag{4.5}$$

Indeed, this equation can also be obtained by straightforward differentiation of (4.3) with respect to t . In terms of the function h , (4.5) takes the following form:

$$\begin{aligned}
(1-t)t \left(6h^2 - (1-2t)h'' - (1-t)th^{(3)} \right) &= 4h^2 + 8(1-2t)hh' - 2 \left(b_1^2 + b_2^2 + b_3^2 + b_4^2 \right) h' \\
&- \prod_{i < j} b_i^2 b_j^2 + 2(1-2t)b_1 b_2 b_3 b_4,
\end{aligned} \tag{4.6}$$

equivalently obtained by derivative of Okamoto's equation (1.11).

5. Discussion and Outlook

We have shown in this paper that the symmetric form (2.10), (2.14) of the Schlesinger system gives rise to a straightforward algorithm that allows to translate the algebraic $\mathfrak{sl}(2)$ identities (3.10), (3.12) into differential equations for the τ -function of the Schlesinger system. In the simplest case of four singularities, the resulting equations reproduce the known Okamoto equation (1.11). In the case of more singularities, the same equations (3.1), (3.2) give rise to a number of nontrivial differential equations to be satisfied by the τ -function.

Apart from this direct extension of Okamoto's equation, the link between the algebraic structure of $\mathfrak{sl}(2)$ and the Schlesinger system's τ -function gives rise to further generalizations. Note that in the proof of Theorem 3.1, with (3.7), we have already given the analogue of (3.8) to arbitrary values of k, m, n . Combining this equation with identity (3.9) thus gives rise to an entire hierarchy of third-order equations that generalize (3.1). Likewise, the construction leading to the fourth-order equation (3.2) can be generalized straightforwardly upon applying (3.11) to other Virasoro descendants of the cubic equation.

As an illustration, we give the first three equations of the hierarchy generalizing (3.2):

$$\begin{aligned}
L_3L_2L_2\widehat{\mathcal{H}}_2 &= 6\widehat{\mathcal{H}}_9 - 6L_5\widehat{\mathcal{H}}_4 + 10L_6\widehat{\mathcal{H}}_3 - 5L_7\widehat{\mathcal{H}}_2 - L_3L_3\widehat{\mathcal{H}}_3 + L_4L_3\widehat{\mathcal{H}}_2 + 6L_5L_2\widehat{\mathcal{H}}_2 \\
&\quad + 8\widehat{\mathcal{H}}_2\left(2L_4\widehat{\mathcal{H}}_3 - 3L_5\widehat{\mathcal{H}}_2 - \widehat{\mathcal{H}}_7\right) + 4\widehat{\mathcal{H}}_3\left(L_3\widehat{\mathcal{H}}_3 - L_4\widehat{\mathcal{H}}_2\right) - 8\widehat{\mathcal{H}}_4L_3\widehat{\mathcal{H}}_2 + 12L_2\widehat{\mathcal{H}}_2L_3\widehat{\mathcal{H}}_2, \\
L_4L_2L_2\widehat{\mathcal{H}}_2 &= 2\widehat{\mathcal{H}}_{10} + 3L_5\widehat{\mathcal{H}}_5 - 8L_6\widehat{\mathcal{H}}_4 + 9L_7\widehat{\mathcal{H}}_3 - 9L_8\widehat{\mathcal{H}}_2 - L_4L_3\widehat{\mathcal{H}}_3 \\
&\quad + 4L_4L_4\widehat{\mathcal{H}}_2 - 3L_5L_3\widehat{\mathcal{H}}_2 + 6L_6L_2\widehat{\mathcal{H}}_2 + 8\widehat{\mathcal{H}}_2\left(L_4\widehat{\mathcal{H}}_4 + 3L_5\widehat{\mathcal{H}}_3 - 3L_6\widehat{\mathcal{H}}_2\right) \\
&\quad - 4\widehat{\mathcal{H}}_3\left(L_4\widehat{\mathcal{H}}_3 - 3L_5\widehat{\mathcal{H}}_2\right) + 8\widehat{\mathcal{H}}_4L_4\widehat{\mathcal{H}}_2 - 12L_2\widehat{\mathcal{H}}_2L_4\widehat{\mathcal{H}}_2, \\
L_3L_3L_2\widehat{\mathcal{H}}_2 &= 8\widehat{\mathcal{H}}_{10} - 6L_5\widehat{\mathcal{H}}_5 + 4L_6\widehat{\mathcal{H}}_4 + 6L_7\widehat{\mathcal{H}}_3 - 6L_8\widehat{\mathcal{H}}_2 - L_4L_3\widehat{\mathcal{H}}_3 - 2L_4L_4\widehat{\mathcal{H}}_2 + 6L_5L_3\widehat{\mathcal{H}}_2 \\
&\quad + 4L_6L_2\widehat{\mathcal{H}}_2 - 4\left(\widehat{\mathcal{H}}_5 - 2L_3\widehat{\mathcal{H}}_2\right)L_3\widehat{\mathcal{H}}_2 + 4L_2\widehat{\mathcal{H}}_2L_3\widehat{\mathcal{H}}_3 - 8\widehat{\mathcal{H}}_4L_4\widehat{\mathcal{H}}_2 \\
&\quad + 4\widehat{\mathcal{H}}_3\left(L_4\widehat{\mathcal{H}}_3 - 3L_5\widehat{\mathcal{H}}_2\right) - 8\widehat{\mathcal{H}}_2\left(2\widehat{\mathcal{H}}_8 - 2L_4\widehat{\mathcal{H}}_4 + L_5\widehat{\mathcal{H}}_3 + 2L_6\widehat{\mathcal{H}}_2\right).
\end{aligned} \tag{5.1}$$

Obviously, these equations are not all independent, but related by the action of the lowest Virasoro generators $L_{\pm 1}$, using that

$$L_1\left(L_2L_2L_2\widehat{\mathcal{H}}_2\right) = 4\left(L_3L_2L_2\widehat{\mathcal{H}}_2\right) + \dots, \tag{5.2}$$

and so forth, since $\widehat{\mathcal{H}}_1 = 0$. The number and structure of the independent equations in this hierarchy is thus organized by the structure of representations of the Virasoro algebra. For the case of $N = 4$ singularities, the explicit form of all the equations of the hierarchy reduces to equivalent forms of (1.11) and (4.6). With growing number of simple poles, the number of independent differential equations induced by the hierarchy increases.

Therefore, we arrive to a natural question: which set of derived equations for the tau-function is equivalent to the original Schlesinger system? We stress that all differential equations for the tau-function are PDE with respect to the variables $\lambda_1, \dots, \lambda_N$. However, if one gets a sufficiently high number of independent equations, one can actually come to a set of ODEs for the tau-function. This situation resembles the situation with the original form of the Schlesinger system (1.1): if one ignores the second set of equations in (1.1); one gets a system of PDEs for the residues A_j ; only upon adding the equations for $\partial A_j / \partial \lambda_j$ one gets a system of ODEs with respect to each λ_j (the flows with respect to different λ_j commute).

Let us finally note that the construction we have presented in order to derive the differential equations (3.1), (3.2) suggests a number of interesting further generalizations that deserve further study.

- (i) At the origin of our derivation have figured the algebraic $\mathfrak{sl}(2)$ identities (3.10), (3.12) that we have translated into differential equations. Similar identities exist also for higher rank groups (e.g., $M > 2$, or the Schlesinger system for orthogonal, symplectic, and exceptional groups), where the number of independent tensors may be larger. It would be highly interesting to understand if equations analogous to (3.1), (3.2) can be derived from such higher rank algebraic identities. As those identities will be built from a larger number of invariant tensors (structure constants, etc.), the corresponding differential equations would be of higher order in derivatives.
- (ii) Is it possible to combine our present construction applicable to Schlesinger systems with simple poles only with construction of [8] which requires the presence of higher-order poles? What would be the full set of equations for the tau-function with respect to the full set of deformation parameters in presence of higher-order poles?
- (iii) The Schlesinger system (1.1) has also been constructed for various higher genus Riemann surfaces [10–13]. It would be interesting to first of all find the proper generalization of the symmetric form (2.10), (2.14) of the Schlesinger system to higher genus surfaces which in turn should allow to derive by an analogous construction the nontrivial differential equations satisfied by the associated τ -function. We conjecture that in some sense the form (2.10) should be universal; it should remain the same, although the definition of the Virasoro generators L_m and the variables \mathcal{B}_m may change.
- (iv) As we have mentioned above, the extra term $n\mathcal{B}_{m+n}$ in the Hamiltonian dynamics of the symmetrised Schlesinger system (2.19) can be absorbed into the symplectic action upon replacing the standard affine Lie-Poisson bracket (2.18) by its centrally extended version. However, this central extension is not seen in any of the finite- N Schlesinger systems. This seems to suggest that system (2.10) should be considered not just as a symmetric form of the usual Schlesinger system with finite number of poles, but as a “universal” Schlesinger system which involves an infinite set of independent variables \mathcal{B}_n . Presumably, this full system involves the generators L_n and coefficients \mathcal{B}_n not only for positive, but also for negative n . In this setting, the centrally extended version of the bracket (2.18) should appear naturally. The most interesting problem would be to find the geometric origin of such a generalized system; a possible candidate could be the isomonodromic deformations on higher genus curves.

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