Hindawi Publishing Corporation
Advances in Mathematical Physics
Volume 2010, Article ID 501521, 11 pages
doi:10.1155/2010/501521

Research Article

Partial Bell-State Analysis with Parametric down Conversion in the Wigner Function Formalism

A. Casado,[1] S. Guerra,[2] and J. Plácido[3]

[1] *Departamento de Física Aplicada III, Escuela Superior de Ingenieros, Universidad de Sevilla, 41092 Sevilla, Spain*
[2] *Centro Asociado de la Universidad Nacional de Educación a Distancia de Las Palmas de Gran Canaria, 35004 Las Palmas de Gran Canaria, Spain*
[3] *Departamento de Física, Universidad de Las Palmas de Gran Canaria, 35017 Las Palmas de Gran Canaria, Spain*

Correspondence should be addressed to A. Casado, acasado@us.es

Received 2 October 2009; Accepted 11 October 2009

Academic Editor: Adan Cabello

We apply the Wigner function formalism to partial Bell-state analysis using polarization entanglement produced in parametric down conversion. Two-photon statistics at a beam-splitter are reproduced by a wave-like description with zeropoint fluctuations of the electromagnetic field. In particular, the fermionic behaviour of two photons in the singlet state is explained from the invariance on the correlation properties of two light beams going through a balanced beam-splitter. Moreover, we show that a Bell-state measurement introduces some fundamental noise at the idle channels of the analyzers. As a consequence, the consideration of more independent sets of vacuum modes entering the crystal appears as a need for a complete Bell-state analysis.

1. Introduction

The theory of parametric down conversion (PDC) in the Wigner formalism, along with the theory of detection, was treated in a series of papers [1–4]. The formalism was applied to experiments exhibiting relevant aspects of quantum mechanics, such as entanglement, nonlocality, and other nonclassical features of light. In contrast to the usual Hilbert space formulation, where the corpuscular nature of light is stressed, the Wigner formalism resembles classical optics. More specifically, this alternative approach takes into account the coupling between the zeropoint field (ZPF) and the laser beam entering the nonlinear crystal. Moreover, the propagation of the light fields through the different optical devices is completely classical. A formal bridge between classical nonlinear optics and the quantum theory within the Wigner approach involves two elements without classical counterpart, such as the

ZPF itself, entering into the crystal and the rest of optical devices, and the detection process, in which those vacuum fluctuations are substracted, giving rise to the typically quantum results.

The development of quantum information in the recent years, alongside with the important role of parametric down conversion for experimental schemes, has motivated the application of the Wigner approach to some relevant contexts, up to now almost exclusively linked to the Hilbert domain. Examples of these are quantum cryptography [5], dense coding [6], and teleportation [7, 8]. This research program seeks to apply an alternative explanation for that kind of phenomena, an explanation nevertheless consistent with quantum theory, and therefore with the usual Hilbert space formulation.

Recently, the Wigner formalism has been applied to experiments on quantum cryptography based on the Ekert's protocol [9], also including the presence of eavesdropping in the case of projective measurements. There, it was shown that the Heisenberg uncertainty principle, a key aspect in secure quantum cryptography, is related to the change of the correlation properties of light fields, after going through the optical devices in Alice's and Bob's setups. These correlation properties are affected by vacuum modes activated in the crystal, which give rise to quantum entanglement. Furthermore, the action of Eve introduces some noise, that also turns out to be fundamental to reproduce the quantum results [10].

Bell-state measurements constitute another key aspect in the field of quantum information, posing a relevant problem in quantum dense coding and teleportation schemes. In this context, entangled photon pairs produced in parametric down conversion have also been used in the last decades for experiments on partial Bell-state measurement [11], in which entanglement involves only one degree of freedom, and complete Bell-state measurement, in which hyperentanglement (entanglement between two or more degrees of freedom) takes part [12].

The paper is organized as follows. In Section 2, we introduce the general description of the four Bell-states within the Wigner framework. This description involves the manipulation of only one beam. In this same section, we also study two-photon statistics at a balanced beam-splitter. In Section 3 we study an experiment on partial Bell-state analysis [11]. Finally, in Section 4, we discuss the results, making a comparison with the Hilbert space description.

2. Two-Photon Statistics at a Beam-Splitter in the Wigner Approach

We will start this section by reviewing the basic concepts in the Hilbert framework [13]. The four Bell-states (entangled in polarization) produced in the process of PDC are

$$\begin{aligned} \left|\Psi^{\pm}\right\rangle &= \frac{1}{\sqrt{2}}[|H\rangle_1|V\rangle_2 \pm |V\rangle_1|H\rangle_2], \\ \left|\Phi^{\pm}\right\rangle &= \frac{1}{\sqrt{2}}[|H\rangle_1|H\rangle_2 \pm |V\rangle_1|V\rangle_2], \end{aligned} \tag{2.1}$$

where $H(V)$ represents linear horizontal (vertical) polarization. Let us suppose that the two beams are recombined at a balanced beam splitter (BS). If $|a\rangle$ and $|b\rangle$ represent the input modes of the BS, the possible spatial states are

$$\left|\psi_A\right\rangle = \frac{1}{\sqrt{2}}[|a\rangle_1|b\rangle_2 - |b\rangle_1|a\rangle_2], \qquad \left|\psi_S\right\rangle = \frac{1}{\sqrt{2}}[|a\rangle_1|b\rangle_2 + |b\rangle_1|a\rangle_2], \tag{2.2}$$

where $A(S)$ denotes antisymmetric (symmetric). Due to the fact that the particles carrying the information are photons, the total state must obey the bosonic symmetry, so that the total two photon states are $|\Psi^+\rangle|\psi_S\rangle$, $|\Psi^-\rangle|\psi_A\rangle$, $|\Phi^+\rangle|\psi_S\rangle$, $|\Phi^-\rangle|\psi_S\rangle$. Assuming that the BS does not influence the internal state (polarization), the two-photon state can only be changed in the spatial part, via the Hadamard transformation, as $\widehat{H}|a\rangle = (1/\sqrt{2})(|a\rangle + |b\rangle)$ and $\widehat{H}|b\rangle = (1/\sqrt{2})(|a\rangle - |b\rangle)$. Owing to $\widehat{H}|\psi_A\rangle = |\psi_A\rangle$, only in this case the two photons emerge at the different outputs from the BS. In the other three cases, the two photons emerge together in one of the two outputs ports [14].

Let us now go to the Wigner formalism. The Wigner transformation stablishes a correspondence between a field operator acting on a vector in the Hilbert space and a (complex) amplitude of the field. In the case of zeropoint field, these amplitudes follow a particular stochastic distribution, given by the Wigner function of the vacuum.

Quantum predictions corresponding to the state $|\Psi^+\rangle$ are reproduced in the Wigner framework by considering the following two correlated beams outgoing the crystal [2]:

$$\begin{aligned}
\mathbf{F}_1^{(+)}(\mathbf{r},t) &= F_s^{(+)}\left(\mathbf{r},t;\left\{\alpha_{\mathbf{k}_1,H};\alpha^*_{\mathbf{k}_2,V}\right\}\right)\mathbf{i} + F_p^{(+)}\left(\mathbf{r},t;\left\{\alpha_{\mathbf{k}_1,V};\alpha^*_{\mathbf{k}_2,H}\right\}\right)\mathbf{j}, \\
\mathbf{F}_2^{(+)}(\mathbf{r},t) &= F_q^{(+)}\left(\mathbf{r},t;\left\{\alpha_{\mathbf{k}_2,H};\alpha^*_{\mathbf{k}_1,V}\right\}\right)\mathbf{i}' + F_r^{(+)}\left(\mathbf{r},t;\left\{\alpha_{\mathbf{k}_2,V};\alpha^*_{\mathbf{k}_1,H}\right\}\right)\mathbf{j}',
\end{aligned} \tag{2.3}$$

where $\mathbf{i}$ and $\mathbf{i}'$ ($\mathbf{j}$ and $\mathbf{j}'$) are unit vectors representing horizontal (vertical) linear polarization at beams "1" and "2" and $\{\alpha_{\vec{k}_i,V};\alpha_{\vec{k}_i,H}\}$ $(i = 1,2)$ represent four sets of relevant zeropoint amplitudes entering the crystal. The four set of modes $\{\vec{k}_{i,\lambda}\}$ $(i = 1,2;\ \lambda \equiv H,V)$ are "activated" and coupled with the laser beam inside the nonlinear medium.

As said before, amplitudes $\{\alpha_{\vec{k},\lambda}\}$ follow a distribution given by the Wigner function of the vacuum field [1]:

$$W_{\mathrm{ZPF}}(\{\alpha\}) = \prod_{[\mathbf{k}],\lambda} \frac{2}{\pi} \mathrm{e}^{-2|\alpha_{\mathbf{k},\lambda}|^2}. \tag{2.4}$$

If $A(\mathbf{r},t;\{\alpha\})$ and $B(\mathbf{r}',t';\{\alpha\})$ are two complex amplitudes, the correlation between them is given by:

$$\langle AB\rangle = \int W_{\mathrm{ZPF}}(\{\alpha\})A(\mathbf{r},t;\{\alpha\})B(\mathbf{r}',t';\{\alpha\})d\{\alpha\}. \tag{2.5}$$

In expressions (2.3), the only nonvanishing correlations are those involving the combinations $r \leftrightarrow s$ and $p \leftrightarrow q$. These correlations are directly related to the way in which the vacuum components are distributed inside the total field amplitudes.

The four Bell-states can be generated by manipulating only one beam, and this is related to the possibility of sending two bits of classical information via the manipulation of only one particle [6]. In the Wigner framework, the effect of a linear optical device on a beam accounts for a change on the distribution of zeropoint amplitudes inside the field components. Therefore, correlation properties are also changed. In [10] we performed the same analysis on the four Bell-states, this time considering a modification of the two beams, initially departing from the description of $|\Psi^+\rangle$. In this paper, in order to keep consistency with the essence of dense coding, we will consider that the optical devices modify only one of the beams, while the other beam remains unchanged from its generation at Bob's station.

Let us now focus on the experimental setup in Figure 1. The transformations at Bob's station are performed by a polarization rotator and a wave retarder. For instance, if we place a polarization rotator acting on beam "1", the plane of polarization of $\mathbf{F}_1^{(+)}$ will be rotated by an angle β. We can compute the field components behind the rotator in the following way:

$$\begin{aligned}\mathbf{F}'^{(+)}_1(\mathbf{r},t) &= \begin{pmatrix}\cos\beta & -\sin\beta\\ \sin\beta & \cos\beta\end{pmatrix}\begin{pmatrix}F_s^{(+)}(\mathbf{r},t)\\ F_p^{(+)}(\mathbf{r},t)\end{pmatrix}\\ &= \begin{pmatrix}F_s^{(+)}(\mathbf{r},t)\cos\beta - F_p^{(+)}(\mathbf{r},t)\sin\beta\\ F_s^{(+)}(\mathbf{r},t)\sin\beta + F_p^{(+)}(\mathbf{r},t)\cos\beta\end{pmatrix}.\end{aligned} \tag{2.6}$$

Now, let the wave retarder introduce a phase shift κ between the horizontal and vertical field components of beam "1". Taking into account the action of both optical devices, the expressions of the beams are

$$\begin{aligned}\mathbf{F}''^{(+)}_1(\mathbf{r},t) &= \left[F_s^{(+)}(\mathbf{r},t)\cos\beta - F_p^{(+)}(\mathbf{r},t)\sin\beta\right]\mathbf{i}\\ &\quad + e^{i\kappa}\left[F_s^{(+)}(\mathbf{r},t)\sin\beta + F_p^{(+)}(\mathbf{r},t)\cos\beta\right]\mathbf{j},\\ \mathbf{F}_2^{(+)} &= F_q^{(+)}(\mathbf{r},t)\mathbf{i}' + F_r^{(+)}(\mathbf{r},t)\mathbf{j}'.\end{aligned} \tag{2.7}$$

The combination $\beta = 0$, $\kappa = 0$ gives (2.3), corresponding to the state $|\Psi^+\rangle$. On the other hand, for $\beta = 0$ and $\kappa = \pi$ we obtain the description of $|\Psi^-\rangle$:

$$\mathbf{F}_1^{(+)}(\mathbf{r},t) = F_s^{(+)}(\mathbf{r},t)\mathbf{i} - F_p^{(+)}(\mathbf{r},t)\mathbf{j}, \qquad \mathbf{F}_2^{(+)}(\mathbf{r},t) = F_q^{(+)}(\mathbf{r},t)\mathbf{i}' + F_r^{(+)}(\mathbf{r},t)\mathbf{j}'. \tag{2.8}$$

In both cases the horizontal component of one beam is correlated with the vertical component of the other, the only difference being the minus sign that appears in $\mathbf{F}''^{(+)}_1$ in the case of $|\Psi^-\rangle$.

Finally, the case $\beta = -\pi/2$ and $\kappa = \pi$ corresponds to the description of $|\Phi^+\rangle$

$$\mathbf{F}_1^{(+)}(\mathbf{r},t) = F_p^{(+)}(\mathbf{r},t)\mathbf{i} + F_s^{(+)}(\mathbf{r},t)\mathbf{j}, \qquad \mathbf{F}_2^{(+)}(\mathbf{r},t) = F_q^{(+)}(\mathbf{r},t)\mathbf{i}' + F_r^{(+)}(\mathbf{r},t)\mathbf{j}', \tag{2.9}$$

and $\beta = -\pi/2$, $\kappa = 0$ corresponds to the description of $|\Phi^-\rangle$:

$$\mathbf{F}_1^{(+)}(\mathbf{r},t) = F_p^{(+)}(\mathbf{r},t)\mathbf{i} - F_s^{(+)}(\mathbf{r},t)\mathbf{j}, \qquad \mathbf{F}_2^{(+)}(\mathbf{r},t) = F_q^{(+)}(\mathbf{r},t)\mathbf{i}' + F_r^{(+)}(\mathbf{r},t)\mathbf{j}'. \tag{2.10}$$

In these two cases we observe that the horizontal (vertical) component of one beam is correlated with the horizontal (vertical) component of the other, the difference being the minus sign that appears in $\mathbf{F}_1^{(+)}$ in the case of $|\Phi^-\rangle$.

This description of the four Bell-states is equivalent to the one in [10]. Nevertheless, as we already pointed out, in this case we have modified only one of the two beams. The net effect of the polarization rotator and the wave retarder is similar to the one of a half-wave plate and a quarter-wave plate, used in [11].

The general expressions (2.7), where the values of β and κ are undetermined, correspond, in the Hilbert space, to a superposition of the base states $|\Psi^+\rangle$, $|\Psi^-\rangle$, $|\Phi^+\rangle$, and $|\Phi^-\rangle$.

We will now study the action of a balanced beam-splitter on the correlation properties of the light beams. For the sake of clarity, we suppose an identical distance separating the source from the BS's, so the contribution of the phase shift in [2, equation (16)] can be ignored.

Because there is one beam at each input port, it is not necessary to consider the vacuum field at the beam-splitter [15]. This time the BS does not introduce any additional noise to the one provided by the zeropoint field entering the crystal. The beams are represented by (2.7), $\mathbf{r} = \mathbf{r}_{\mathrm{BS}}$, with $\mathbf{r}_{\mathrm{BS}}$ being the position of the beam-splitter where the two beams are recombined.

For the light beams at the outgoing channels we have

$$
\begin{aligned}
\mathbf{F}_{1,\mathrm{out}}^{(+)}(\mathbf{r}_{\mathrm{BS}},t) &= F_{1H}^{(+)}(\mathbf{r}_{\mathrm{BS}},t)\mathbf{i} + F_{1V}^{(+)}(\mathbf{r}_{\mathrm{BS}},t)\mathbf{j}, \\
\mathbf{F}_{2,\mathrm{out}}^{(+)}(\mathbf{r}_{\mathrm{BS}},t) &= F_{2H}^{(+)}(\mathbf{r}_{\mathrm{BS}},t)\mathbf{i}' + F_{2V}^{(+)}(\mathbf{r}_{\mathrm{BS}},t)\mathbf{j}',
\end{aligned}
\tag{2.11}
$$

where

$$
F_{1H}^{(+)}(\mathbf{r}_{\mathrm{BS}},t) = \frac{1}{\sqrt{2}}\left[iF_q^{(+)}(\mathbf{r}_{\mathrm{BS}},t) + F_s^{(+)}(\mathbf{r}_{\mathrm{BS}},t)\cos\beta - F_p^{(+)}(\mathbf{r}_{\mathrm{BS}},t)\sin\beta\right], \tag{2.12}
$$

$$
F_{1V}^{(+)}(\mathbf{r}_{\mathrm{BS}},t) = \frac{1}{\sqrt{2}}\left[iF_r^{(+)}(\mathbf{r}_{\mathrm{BS}},t) + \mathrm{e}^{i\kappa}\left[F_s^{(+)}(\mathbf{r}_{\mathrm{BS}},t)\sin\beta + F_p^{(+)}(\mathbf{r}_{\mathrm{BS}},t)\cos\beta\right]\right], \tag{2.13}
$$

$$
F_{2H}^{(+)}(\mathbf{r}_{\mathrm{BS}},t) = \frac{1}{\sqrt{2}}\left[F_q^{(+)}(\mathbf{r}_{\mathrm{BS}},t) + iF_s^{(+)}(\mathbf{r}_{\mathrm{BS}},t)\cos\beta - iF_p^{(+)}(\mathbf{r}_{\mathrm{BS}},t)\sin\beta\right], \tag{2.14}
$$

$$
F_{2V}^{(+)}(\mathbf{r}_{\mathrm{BS}},t) = \frac{1}{\sqrt{2}}\left[F_r^{(+)}(\mathbf{r}_{\mathrm{BS}},t) + ie^{i\kappa}\left[F_s^{(+)}(\mathbf{r}_{\mathrm{BS}},t)\sin\beta + F_p^{(+)}(\mathbf{r}_{\mathrm{BS}},t)\cos\beta\right]\right]. \tag{2.15}
$$

We now calculate the cross-correlations between the components of $\mathbf{F}_{1,\mathrm{out}}^{(+)}(\mathbf{r}_{\mathrm{BS}},t)$ and those of $\mathbf{F}_{2,\mathrm{out}}^{(+)}(\mathbf{r}_{\mathrm{BS}},t)$.

(1) The correlation between the field components corresponding to the same polarization at the outgoing channels vanishes, independently of the values of κ and β. This

is due to the fact that the contribution to the correlation of the transmitted components of the field is cancelled by the contribution of the reflected components:

$$\left\langle F_{1H}^{(+)}(\mathbf{r}_{\mathrm{BS}},t)F_{2H}^{(+)}(\mathbf{r}_{\mathrm{BS}},t)\right\rangle = -\frac{1}{2}\sin\beta\left\langle F_{p}^{(+)}(\mathbf{r}_{\mathrm{BS}},t)F_{q}^{(+)}(\mathbf{r}_{\mathrm{BS}},t)\right\rangle\left[1+i^{2}\right] = 0, \tag{2.16}$$

$$\left\langle F_{1V}^{(+)}(\mathbf{r}_{\mathrm{BS}},t)F_{2V}^{(+)}(\mathbf{r}_{\mathrm{BS}},t)\right\rangle = \frac{1}{2}e^{i\kappa}\sin\beta\left\langle F_{r}^{(+)}(\mathbf{r}_{\mathrm{BS}},t)F_{s}^{(+)}(\mathbf{r}_{\mathrm{BS}},t)\right\rangle\left[1+i^{2}\right] = 0. \tag{2.17}$$

(2) For the correlation between the field components corresponding to different polarization and different outgoing channels, $\langle F_{2H}^{(+)}(\mathbf{r}_{\mathrm{BS}},t)F_{1V}^{(+)}(\mathbf{r}_{\mathrm{BS}},t)\rangle$ and $\langle F_{2V}^{(+)}(\mathbf{r}_{\mathrm{BS}},t)F_{1H}^{(+)}(\mathbf{r}_{\mathrm{BS}},t)\rangle$, we have

$$\begin{aligned}\left\langle F_{1H}^{(+)}(\mathbf{r}_{\mathrm{BS}},t)F_{2V}^{(+)}(\mathbf{r}_{\mathrm{BS}},t)\right\rangle &= -\left\langle F_{1V}^{(+)}(\mathbf{r}_{\mathrm{BS}},t)F_{2H}^{(+)}(\mathbf{r}_{\mathrm{BS}},t)\right\rangle \\ &= \frac{\cos\beta}{2}\left(\left\langle F_{s}^{(+)}(\mathbf{r}_{\mathrm{BS}},t)F_{r}^{(+)}(\mathbf{r}_{\mathrm{BS}},t)\right\rangle + i^{2}e^{i\kappa}\left\langle F_{q}^{(+)}(\mathbf{r}_{\mathrm{BS}},t)F_{p}^{(+)}(\mathbf{r}_{\mathrm{BS}},t)\right\rangle\right).\end{aligned} \tag{2.18}$$

As there is no path difference before the beam-splitter between $\mathbf{F}_{1,\mathrm{out}}^{(+)}$ and $\mathbf{F}_{2,\mathrm{out}}^{(+)}$, the cross correlations $\langle F_{s}^{(+)}(\mathbf{r},t)F_{r}^{(+)}(\mathbf{r}',t')\rangle$ and $\langle F_{q}^{(+)}(\mathbf{r}',t')F_{p}^{(+)}(\mathbf{r},t)\rangle$, computed at the same position $\mathbf{r}_{\mathrm{BS}}$ and time t, have the same value. With all this, we have

$$\begin{aligned}\left\langle F_{1H}^{(+)}(\mathbf{r}_{\mathrm{BS}},t)F_{2V}^{(+)}(\mathbf{r}_{\mathrm{BS}},t)\right\rangle &= -\left\langle F_{1V}^{(+)}(\mathbf{r}_{\mathrm{BS}},t)F_{2H}^{(+)}(\mathbf{r}_{\mathrm{BS}},t)\right\rangle \\ &= \frac{\cos\beta}{2}\left\langle F_{s}^{(+)}(\mathbf{r}_{\mathrm{BS}},t)F_{r}^{(+)}(\mathbf{r}_{\mathrm{BS}},t)\right\rangle\left(1+i^{2}e^{i\kappa}\right).\end{aligned} \tag{2.19}$$

We can see that if $\beta = \pi/2$, that is, the states $|\Phi^{\pm}\rangle$, the latter correlations vanish. On the other hand, in the case of $|\Psi^{+}\rangle$ ($\kappa = 0$ and $\beta = 0$) these correlations are also null. Finally, only when $\beta = 0$ and $\kappa = \pi$, that is, the state $|\Psi^{-}\rangle$, these correlations are different from zero:

$$\begin{aligned}\left\langle F_{1H}^{(+)}(\mathbf{r}_{\mathrm{BS}},t)F_{2V}^{(+)}(\mathbf{r}_{\mathrm{BS}},t)\right\rangle &= -\left\langle F_{1V}^{(+)}(\mathbf{r}_{\mathrm{BS}},t)F_{2H}^{(+)}(\mathbf{r}_{\mathrm{BS}},t)\right\rangle \\ &= \left\langle F_{s}^{(+)}(\mathbf{r}_{\mathrm{BS}},t)F_{r}^{(+)}(\mathbf{r}_{\mathrm{BS}},t)\right\rangle.\end{aligned} \tag{2.20}$$

(3) To conclude, we compute the correlations between the two field components of different polarization, corresponding to the same outgoing beam,

$$\begin{aligned}\left\langle F_{1H}^{(+)}(\mathbf{r}_{\mathrm{BS}},t)F_{1V}^{(+)}(\mathbf{r}_{\mathrm{BS}},t)\right\rangle &= \left\langle F_{2H}^{(+)}(\mathbf{r}_{\mathrm{BS}},t)F_{2V}^{(+)}(\mathbf{r}_{\mathrm{BS}},t)\right\rangle \\ &= \frac{i\cos\beta}{2}\left(\left\langle F_{s}^{(+)}(\mathbf{r}_{\mathrm{BS}},t)F_{r}^{(+)}(\mathbf{r}_{\mathrm{BS}},t)\right\rangle + e^{i\kappa}\left\langle F_{q}^{(+)}(\mathbf{r}_{\mathrm{BS}},t)F_{p}^{(+)}(\mathbf{r}_{\mathrm{BS}},t)\right\rangle\right) \\ &= \frac{i\cos\beta}{2}\left\langle F_{s}^{(+)}(\mathbf{r}_{\mathrm{BS}},t)F_{r}^{(+)}(\mathbf{r}_{\mathrm{BS}},t)\right\rangle\left(1+e^{i\kappa}\right).\end{aligned} \tag{2.21}$$

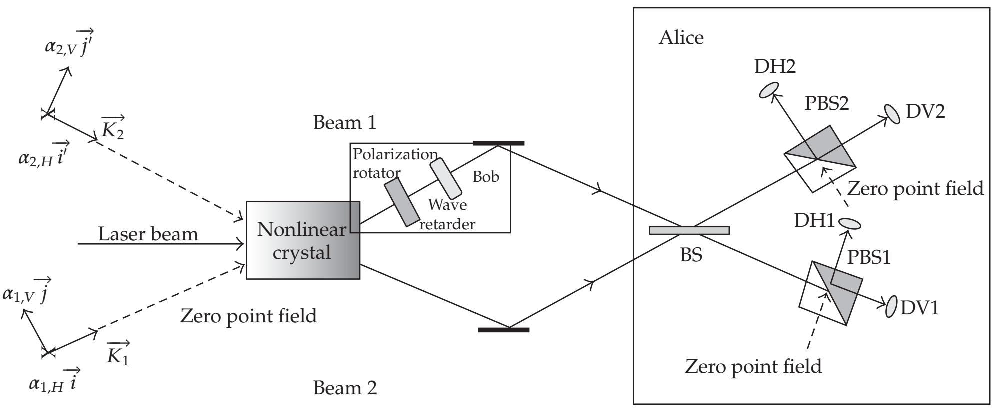

Figure 1: Setup for quantum dense coding. The polarization rotator and the wave retarder used by Bob allow a transmission to Alice of any of the Bell-states. Alice's station consists on a balanced beam-splitter, two polaryzing beam-splitters, and detectors DH1, DV1, DH2, and DV2.

It can be easily seen that the correlations above are different from zero only for the state $|\Psi^{+}\rangle$ ($\kappa = 0$ and $\beta = 0$). In this situation, we have

$$\begin{aligned}\left\langle F_{1H}^{(+)}(\mathbf{r}_{\mathrm{BS}}, t) F_{1V}^{(+)}(\mathbf{r}_{\mathrm{BS}}, t)\right\rangle &= \left\langle F_{2H}^{(+)}(\mathbf{r}_{\mathrm{BS}}, t) F_{2V}^{(+)}(\mathbf{r}_{\mathrm{BS}}, t)\right\rangle \\ &= i\left\langle F_{s}^{(+)}(\mathbf{r}_{\mathrm{BS}}, t) F_{r}^{(+)}(\mathbf{r}_{\mathrm{BS}}, t)\right\rangle.\end{aligned} \tag{2.22}$$

3. Partial Bell-State Analysis in the Wigner Approach

Let us again consider the situation in Figure 1. The nonlinear medium is an element of a quantum-optical experimental scheme for dense coding [11]. The two beams (2.3) are correlated through the zeropoint field entering the crystal, which is "amplified" via the activation of the four set of vacuum modes $\{\overrightarrow{k}_{i,\lambda}\}$ ($i = 1, 2$; $\lambda \equiv H, V$). The beam "1" can be modified at Bob's station, who can activate the polarization rotator and/or the wave retarder. Manipulation of beam "1" allows for the possibility of distributing the vacuum amplitudes in four different ways, so that the correlation properties of beams "1" and "2" can be modified and used for information encoding. In our framework, the possibility of sending two bits of classical information via the manipulation of one particle is explained through the change on the correlation properties of two beams when one of them is modified at Bob's station. Such correlations have their origin in the crystal, where the zeropoint modes are coupled with the laser field, and the information is carried by the amplified vacuum fluctuations [16, 17].

The two beams are recombined at Alice's Bell-state analyser by means of a balanced beam-splitter (BS), and the horizontal and vertical polarization components of each outgoing beams are separated at the polarizing beam-splitters PBS1 and PBS2. Finally, all coincidence detection probabilities can be measured with the detectors DH1, DH2, DV1, and DV2.

We focus now on the fields at the detectors. We will suppose that there is the same distance between the BS and any of the detectors, and again the phase factor corresponding to the propagation of the different amplitudes is irrelevant. Owing to the fact that each polaryzing beam-splitter reflects (transmits) the horizontal (vertical) polarization, the electric field at the detector (DH1, DV1, DH2, DV2) is the superposition of ($iF_{1H}^{(+)}$, $F_{1V}^{(+)}$, $iF_{2H}^{(+)}$, $F_{2V}^{(+)}$) and a vacuum field amplitude which has been (reflected, transmitted, reflected, transmitted) at the other idle channel of the corresponding PBS. These vacuum amplitudes have no correlation with the signals, nor they have with each other. If $\mathbf{F}_{\mathrm{ZPF,Alice1}}^{(+)}$ ($\mathbf{F}_{\mathrm{ZPF,Alice2}}^{(+)}$) is the vacuum field entering PBS1 (PBS2), for the field amplitudes at the detectors we have

$$
\begin{aligned}
\mathbf{F}_{\mathrm{DH1}}^{(+)}(\mathbf{r}_{\mathrm{DH1}},t_1) &= iF_{1H}^{(+)}(\mathbf{r}_{\mathrm{DH1}},t_1)\mathbf{i} + \left[\mathbf{F}_{\mathrm{ZPF,Alice1}}^{(+)}(\mathbf{r}_{\mathrm{DH1}},t_1)\cdot\mathbf{j}\right]\mathbf{j},\\
\mathbf{F}_{\mathrm{DV1}}^{(+)}(\mathbf{r}_{\mathrm{DV1}},t_1') &= F_{1V}^{(+)}(\mathbf{r}_{\mathrm{DV1}},t_1')\mathbf{j} + i\left[\mathbf{F}_{\mathrm{ZPF,Alice1}}^{(+)}(\mathbf{r}_{\mathrm{DV1}},t_1')\cdot\mathbf{i}\right]\mathbf{i},\\
\mathbf{F}_{\mathrm{DH2}}^{(+)}(\mathbf{r}_{\mathrm{DH2}},t_2) &= iF_{2H}^{(+)}(\mathbf{r}_{\mathrm{DH2}},t_2)\mathbf{i}' + \left[\mathbf{F}_{\mathrm{ZPF,Alice2}}^{(+)}(\mathbf{r}_{\mathrm{DH2}},t_2)\cdot\mathbf{j}'\right]\mathbf{j}',\\
\mathbf{F}_{\mathrm{DV2}}^{(+)}(\mathbf{r}_{\mathrm{DV2}},t_2') &= F_{2V}^{(+)}(\mathbf{r}_{\mathrm{DV2}},t_2')\mathbf{j}' + i\left[\mathbf{F}_{\mathrm{ZPF,Alice2}}^{(+)}(\mathbf{r}_{\mathrm{DV2}},t_2')\cdot\mathbf{i}'\right]\mathbf{i}',
\end{aligned}
\tag{3.1}
$$

where $F_{1H}^{(+)}$, $F_{1V}^{(+)}$, $F_{2H}^{(+)}$, $F_{2V}^{(+)}$ are given by (2.12), (2.13), (2.14), and (2.15), respectively, owing to the path length from the BS to each detector is the same, so that the related phase shifts in [2, equation (16)] can be discarded.

To calculate joint detection probabilities we use (see [2, equation (28)])

$$
P_{AB}(\mathbf{r},t;\mathbf{r}',t') \propto \sum_{\lambda}\sum_{\lambda'}\left|\left\langle F_{\lambda}^{(+)}(\phi_A;\mathbf{r},t)F_{\lambda'}^{(+)}(\phi_B;\mathbf{r}',t')\right\rangle\right|^2,
\tag{3.2}
$$

where λ and λ' are polarization indices, and ϕ_A and ϕ_B represent controllable parameters of the experimental setup.

For instance, let us show the calculation of $P_{\mathrm{DH1,DH2}}$. For simplicity we focus on the ideal situation with $t_1 = t_2$, and discard the dependence on position and time. We have

$$
\begin{aligned}
P_{\mathrm{DH1,DH2}} \propto{}& \left|\left\langle F_{1H}^{(+)}F_{2H}^{(+)}\right\rangle\right|^2 + \left|\left\langle F_{1H}^{(+)}\left[\mathbf{F}_{\mathrm{ZPF,Alice2}}^{(+)}\cdot\mathbf{j}'\right]\right\rangle\right|^2\\
&+\left|\left\langle\left[\mathbf{F}_{\mathrm{ZPF,Alice1}}^{(+)}\cdot\mathbf{j}\right]F_{2H}^{(+)}\right\rangle\right|^2 + \left|\left\langle\left[\mathbf{F}_{\mathrm{ZPF,Alice1}}^{(+)}\cdot\mathbf{j}\right]\left[\mathbf{F}_{\mathrm{ZPF,Alice2}}^{(+)}\cdot\mathbf{j}'\right]\right\rangle\right|^2 = \left|\left\langle F_{1H}^{(+)}F_{2H}^{(+)}\right\rangle\right|^2,
\end{aligned}
\tag{3.3}
$$

where we have taken into account that the ZPF inputs at the PBS's are uncorrelated with the signals and with each other. With (2.16), we finally obtain

$$P_{\mathrm{DH1,DH2}} = 0. \tag{3.4}$$

The rest of the probabilities can be obtained similarly. By using (2.17), (2.19), and (2.21), we find

$$\begin{aligned}
&P_{\mathrm{DV1,DV2}} = 0,\\
&\frac{P_{\mathrm{DH1,DV2}}}{k_{\mathrm{DH1,DV2}}} = \frac{P_{\mathrm{DV1,DH2}}}{k_{\mathrm{DV1,DH2}}} = \frac{\cos^2\beta}{2}[1+\cos(\kappa+\pi)]\left|\left\langle F_s^{(+)}F_r^{(+)}\right\rangle\right|^2,\\
&\frac{P_{\mathrm{DH1,DV1}}}{k_{\mathrm{DH1,DV1}}} = \frac{P_{\mathrm{DH2,DV2}}}{k_{\mathrm{DH2,DV2}}} = \frac{\cos^2\beta}{2}(1+\cos\kappa)\left|\left\langle F_s^{(+)}F_r^{(+)}\right\rangle\right|^2,
\end{aligned} \tag{3.5}$$

where $k_{\mathrm{DH1,DV2}}$, $k_{\mathrm{DV1,DH2}}$, $k_{\mathrm{DH1,DV1}}$, and $k_{\mathrm{DH2,DV2}}$ are constants related to the effective efficiency of the detection processes.

From (3.4) and (3.5) we conclude that

(i) recording a coincidence of DH1 and DH2 (DV1 and DV2) is not possible, for whatever κ and β,

(ii) recording a coincidence of DH1 and DV2 (DV1 and DH2) is possible only in the case ($\beta = 0$, $\kappa = \pi$), that is, the state $|\Psi^-\rangle$,

(iii) recording a coincidence of DH1 and DV1 (DH2 and DV2) is possible only for ($\beta = 0$, $\kappa = 0$), that is, the state $|\Psi^+\rangle$,

(iv) when $\beta = \pi/2$, that is, the states $|\Phi^\pm\rangle$, all coincidence probabilities vanish. Hence, these Bell-states cannot be distinguished.

4. Discussion

We have applied the Wigner approach to study two-photon statistics at a balanced beam splitter. We have also treated an experimental setup for partial Bell-state analysis. The Wigner formalism allows for an interpretation of these experiments in terms of waves, but, however, the whole formalism lies inside the quantum domain, the zeropoint field being an alternative to the role of vacuum fluctuations in the Hilbert space.

As we already pointed out, once in the Wigner framework, the typical quantum results appear precisely as a consequence of the introduction of the zeropoint field. This vacuum field enters in the crystal and also in the rest of optical devices. Finally, it is substracted in the detection process. Quantum correlations can be then explained solely in terms of the propagation of those vacuum amplitudes through the experimental setup, and their subsequent substraction at the detectors.

At the beginning of Section 2 we presented the fundamental ideas on two-photon statistics at a beam-splitter in the Hilbert space formalism, which account for the usual corpuscular description of light. In the Wigner framework, a clear counterpart to that description is found when, in order to preserve the bosonic character of the photons, the spatial part of the quantum state is forced to remain antisymmetric. This happens only for

$|\Psi^-\rangle$, and, in this case, the pair of photons behaves as fermions at the beam-splitter, emerging at different output ports. For the other three Bell-states, in which both the polarization and spatial parts of the two-photon state are symmetric, both particles emerge together at the same output port of the BS.

For the Wigner representation, the corpuscular aspect of light appears as just an interplay of (Maxwell) waves, including a zeropoint vacuum field. The action of the beam-splitter must be treated as in the classical framework: a part of each entering beam is transmitted, and the other part is reflected, without any change in their polarization properties. From (2.19), it can be seen that the correlation between the field components corresponding to different polarization and different outgoing channels vanishes for $\beta = \pi/2$ (states $|\Phi^\pm\rangle$), and also for $\kappa = 0$ (state $|\Psi^+\rangle$). In this last case, the factor i^2 indicates that the contribution to the correlation of the transmitted components is cancelled by the contribution of the reflected components. Nevertheless, in the case of $|\Psi^-\rangle$ there is a constructive superposition of the two terms in (2.19). In other words, the net effect of the phase shift $\kappa = \pi$ and the beam-splitter in the field amplitudes is to leave correlations unchanged. This is equivalent, in the Hilbert space, to the fact that $|\psi_A\rangle$ (see (2.2)) is an eigenstate of the Hadamard operator.

At this point it is worth to stop again at the question: how can we explain a typical particle behaviour (the bosonic nature of photons in the Hilbert space description), from a wave-like description, just with the inclusion of vacumm fluctuations of the electromagnetic field? By inspection of (2.7), it can easily be seen that, only in the case of $|\Psi^-\rangle$, the exchange $1 \leftrightarrow 2$ implies a sign flip in the correlation properties of the beams (they remain equal in the other three cases). The bosonic nature of photons is completely represented by the correlation properties that characterize each of the four Bell-states. If now Bob activates just the wave retarder, this gives rise to a change, not only in the internal (polarization) state, but also in the spatial part of the quantum state, in order to keep the requirement of boson symmetry. This "double" effect is explained in the Wigner formalism by taking into account the way in which the zeropoint field is coupled inside the crystal [2].

The Bell-state measurement performed at Alice's station only identifies the states $|\Psi^-\rangle$ and $|\Psi^+\rangle$. We complete our picture here by pointing out that the analyzer introduces some fundamental noise at the idle channels of the polaryzing beam splitters. These zeropoint fluctuations, although they will (again) be finally substracted at the detectors, and (again) hold no correlation with the signals, turn out to be the fundamental ingredient of the Wigner approach to this experimental setups. A quick look at Figure 1 shows that there are four relevant sets of vacuum modes entering the crystal, in which the quantum information is "carried," and another two sets of vacuum modes entering the PBS's, containing only zeropoint field. This image constrasts to the usual "qubit language," where "particles carry the information" [13].

Recently, the problem of performing complete Bell-state measurements has been solved by considering a higher number of degrees of freedom (hyperentanglement). Hyperentanglement is also a convenient resource for some other recent and important applications in the field of quantum computing. An example of these is one-way quantum computation using clusters states [18]. The use of Hilbert spaces of higher dimension is related, within the Wigner formalism, to the inclusion of more sets of vacuum modes entering the crystal. With increasing number of vacumm inputs, the possibility for extracting more information from the zeropoint field also increases. For instante, the momentum-polarization hyperentanglement needs, in the Wigner framework, the consideration of 8 sets of relevant modes of the zeropoint field entering the crystal [19].

Acknowledgments

The authors thank Professor E. Santos for helpful suggestions and comments on the work. They also thank R. Risco and D. Rodríguez for their comments and careful reading of the manuscript. A. Casado acknowledge the support from the Spanish MCI Project no. FIS2008-05596.

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