

## Research Article

# A Lie Algebroid on the Wiener Space

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We define a Lie algebroid on the space of smooth 1-forms in the Nualart-Pardoux sense on the Wiener space associated to the stochastic linear Poisson structure on the Wiener space defined Léandre (2009).

## 1. Introduction

Infinite dimensional Poisson structures play a big role in the theory of infinite dimensional Lie algebras [1], in the theory of integrable system [2], and in field theory [3]. But for instance, in [2], the test functional space where the hydrodynamic Poisson structure acts continuously is not conveniently defined. In [4, 5] we have defined such a test functional space in the case of a linear Poisson bracket of hydrodynamic type. On the other hand, it is very well known [6] that the theories of Lie groupoids and Lie algebroids play a key role in Poisson geometry. It is interesting to study a Lie algebroid for the Poisson structure [4] defined analytically in the framework of [4]. We postpone until later the study the Lie groupoid associated to the same Poisson structure but in the algebraic framework of [5]. The definition of this Lie groupoid in the framework of [4] presents, namely, some difficulties. Moreover some deformation quantizations for symplectic structures in infinite dimensional analysis were recently performed (see the review of Léandre [7] on that). The theory of groupoids is related [8] to Kontsevich deformation quantization [9].

Let us recall what a Lie algebroid is [6, 10–13]. We consider a bundle  $E$  on a smooth finite dimensional manifold  $M$ .  $TM$  is the tangent bundle of  $M$ .  $\Gamma^\infty(E)$  and  $\Gamma^\infty(TM)$  denote the space of smooth section of  $E$  and  $TM$ . A Lie algebroid on  $E$  is given by the following data.

(i) A Lie bracket structure  $[\cdot, \cdot]_E$  on  $\Gamma^\infty(E)$  has in particular to satisfy the Jacobi relation

$$[[X_1, X_2]_E, X_3]_E + [[X_2, X_3]_E, X_1]_E + [[X_3, X_1]_E, X_2]_E = 0. \quad (1.1)$$

- (ii) A smooth fiberwise linear map  $\rho_E$ , called the anchor map, from  $E$  into  $TM$  satisfies the relation

$$[X, fY]_E = f[X, Y]_E + \langle df, \rho_E(X) \rangle Y, \quad (1.2)$$

for any smooth sections  $X, Y$  of  $E$  and any element  $f$  of  $C^\infty(M)$ , the space of smooth functions on  $M$ .

Let us recall the definition of a Poisson structure on  $M$ . It is an antisymmetric  $\mathbb{R}$ -bilinear map  $\{\cdot, \cdot\}$  from  $C^\infty(M) \times C^\infty(M)$  into  $C^\infty(M)$ , which is a derivation on each component, vanishes on the constant and satisfies the Jacobi relation

$$\{\{f_1, f_2\}, f_3\} + \{\{f_2, f_3\}, f_1\} + \{\{f_3, f_1\}, f_2\} = 0. \quad (1.3)$$

A Poisson structure is given by a bivector  $\pi$ , that is, an element of  $\Gamma^\infty(\Lambda^2 TM)$  as

$$\{f_1, f_2\} = \langle \pi, df_1, df_2 \rangle = i_{df_2} i_{df_1} \pi, \quad (1.4)$$

where  $df_1$  and  $df_2$  can be seen as 1-forms and the dual of  $T_x^*(M)$  is  $T_x(M)$ .  $i_\alpha \pi$  denotes the interior product of the bivector  $\pi$  by the 1-form  $\alpha$ . If  $\pi$  is a bivector, then we can define a fiberwise linear smooth map from  $T^*M$ , the cotangent bundle of  $M$ , into  $TM$ , called  $\tilde{\pi}$ .

If  $\alpha$  is a smooth section of  $T^*M$ , then

$$\tilde{\pi}\alpha = i_\alpha \pi. \quad (1.5)$$

This allows us to define a Lie algebroid structure on  $T^*M$  [11, 14–18] as follows.

- (i) The Bracket is defined by

$$[\alpha, \beta]_{T^*M} = L_{\tilde{\pi}\alpha}\beta - L_{\tilde{\pi}\beta}\alpha - d(\pi(\alpha, \beta)), \quad (1.6)$$

where  $L$  is the usual Lie derivative of a 1-form: if  $X$  is a vector field and  $\alpha$  a  $k$ -form, then  $L_X \alpha$  is given by the Cartan formula

$$L_X \alpha = i_X d\alpha + di_X \alpha. \quad (1.7)$$

- (ii) The anchor map  $\rho_{T^*M}$  is the map  $\tilde{\pi}$ .

Infinite dimensional symplectic structures and their related Poisson structure were introduced by Dito and Léandre [19], Léandre [7, 20–22], and Léandre and Obame [23] in the infinite dimensional analysis, motivated by the theory of deformation quantization in infinite dimension. We refer to the review of Léandre on that in [7].

The infinite dimensional Poisson structure is a tool in the theory of integrable system [24]. We refer to the review of Mokhov in [3] and Dubrovin and Novikov in [2] on that. In particular, some partial differential equations of the theory of integrable systems are

described as Hamiltonian systems associated to some Poisson structure. For instance, the Gardner-Zakharov-Faddeev (ultralocal) bracket (see [2, pages 52-53])

$$\{x(s), x(t)\} = \delta'_0(s - t) \quad (1.8)$$

is used to describe the KdV equation as a Hamiltonian system.

Another simple Poisson structure of Dubrovin and Novikov [2] is given as follows. We consider the set of smooth paths  $t \rightarrow (x_i(t))$  into  $(\mathbb{R}^m)^*$ , the dual of a Lie algebra structure on  $\mathbb{R}^m$  with structural constants  $c_{i,j}^k$ . In such a case,

$$\{x_i(s), x_j(t)\} = \sum_k \delta_0(s - t) c_{i,j}^k x_k(t). \quad (1.9)$$

It is useful to avoid the presence of the Dirac mass, and Léandre [4] has given an appropriate definition of the Poisson structure of the previous formula in the framework of Malliavin Calculus.

The goal of infinite dimensional analysis is to give a rigorous meaning to some formal considerations of mathematical physics. The formal operations of mathematical physics are defined consistently on some functional spaces. It is very well known, for instance, that the vacuum expectation of some operator algebras [25] is given by formal path integrals on the fields. Infinite dimensional analysis deals in the simplest case where these objects are mathematically well established.

- (i) The functional integral side is given by the Malliavin Calculus [26].
- (ii) The operator algebra side is given by white noise analysis and quantum probability [27, 28].

Let us recall basically the objects of these Calculi.

- (i) The main object of white noise analysis and quantum probability is given by the Bosonic Fock space  $\text{Fock}(\mathbb{H}_2)$  associated to the Hilbert space  $\mathbb{H}_2$  of  $L^2$  maps from  $[0, 1]$  into  $\mathbb{R}$ .  $\text{Fock}(\mathbb{H}_2)$  is constituted of series  $\sigma = \sum h^n$  where  $h^n$  belongs to  $\mathbb{H}_2^{\otimes n}$ , the symmetric  $n$ -tensor product of  $\mathbb{H}_2$  such that

$$\|\sigma\|^2 = \sum n! \|h^n\|_{\mathbb{H}_2^{\otimes n}}^2 < \infty. \quad (1.10)$$

The operator algebra is the algebra of annihilation and creation operator on the Fock space submitted to the canonical commutation relations

$$\begin{aligned} [a(s), a(t)] &= 0, \\ [a^*(s), a^*(t)] &= 0, \\ [a(s), a^*(t)] &= \delta_0(s - t), \end{aligned} \quad (1.11)$$

where  $a(s)$  is an elementary annihilation operator.  $a^*(t)$  is an elementary creation operator. The presence of a Dirac mass leads to the same difficulties as in (1.8)

and (1.9) and leads white noise analysis to consider an improvement of the Fock space (called the Hida Fock space) such that these operators act continuously on it.  $a(s) + a^*(s)$  is called the white noise and can be interpreted in the measure theory.

(ii) The main object of the Malliavin Calculus is the  $L^p$  space of the Wiener measure.

If we consider the Brownian motion  $t \rightarrow B(t)$  on  $\mathbb{R}$  (see part 2 of this work), then we can introduce the Brownian functional associated to  $\sigma$  in the Bosonic Fock space:

$$\varphi(\sigma) = \sum \int_{[0,1]^n} h^n(s_1, \dots, s_n) \delta B(s_1) \cdots \delta B(s_n). \quad (1.12)$$

An element  $h^n$  of  $\mathbb{H}_2^{\otimes n}$  can be realized as a symmetric map from  $[0,1]^n$  into  $\mathbb{R}$  and  $\int_{[0,1]^n} h^n(s_1, \dots, s_n) \delta B(s_1) \cdots \delta B(s_n)$  denotes a Wiener chaos [28]. This map  $\varphi$  realizes an isometry between the Fock space and the  $L^2$  of the Wiener measure. The main ingredient of the Malliavin Calculus is to take the derivative (*almost surely defined!*) in the direction of an element  $t \rightarrow \int_0^t h(s) ds$ ;  $h \in \mathbb{H}_2$ . An element  $t \rightarrow \int_0^t h(s) ds$  is called an element of the Cameron-Martin space  $\mathbb{H}$ . This operation can be interpreted as a “nonelementary” annihilation operator on the Fock space. Through this isomorphism,  $a(t) + a^*(t)$  can be interpreted as  $d/dtB(t)$ , the white noise associated to the Brownian motion, which does not exist in the traditional sense because the Brownian motion is *only continuous!* Since there are integrations by parts associated to a derivatation along an element of the Cameron-Martin space of a cylindrical functional, this operation is closable. It is the generalization in infinite dimension of the traditional definition of Sobolev spaces on finite dimensional spaces. But in infinite dimension, we consider Gaussian measures and not Lebesgue measure, which does not exist as a measure in infinite dimension! But since there is no Sobolev imbedding in infinite dimension, functionals which belong to all the Sobolev spaces of the Malliavin Calculus (these functionals are said to be smooth in the Malliavin sense) are in general only *almost surely defined!*

The study of Poisson structures requests that the test functional space where this Poisson structure acts is an *algebra*.

- (i) In the case of the Malliavin Calculus, there is a natural way to choose an algebra starting from the considerations of measure theory. With the intersection of all the  $L^p$ ,  $p < \infty$  is indeed an algebra through the Hoelder inequality. We consider the Wiener product on the Wiener space which is the classical product of functionals.
- (ii) In white noise analysis, there is on the Fock space another product called the standard Wick product. The traditional product of a Wiener chaos of length  $n$  and of length  $m$  is not a chaos of length  $n + m$  by the help of the Itô formula. It is an infinite dimensional generalization of the fact that the product of two Hermite polynomials in finite dimension is not a Hermite polynomial. The classical Wick product consists to keep in the product of these two chaoses the chaoses of length  $n+m$ . Reference [29] has defined another Wick product (called the normalized Wick product), which fits well with Stratonovitch chaos. We consider now

$$\varphi^{st}(\sigma) = \sum \int_{[0,1]^n} h(s_1, \dots, s_n) dB(s_1) \cdots dB(s_n). \quad (1.13)$$

We consider this time multiple Stratonovitch integrals. They are the limit when  $k \rightarrow \infty$  of the classical random multiple integrals  $\int_{[0,1]^n} h(s_1, \dots, s_n) dB^k(s_1) \cdots dB^k(s_n)$  where  $t \rightarrow B^k(t)$  is the polygonal approximation of the Brownian motion. The Itô-Stratonovitch integral is the classical one. The normalized Wick product of Léandre and Rogers [29] :  $\sigma_1 \cdot \sigma_2$  : of  $\sigma_1$  and  $\sigma_2$  belonging to the Hida Fock space is done in order

$$\psi^{st}(\sigma_1 \cdot \sigma_2) = \psi^{st}(\sigma_1)\psi^{st}(\sigma_2), \quad (1.14)$$

using the Itô-Stratonovitch formula [23]. This reflects in an infinite dimensional sense the classical fact that the product of two monomials is still a monomial in a finite dimensional polynomial algebra.

Léandre [4] has given an appropriate definition to the Poisson structure of the previous formula on an algebra of functional on the Wiener space of the Malliavin type. The main difficulty to overcome is that the good understanding of this Poisson structure leads to the study of some anticipative Stratonovitch integrals. The traditional Malliavin Calculus [26] is not suitable to study some anticipative Stratonovitch integrals. This means that if we consider a random element  $s \rightarrow h(s)$  of  $\mathbb{H}_2$  which belongs to all the Sobolev spaces of the Malliavin Calculus, then the anticipative Stratonovitch integral

$$\int_0^1 h(s)dB(s) = \lim_{k \rightarrow \infty} \int_0^1 h(s)dB^k(s) \quad (1.15)$$

does not exist. This pathology is not true if we consider a refinement of the Itô integral called the Hitsuda-Skorokhod integral [30]. Let us recall that the first authors who have studied anticipative Stratonovitch integrals are Nualart and Pardoux [30]. Léandre has defined conveniently some Sobolev spaces on the Wiener space such that the map anticipative Stratonovitch integral acts continuously on them [31–37]. This means that if  $h \in \mathbb{H}_2$  belongs to all the Sobolev spaces in the Nualart-Pardoux sense, then the anticipative Stratonovitch integral (1.15) exists. The main difference between the classical definition of the Sobolev space in the Malliavin sense and the Sobolev spaces in the Nualart-Pardoux sense is that some regularity on the kernels on the derivatives on the considered functionals is requested. This is a generalization of the following fact: we can define a nonanticipative Itô integral  $\int_0^1 h(s)\delta B(s)$  without assuming a lot of regularity on the nonanticipative element  $h$  of  $\mathbb{H}_2$ . But in the case of a nonanticipative Stratonovitch integral,  $\int_0^1 h(s)dB(s)$ , we have to assume that  $h$  is a semimartingale; this means some regularity on  $h$ ! The Sobolev spaces of Nualart-Pardoux type were introduced by Léandre in [31–36] in order to study some Sobolev cohomology theories of some loop space endowed with the Brownian bridge measure on a compact Riemannian manifold. So the Poisson structure (1.9) can be defined consistently on the Nualart-Pardoux test algebra [4].

Let us recall that in white noise analysis, the algebraic counterpart of the Malliavin Calculus, the main tool is the Fock space and the algebra of creation and annihilation operators on the Fock space. The Bosonic Fock space is transformed into the  $L^2$  of an infinite dimensional Gaussian measure by the help of the map Wiener chaos. The Poisson structure (1.9) was defined by Léandre in [5] on the Hida test algebra endowed with the normalized Wick product.

The goal of this paper is to define a Lie algebroid associated to the Poisson structure (1.9) on the Nualart-Pardoux test algebra. The main remark is that the map  $\tilde{\pi}$  transforms a 1-form on the Wiener space smooth in the Nualart-Pardoux sense in a generalized vector field on the Wiener space, whose theory was done by Léandre in [32, 35], and not in an ordinary vector field on the Wiener space! Classical vector field on the Wiener space are random elements of the Cameron-Martin space which belongs to all the Sobolev spaces of the Malliavin Calculus. Generalized vectorfield is  $H(t) = \int_0^t h(s)dB(s) + \int_0^t h^1(s)ds$  where  $h(s)$  is chosen well such that we can define the anticipative Stratonovitch integral  $\int_0^t h(s)dB(s)$ . In general, we cannot define the derivative of a functional which belongs to all the Sobolev spaces of the Malliavin Calculus along a generalized vector field. But we can do that if the functional belongs to all the Sobolev spaces in the Nualart-Pardoux sense. In this paper, since we consider smooth 1-forms in the Nualart-Pardoux sense, we can define still their interior product by a generalized vector field through our theory of anticipative Stratonovitch integral. So the formulas (1.1) and (1.2) are still true, but almost surely!

## 2. The Linear Stochastic Poisson Structure

We consider the set of continuous paths  $C([0,1];\mathbb{R}^m)$  from  $[0,1]$  into  $\mathbb{R}^m$  endowed with the uniform topology. A typical path is denoted by  $t \rightarrow B(t) = (B_i(t))$ , on which we consider the Brownian motion measure  $dP$  [38].

Let us recall how we construct  $dP$ . We consider the Cameron-Martin Hilbert space  $\mathbb{H}$  [39] of maps from  $[0,1]$  into  $\mathbb{R}^m$  such that

$$\int_0^1 |h'_s|^2 ds = \|h\|^2 < \infty, \quad (2.1)$$

and  $dP$  is formally the Gaussian probability measure

$$\frac{1}{Z} \exp \left[ -\frac{\|h\|^2}{2} \right] dD(h), \quad (2.2)$$

and  $dD$  is the formal Lebesgue measure on  $\mathbb{H}$  which does not exist as a measure (we refer to the works of Léandre [40–42], Asada [43], and Pickrell [44] for various approaches to the Lebesgue measure in infinite dimension). Let  $\mathbb{H}_1$  be a finite dimensional real Hilbert space ( $h_1 \in \mathbb{H}_1$ ) with Hilbert norm  $\|\cdot\|_1$ . Let us consider the centered normalized Gaussian measure on  $\mathbb{H}_1$ . It is classically represented by  $\sum e_i N_i$  where  $N_i$  are centered normalized independent one-dimensional Gaussian variables and the system of  $e_i$  constitutes an orthonormal basis of the real Hilbert space  $\mathbb{H}_1$ .

It should be tempting to represent  $dP$  by using the same procedure. We consider an orthonormal basis  $e_i$  of  $\mathbb{H}$ . The law of the Brownian motion is represented by the series  $\sum e_i N_i$  where the  $N_i$  is a collection of independent centered one-dimensional Gaussian variables. This series *does not converge* in  $\mathbb{H}$  but in  $C([0,1];\mathbb{R}^m)$  [45]. We refer to the textbook of Kuo [46] for the theory of infinite dimensional Gaussian measures.

Let us consider a functional  $F$  on  $C([0,1];\mathbb{R}^m)$ . Its  $r$ th stochastic derivative  $\nabla^r F$ , according to the framework of the Malliavin Calculus [26, 47, 48], is defined if it exists by

$$\langle \nabla^r F, h_1, \dots, h_r \rangle = \int_{[0,1]^r} \left\langle \nabla^r F(s_1, \dots, s_r), h'_{1,s_1}, \dots, h'_{r,s_r} \right\rangle ds_1 \cdots ds_r, \quad (2.3)$$

where  $h_i$  belongs to the Hilbert space of paths  $\mathbb{H}$  from  $[0,1]$  into  $\mathbb{R}^m$  satisfying

$$\int_0^1 |h'_s|^2 ds = \|h\|^2 < \infty. \quad (2.4)$$

The Sobolev norms of the Malliavin Calculus are defined by the following formula. If  $F$  is a Brownian functional, then

$$\left\{ E \left[ \left( \int_{[0,1]^r} |\nabla^r F(s_1, \dots, s_r)|^2 ds_1 \cdots ds_r \right)^{p/2} \right] \right\}^{1/p} = \|F\|_{r,p}. \quad (2.5)$$

The Malliavin test algebra consists of Brownian functionals  $F$  all of whose Sobolev norms  $\|F\|_{r,p}$  are finite  $r, p < \infty$ . Let us recall how we construct these Sobolev spaces. Let  $f$  be a smooth function from  $(\mathbb{R}^m)^d$  into  $\mathbb{R}$  with compact support and some times  $0 < t_1 < \cdots < t_d \leq 1$ . We introduce the cylindrical functional  $F(B(\cdot)) = f(B(t_1), \dots, B(t_d))$ . We consider the Gateaux derivative of  $F$  along a deterministic direction  $h$  of  $\mathbb{H}$ :

$$\langle \nabla F, h \rangle = \left\langle \sum \frac{\partial}{\partial x_i} f(B(t_1), \dots, B(t_d)), h_{t_i} \right\rangle. \quad (2.6)$$

There is absolutely no problem to define it. We use the integration by parts formula for a cylindrical functional

$$E[\langle \nabla F, h \rangle] = E \left[ F \int_0^1 \langle h'_s, \delta B(s) \rangle \right], \quad (2.7)$$

where  $\delta B(s)$  is the Itô differential. The Itô integral is the limit in all the  $L^p(dP)$ ,  $p < \infty$  of the sum  $\sum \langle h'_{s_i}, B(s_{i+1}) - B(s_i) \rangle$  where  $0 < s_1 < \cdots < s_i < s_{i+1} < \cdots < s_{2^n-1} < 1 = s_{2^n}$  is a dyadic subdivision of  $[0,1]$  of length  $2^n$ . The convergence does not pose any problem because  $h$  is deterministic. Since we have the integration by parts formula (2.7), we can extend the operation of taking the stochastic derivative of a Brownian functional  $F$  consistently, as we establish classically the definition of Sobolev spaces in finite dimension. The main novelty of the Malliavin Calculus with respects of [49–52] motivated by mathematical physics is that the algebra of functionals which belong to all the Sobolev spaces of the Malliavin Calculus (these functionals are said to be smooth in the Malliavin sense) is constituted of functionals almost surely defined. The reader interested in the Malliavin Calculus can see the books of [26, 47].

If we consider the same dyadic subdivision as before, we can introduce the polygonal approximation  $B^n(t)$  of  $B(t)$ . Let us consider a “nondeterministic!” map from  $[0,1]$  into



$\mathbb{R}^m t \rightarrow \beta_t$  which belongs to  $L^2([0, 1]; \mathbb{R}^m)$ . We can consider the random ordinary integral  $\int_0^1 \langle \beta_t, dB(t) \rangle$ . It can also be easily defined. In general, the limit may not exist when  $n$  tends to infinity, because the Brownian motion is only continuous. If we can pass to the limit, we say that the limit  $\int_0^1 \langle \beta_t, dB(t) \rangle$  is an anticipative Stratonovitch integral. Nualart and Pardoux [30] are the first authors who have defined some anticipative Stratonovitch integrals. An appropriate theory was established by Léandre [31–36] in order to understand some Sobolev cohomology theories on the loop space. Let us recall it quickly.

We consider another set of Sobolev norms [31]. We suppose that outside the diagonals of  $[0, 1]^r$

$$\{E[|\nabla^r F(s_1, \dots, s_r) - \nabla^r F(s'_1, \dots, s'_r)|^p]\}^{1/p} \leq C_{r,p} \sum |s_i - s'_i|^{1/2}. \quad (2.8)$$

The smallest  $C_{r,p}$  such that the previous inequality is satisfied is called the first Nualart-Pardoux Sobolev norm. The second Nualart-Pardoux Sobolev norm is the smallest  $C_{r,p}^1$  such that, for all  $(s_i) \in [0, 1]^r$ ,

$$\{E[|\nabla^r F(s_1, \dots, s_r)|^p]\}^{1/p} \leq C_{r,p}^1. \quad (2.9)$$

*Definition 2.1.* The Nualart-Pardoux test algebra  $N.P_{\infty-}$  consists of functionals  $F$  whose all Nualart-Pardoux Sobolev norms of first type and second type are finite. The elements of  $N.P_{\infty-}$  are said to be smooth in the Nualart-Pardoux sense.

Let us recall that  $N.P_{\infty-}$  is an algebra [31].

We can consider a random element of  $L^2([0, 1]; \mathbb{R}^m)$ ,  $t \rightarrow \beta_t$ . We can consider its  $r$ th stochastic derivative

$$\langle \nabla^r \beta_t, h_1, \dots, h_r \rangle = \int_{[0,1]^r} \langle \nabla^r \beta_t(s_1, \dots, s_r), h'_{1,s_1}, \dots, h'_{r,s_r} \rangle ds_1 \cdots ds_r. \quad (2.10)$$

Its first Nualart-Pardoux Sobolev norm  $C_{r,p}$  is the smallest number such that outside the diagonals of  $[0, 1] \times [0, 1]^r$

$$\{E[|\nabla^r \beta_t(s_1, \dots, s_r) - \nabla^r \beta_{t'}(s'_1, \dots, s'_r)|^p]\}^{1/p} \leq C_{r,p} (|t - t'|^{1/2} + \sum |s_i - s'_i|^{1/2}). \quad (2.11)$$

The second type of Nualart-Pardoux Sobolev norm  $C_{r,p}^1$  of  $\beta_{(\cdot)}$  is the smallest number such that for all  $(t, s_1, \dots, s_r) \in [0, 1] \times [0, 1]^r$

$$\{E[|\nabla^r \beta_t(s_1, \dots, s_r)|^p]\}^{1/p} \leq C_{r,p}^1. \quad (2.12)$$

Let us recall the theorem of Léandre [31].

**Theorem 2.2.** *Let  $\beta$  be a random element of  $L^2([0, 1]; \mathbb{R}^m)$  such that all its Nualart-Pardoux Sobolev norms are finite. Then the anticipative Stratonovitch integral*

$$\int_0^1 \langle \beta_t, dB(t) \rangle \quad (2.13)$$



is smooth in the Nualart-Pardoux sense and its Nualart-Pardoux Sobolev norms can be estimated in terms of the Nualart-Pardoux norms of  $\beta$ .

In such a case,  $\int_0^1 \langle \beta_t, dB(t) \rangle$  is the limit in all the  $L^p(dP)$ ,  $p < \infty$  of  $\int_0^1 \langle \beta_t, dB^n(t) \rangle$ . Moreover,

$$\left\langle \nabla \left( \int_0^1 \langle \beta_t, dB(t) \rangle \right), \tilde{h} \right\rangle = \int_0^1 \left\langle \langle \nabla \beta_t, \tilde{h} \rangle, dB(t) \right\rangle + \int_0^1 \left\langle \beta_t, \frac{d}{dt} \tilde{h}_t \right\rangle dt. \quad (2.14)$$

This means that the kernel of the stochastic derivative of  $\int_0^1 \langle \beta_t, dB(t) \rangle$  is  $\int_0^1 \langle \nabla \beta_t(s), dB(t) \rangle + \beta_s$ . Let us explain this formula; in order to take the stochastic derivative of  $\int_0^1 \langle \beta_t, dB(t) \rangle$ , we do the same formal computations as if the anticipative Stratonovitch integral had been a classical integral; we take first of all derivatives of  $\beta_t$  which lead to the term  $\langle \nabla \beta_t, \tilde{h} \rangle$  and derivatives of  $dB_t$  which lead to  $d/dt \tilde{h}_t dt$ .

Let us recall the notion of a Poisson bracket  $\{\cdot, \cdot\}$ . We consider a commutative Frechet unital real algebra endowed with a family of Banach norms  $\|\cdot\|_p$ . This means that for all  $p$ , there exists  $p^1$  such that for all  $F^1, F^2$  in  $A$

$$\|F^1 F^2\|_p \leq C_p \|F^1\|_{p^1} \|F^2\|_{p^1}. \quad (2.15)$$

A Poisson Bracket is a bilinear map from  $A \times A$  into  $A$ , which is a derivation in each argument, vanishes on the unit. The derivation property means that for all  $F^1, F^2, F^3$  in  $A$

$$\{F^1 F^2, F^3\} = F^1 \{F^2, F^3\} + \{F^1, F^3\} F^2. \quad (2.16)$$

Moreover, it satisfies the following properties: if  $F^1, F^2, F^3$  belong to  $A$ , then

$$\begin{aligned} \{F^1, F^2\} &= -\{F^2, F^1\}, \\ \{\{F^1, F^2\}, F^3\} + \{\{F^2, F^3\}, F^1\} + \{\{F^3, F^1\}, F^2\} &= 0. \end{aligned} \quad (2.17)$$

Moreover, for all  $p$ , there exist  $p'$  and  $C_p$  such that

$$\|\{F^1, F^2\}\|_p \leq C_p \|F^1\|_{p'} \|F^2\|_{p'}. \quad (2.18)$$

In the sequel, we will choose  $A = N.P_{\infty}$ . We consider the structural constants  $c_{ij}^k$  of a Lie algebra structure on  $(\mathbb{R}^m)^*$ . The stochastic gradient  $\nabla F$  of a functional  $F$  can be written  $\nabla F = (\nabla F_i)$ . Formula (1.9) reads in this framework as

$$\{F^1, F^2\} = \sum_{i,j,k} \int_0^1 \nabla F_i^1(s) \nabla F_j^2(s) c_{ij}^k dB_k(s), \quad (2.19)$$

where we consider a Stratonovitch anticipative integral. This defines a Poisson structure on  $N.P_{\infty-}$  in our framework [4, Theorem 1].

*Remark 2.3.* Let us motivate (2.19). Let us consider the Hilbert space  $\mathbb{H}_2$  of  $L^2$  maps from  $[0, 1]$  into  $\mathbb{R}^m$ . The  $c_{i,j}^k$  define a structure of Lie algebra on  $\mathbb{R}^{m*}$ , and therefore, on  $\mathbb{H}_2$ . Let us consider two functionals  $F^1$  and  $F^2$  Frechet smooth on  $\mathbb{H}_2^*$ . Their derivatives are given by kernels

$$\langle \nabla F^i, h \rangle = \int_0^1 \langle \nabla F^i(s), h(s) \rangle ds. \quad (2.20)$$

The Lie bracket  $[\nabla F^1, \nabla F^2]$  is given by  $[\nabla F^1(s), \nabla F^2(s)]$  and the classical Lie-Poisson structure is given by

$$\{F^1, F^2\} = \int_0^1 \langle h(s), [\nabla F^1(s), \nabla F^2(s)] \rangle ds. \quad (2.21)$$

These considerations are heuristic because the product of two elements of  $L^2$  is not an element of  $L^2$ . If we replace  $dB(s)$  by  $h(s)ds$  and if we consider the white-noise measure on  $\mathbb{H}_2$   $1/Z \exp[-\|h\|_{\mathbb{H}_2}^2] dD(h)$  instead of the Brownian measure (2.2), then this heuristic formula gives the formula (2.19). This is relevant of the so-called Malliavin transfer principle: a formula becomes almost surely true through the theory of Stratonovitch integrals.

### 3. The Stochastic Lie Algebroid

Smooth vector fields in the Nualart-Pardoux sense on the Wiener space are functions  $\beta_t$  from  $[0, 1]$  into  $\mathbb{R}^m$  such that

$$\{E[|\nabla^r \beta_t(s_1, \dots, s_r) - \nabla^r \beta_{t'}(s'_1, \dots, s'_r)|^p]\}^{1/p} \leq C_{r,p} \left( |t - t'|^{1/2} + \sum |s_i - s'_i|^{1/2} \right) \quad (3.1)$$

on the connected complements of  $[0, 1] \times [0, 1]^r$  where we have removed the diagonals and such that

$$\{E[|\nabla^r \beta_t(s_1, \dots, s_r)|^p]\}^{1/p} \leq C_{r,p}^1. \quad (3.2)$$

The infimums of  $C_{r,p}$  and of  $C_{r,p}^1$  in the previous formula are called the Nualart-Pardoux Sobolev norms of the vector field  $\beta_{(\cdot)}$ .

Smooth 1-forms in the Nualart-Pardoux sense on the Wiener space are functions  $\alpha_{(\cdot)}$  from  $[0, 1]$  into  $\mathbb{R}^{m*}$  such that

$$(E[|\nabla^r \alpha_t(s_1, \dots, s_r) - \nabla^r \alpha_{t'}(s'_1, \dots, s'_r)|^p])^{1/p} \leq C_{r,p} \left( |t - t'|^{1/2} + \sum |s_i - s'_i|^{1/2} \right) \quad (3.3)$$

on the connected complements of  $[0, 1] \times [0, 1]^r$  where we have removed the diagonals and such that

$$\{E[|\nabla^r \alpha_t(s_1, \dots, s_r)|^p]\}^{1/p} \leq C_{r,p}^1. \quad (3.4)$$

The infimums of  $C_{r,p}$  and of  $C_{r,p}^1$  in the previous formula are called the Sobolev norms of the 1-form  $\alpha_{(\cdot)}$ . The pairing between a 1-form  $\alpha_{(\cdot)}$  and a vector field  $\beta_{(\cdot)}$  is realized via the formula

$$\langle \alpha_{(\cdot)}, \beta_{(\cdot)} \rangle = \int_0^1 \langle \alpha_t, \beta_t \rangle dt. \quad (3.5)$$

If  $\alpha^1, \alpha^2$  are two smooth 1-forms in the Nualart-Pardoux sense on the Wiener space, then the bivector  $\pi$  associated to the stochastic Poisson structure is given by

$$\pi(\alpha^1, \alpha^2) = \sum_{i,j,k} \int_0^1 \alpha_{i,s}^1 \alpha_{j,s}^2 c_{i,j}^k dB_k(s). \quad (3.6)$$

The stochastic bivector  $\pi$  realizes a continuous bilinear map on the space of smooth 1-forms smooth into the space of smooth functionals.

A generalized vector field according to our theory [32, 35] is a random application from  $[0, 1]$  into  $\mathbb{R}^m$   $\beta_{(\cdot)}^g$  of the form

$$\beta_t^g = \sum_{i,j} \int_0^t \beta_{(i,j,s)} dB_j(s) e_i + \int_0^t \beta_s ds, \quad (3.7)$$

where  $\beta_{(i,j,\cdot)}$  and  $\beta_{(\cdot)}$  are smooth in the Nualart-Pardoux sense. The Nualart-Pardoux Sobolev norms of a generalized vector field  $\beta_{(\cdot)}^g$  are the collection of Nualart-Pardoux norms of  $\beta_{(i,j,\cdot)}$  and  $\beta_{(\cdot)}$ .

We can define a pairing between smooth 1-form and generalized vector fields by using the formula

$$\langle \alpha, \beta^g \rangle = \sum_{i,j} \int_0^1 \alpha_{(i,s)} \beta_{(i,j,s)} dB_j(s) + \int_0^1 \langle \beta_s, \alpha_s \rangle ds. \quad (3.8)$$

This allows us to define  $\tilde{\pi} \alpha = i_\alpha \pi$  for a smooth 1-form in the Nualart-Pardoux sense as the generalized vector field

$$\tilde{\pi} \alpha_t = \sum_{i,j,k} \int_0^t \alpha_i(s) c_{i,j}^k dB_k(s) e_j. \quad (3.9)$$

This allows us to put the following definition.

*Definition 3.1.* If  $\alpha$  and  $\beta$  are smooth 1-forms in the Nualart-Pardoux sense, then we define

$$[\alpha, \beta] = L_{\tilde{\pi}\alpha}\beta - L_{\tilde{\pi}\beta}\alpha - d\pi(\alpha, \beta), \quad (3.10)$$

where the Lie derivative is defined as usual by the formula

$$L_{\tilde{\alpha}}\beta = i_{\tilde{\pi}\alpha}d\beta + di_{\tilde{\pi}\alpha}\beta, \quad (3.11)$$

and  $d$  is the exterior derivative.

Since  $\tilde{\pi}\alpha$  is a generalized vector field, the introduction of the Lie derivative leads to stochastic integral (we refer to [32, 33, 35] for similar constructions). Let us recall some results of [31, 32]. Let  $\alpha_{(t_1, t_2)}$  be a map from  $[0, 1] \times [0, 1]$  into  $\mathbb{R}^{m^*}$  or later into  $(\mathbb{R}^{m^*})^{\otimes 2}$  such that

$$\begin{aligned} & \left\{ E \left[ \left| \nabla^r \alpha_{(t_1, t_2)}(s_1, \dots, s_r) - \nabla^r \alpha_{(t'_1, t'_2)}(s'_1, \dots, s'_r) \right|^p \right] \right\}^{1/p} \\ & \leq C_{r,p} \left( |t_1 - t'_1|^{1/2} + |t_2 - t'_2| + \sum |s_i - s'_i|^{1/2} \right) \end{aligned} \quad (3.12)$$

on the connected complements of  $[0, 1] \times [0, 1] \times [0, 1]^r$  where we have removed the diagonals and such that

$$\left\{ E \left[ |\nabla^r \alpha_{t_1, t_2}(s_1, \dots, s_r)|^p \right] \right\}^{1/p} \leq C_{r,p}^1. \quad (3.13)$$

The infimums of  $C_{r,p}$  and of  $C_{r,p}^1$  in the previous formula are called the Nualart-Pardoux Sobolev norms of the  $\alpha_{(\cdot)}$ . In such case  $\int_0^1 \alpha_{(t_1, t_2)} dB(t_1)$  is still smooth in the Nualart-Pardoux sense, with  $t_1$  being included as well as  $\int_0^1 \alpha_{t_1, t_2} dB(t_1) dB(t_2)$ .

This allows us to show the following theorem.

**Theorem 3.2.**  $[\cdot, \cdot]$  is a continuous antisymmetric bilinear application acting on the space of smooth 1-forms in the Nualart-Pardoux sense with values in the set of smooth 1-forms in the Nualart-Pardoux sense.

*Proof of Theorem 3.2.* We remark that

$$\pi(\alpha^1, \alpha^2) = \sum_{i,j,k} \int_0^1 \alpha_i^1(s) \alpha_j^2(s) c_{i,j}^k dB_k(s), \quad (3.14)$$

which is smooth and its Sobolev norms can be estimated in terms of the Sobolev norms of  $\alpha^i$  by Theorem 2.2.

Moreover,

$$i_{\tilde{\pi}\alpha^1} \alpha^2 = \sum_{i,j,k} \int_0^1 \alpha_i^1(s) \alpha_j^2(s) c_{j,i}^k dB_k(s), \quad (3.15)$$

which is still smooth. Moreover,

$$d\alpha^2(s, t) = \sum_i \left( \nabla \left( \alpha_i^2 \right)_t (s) - \nabla \left( \alpha_i^2 \right)_s (t) \right) e_i. \quad (3.16)$$

Moreover,

$$i_{\bar{\pi}\alpha^1} d\alpha^2(t) = \sum_{i,j,k} \left( \int_0^1 \nabla \left( \alpha_i^2 \right)_t (s) \alpha_j^1(s) c_{j,i}^k dB_k(s) - \int_0^1 \nabla \left( \alpha_i^2 \right)_s (t) \alpha_j^1(s) c_{j,i}^k dB_k(s) \right), \quad (3.17)$$

which is smooth by the remark preceding the theorem.  $\square$

**Theorem 3.3.**  $[\cdot, \cdot]$  defines a Lie bracket.

Let  $0 < t_1 < \dots < t_r = 1$  be a dyadic subdivision of  $[0, 1]$ . We deduce a partition of  $[0, 1]^r$  in cubes  $I_{n,r}$  of volume  $V_n$ . If  $\beta$  is a function from  $[0, 1]^r$  into some linear space, then we put

$$\chi_n \beta(t_1, \dots, t_r) = V_n^{-1} \sum 1_{I_{n,r}}(t_1, \dots, t_r) \int_{I_{n,r}} \beta(s_1, \dots, s_r) ds_1 \cdots ds_r. \quad (3.18)$$

If we consider the polygonal approximation  $B^n$  of  $B$  and  $F_n$  being the  $\sigma$ -algebra associated to  $B^n$ , We denote by  $\Pi_n$  the operation of taking the conditional expectation of a functional  $f$  by  $F_n$ . The results of Léandre [31, Appendix], allow to state the proposition.

**Proposition 3.4.** *Let one have*

$$\nabla^r \Pi_n F = \Pi_n \chi_n \nabla^r F. \quad (3.19)$$

If  $\alpha(t_1)$  with values in  $\mathbb{R}^{m^*}$  is smooth in the Nualart-Pardoux sense,  $t_1$  being included, then the random ordinary integral  $\int_0^1 \langle \chi_n \Pi_n \alpha(t), dB_t^n \rangle$  tends in all the Sobolev spaces of the Malliavin Calculus to the anticipative Stratonovitch integral  $\int_0^1 \langle \alpha(t), dB_t \rangle$ . If  $\alpha(t_1, t_2)$  which takes its values in  $(\mathbb{R}^{m^*})^{\otimes 2}$  is smooth in the Nualart-Pardoux sense, then the double random ordinary integral  $\int_0^1 \langle \chi_n \Pi_n \alpha(t_1, t_2), dB_{t_1}^n, dB_{t_2}^n \rangle$  tends in all the Sobolev spaces of the Malliavin Calculus to the double anticipative Stratonovitch integral  $\int_0^1 \langle \alpha(t_1, t_2), dB_{t_1}, dB_{t_2} \rangle$ .

*Proof of Theorem 3.3.* Let us consider the finite dimensional Gaussian space  $B_t^n$ . A 1-form  $\alpha_t^n$  is piecewise constant as well as a vector field  $h_t$ . If  $F^n$  is a functional which depends on  $B^n$  only, then  $\nabla^r F^n$  is constant on each  $I_{n,r}$ . We put

$$\left\{ F^{1,n}, F^{2,n} \right\}_n = \sum_{i,j,k} \int_0^1 \nabla_i F_i^{1,n}(s) \nabla_j F_j^{2,n}(s) c_{i,j}^k dB_k^n(s). \quad (3.20)$$

This defines a Poisson structure on the finite dimensional Gaussian space.

We can define  $\pi^n$ ,  $[\cdot, \cdot]_n$ , and  $\tilde{\pi}^n$  according to the line of the introduction. To a 1-form smooth in the Nualart-Pardoux sense  $\alpha$  on the total Wiener space, we consider the 1-form  $\Pi^n \chi_n \alpha = \alpha^n$  on the finite dimensional Gaussian space. We get

$$\left[ [\alpha^{1,n}, \alpha^{2,n}]_n, \alpha^{3,n} \right]_n + \left[ [\alpha^{2,n}, \alpha^{3,n}]_n, \alpha^{1,n} \right]_n + \left[ [\alpha^{3,n}, \alpha^{1,n}]_n, \alpha^{2,n} \right]_n = 0, \quad (3.21)$$

by doing as in finite dimension. By the previous proposition  $[[\alpha^{1,n}, \alpha^{2,n}]_n, \alpha^{3,n}]_n$  tends in all this attained  $L^p$  to  $[[\alpha^1, \alpha^2], \alpha^3]$ . Therefore the result.

Let us give the scheme of the proof of this last result. When we write  $[[\alpha^{1,n}, \alpha^{2,n}]_n, \alpha^{3,n}]_n$ , there are a lot of terms which will appear. All these terms will tend separately to the corresponding term in  $[[\alpha^1, \alpha^2], \alpha^3]$ . Let us treat one of them, which will lead to double anticipative Stratonovitch integral. The other terms will be treated identically. For instance  $i_{\tilde{\pi}[\alpha^{1,n}, \alpha^{2,n}]} d\alpha^{3,n}$  will lead to double Stratonovitch integral which will tend in all the Sobolev spaces of the Malliavin Calculus to  $i_{\tilde{\pi}[\alpha^1, \alpha^2]} d\alpha^3$ . We can consider in these expressions the term  $i_{\tilde{\pi}^n[i_{\tilde{\pi}^n(\alpha^{1,n})} d\alpha^{2,n}]} d\alpha^{3,n}$  which will lead to a double stochastic integral and which will tend to  $i_{\tilde{\pi}^n[i_{\tilde{\pi}(\alpha^1)} d\alpha^2]} d\alpha^3$ . But there are two parts in  $i_{\tilde{\pi}(\alpha^1)} d\alpha^2$  and  $i_{\tilde{\pi}^n(\alpha^{1,n})} d\alpha^{2,n}$ . We will consider the parts  $\langle \nabla \alpha^2, \tilde{\pi}(\alpha^1) \rangle$  and  $\langle \nabla \alpha^{2,n}, \tilde{\pi}^n(\alpha^{1,n}) \rangle$  where we take the covariant derivative of the 1-form  $\alpha^2$  in the direction of the generalized vector field  $\tilde{\pi}(\alpha^1)$ . We will show that  $i_{\tilde{\pi}^n[\langle \nabla \alpha^{2,n}, \tilde{\pi}^n(\alpha^{1,n}) \rangle]} d\alpha^{3,n}$  tends to  $i_{\tilde{\pi}[\langle \nabla \alpha^2, \tilde{\pi}(\alpha^1) \rangle]} d\alpha^3$ . But in these expressions there are still two parts which can be treated similarly. We will show that the expression  $\langle \nabla \alpha^{3,n}, \tilde{\pi}^n[\langle \nabla \alpha^{2,n}, \tilde{\pi}^n(\alpha^{1,n}) \rangle] \rangle$  tends to  $\langle \nabla \alpha^3, \tilde{\pi}[\langle \nabla \alpha^2, \tilde{\pi}(\alpha^1) \rangle] \rangle$  (we consider the covariant derivative of  $\alpha^{3,n}$  in the direction of  $\tilde{\pi}^n[\langle \nabla \alpha^{2,n}, \tilde{\pi}^n(\alpha^{1,n}) \rangle]$ ).

But

$$\tilde{\pi}^n(\alpha^{1,n})_t = \sum_{i,j,k} \int_0^t \alpha_{s,i}^{1,n} c_{i,j}^k dB_k^n(s) e_j. \quad (3.22)$$

Therefore

$$\langle \nabla \alpha_t^{2,n}, \tilde{\pi}^n(\alpha^{1,n}) \rangle = \sum_{i,j,k} \int_0^1 \langle \nabla \alpha_t^{2,n}(s), \alpha_{s,i}^{1,n} c_{i,j}^k dB_k^n(s) e_j \rangle. \quad (3.23)$$

This implies that

$$\begin{aligned} & \tilde{\pi}^n \left[ \langle \nabla \alpha^{2,n}(\cdot), \tilde{\pi}^n(\alpha^{1,n}) \rangle \right]_t \\ &= \sum_{i,j,k,i',j',k'} \int_0^t \left\langle \int_0^1 \langle \nabla \alpha_{u,i'}^{2,n}(s), \alpha_{s,i}^{1,n} c_{i,j}^k dB_k^n(s) e_j \rangle, c_{i',j'}^{k'} dB_{k'}^n(u) e_{j'} \right\rangle, \end{aligned} \quad (3.24)$$

where  $\alpha_t^{2,n}(s) = \sum_{i'} \alpha_{t,i'}^{2,n} e_{i'}^*$  (we consider a 1-form in the  $t$  variable).

Therefore

$$\begin{aligned} & \langle \nabla \alpha^{3,n}, \tilde{\pi}^n [\langle \nabla \alpha^{2,n}, \tilde{\pi}^n (\alpha^{1,n}) \rangle] \rangle \\ &= \sum_{i,j,k,i',j',k'} \int_0^1 \left\langle \nabla \alpha_{(\cdot),j'}^{3,n}(u), \left\langle \int_0^1 \left\langle \nabla \alpha_{u,i'}^{2,n}(s), \alpha_{s,i}^{1,n} c_{i,j}^k dB_k^n(s) e_j \right\rangle, c_{i',j'}^{k'} dB_{k'}^n(u) e_{j'} \right\rangle \right\rangle. \end{aligned} \quad (3.25)$$

By Proposition 3.4, this tends in all the Sobolev spaces of the Malliavin Calculus to

$$\sum_{i,j,k,i',j',k'} \int_0^1 \left\langle \nabla \alpha_{(\cdot),j'}^3(u), \left\langle \int_0^1 \left\langle \nabla \alpha_{u,i'}^2(s), \alpha_{s,i}^1 c_{i,j}^k dB_k(s) e_j \right\rangle, c_{i',j'}^{k'} dB_{k'}(u) e_{j'} \right\rangle \right\rangle. \quad (3.26)$$

We recognize in this quantity  $\langle \nabla \alpha^3, \tilde{\pi} [\langle \nabla \alpha^2, \tilde{\pi}(\alpha^1) \rangle] \rangle$  where we take the covariant derivative of  $\alpha^3$  in the direction of the generalized vector field  $\tilde{\pi} [\langle \nabla \alpha^2, \tilde{\pi}(\alpha^1) \rangle]$  (we had taken the covariant derivative of  $\alpha^2$  in the direction of the generalized vector field  $\tilde{\pi}(\alpha^1)$ ).  $\square$

**Theorem 3.5.**  $\tilde{\pi}$  is an anchor map. This means that for all 1-form  $\alpha, \beta$  which are smooth in the Nualart-Pardoux sense all functional  $F$  are smooth in the Nualart-Pardoux sense; one has the relation:

$$[\alpha, F\beta] = F[\alpha, \beta] + \langle \nabla F, \tilde{\pi}(\alpha) \rangle \beta. \quad (3.27)$$

*Proof of Theorem 3.5.* We get by classical results in finite dimension

$$[\alpha^n, F^n \beta^n]_n = F^n [\alpha_n, \beta_n]_n + \langle \nabla F^n, \tilde{\pi}^n(\alpha^n) \rangle \beta^n. \quad (3.28)$$

By the results of Proposition 3.4, this tends when  $n \rightarrow \infty$  to the formula

$$[\alpha, F\beta] = F[\alpha, \beta] + \langle \nabla F, \tilde{\pi}(\alpha) \rangle \beta. \quad (3.29)$$

Therefore the result is attained.  $\square$

## 4. Conclusion

We can summarize that  $([\cdot, \cdot], \tilde{\pi})$  realizes a stochastic Lie algebroid acting on the space of smooth 1-forms in the Nualart-Pardoux sense on the Wiener space and functional smooth in the Nualart-Pardoux sense on the Wiener space.  $\tilde{\pi}$  takes its values in the space of generalized vector fields.

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