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Research Article

Justification of the NLS Approximation for the KdV Equation Using the Miura Transformation

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It is the purpose of this paper to give a simple proof of the fact that solutions of the KdV equation can be approximated via solutions of the NLS equation. The proof is based on an elimination of the quadratic terms of the KdV equation via the Miura transformation.

1. Introduction

The NLS equation describes slow modulations in time and space of an oscillating and advancing spatially localized wave packet. There exist various approximation results, cf. [1–4] showing that the NLS equation makes correct predictions of the behavior of the original system. Systems with quadratic nonlinearities and zero eigenvalues at the wave number $k = 0$ turn out to be rather difficult for the proof of such approximation results, cf. [5, 6]. The water wave problem falls into this class. Very recently, this long outstanding problem [7] has been solved [8] for the water wave problem in case of no surface tension and infinite depth by using special properties of this problem. Another equation which falls into this class is the KdV equation. The connection between the KdV and the NLS equation has been investigated already for a long time, cf. [9]. In [10, 11] the NLS equation has been derived as a modulation equation for the KdV equation, and its inverse scattering scheme has been related to the one of the KdV equation. It is the purpose of this paper to give a simple proof of the fact that solutions of the KdV equation can be approximated via solutions of the NLS equation. Beyond things this has been shown by numerical experiments in [12]. An analytical approximation result has been given by a rather complicated proof in [5] with a small correction explained in [6]. The much simpler proof of this fact presented here is based on an elimination of the quadratic terms of the KdV equation via the Miura transformation.

Following [13] the KdV equation

$$\partial_t u - 6u\partial_x u + \partial_x^3 u = 0 \tag{1.1}$$

can be transferred with the help of the Miura transformation

$$u = v^2 + \partial_x v \tag{1.2}$$

via direct substitution

$$\begin{aligned} &2v\partial_t v + \partial_x\partial_t v - 6\left(v^2 + \partial_x v\right)\left(2v\partial_x v + \partial_x^2 v\right) + 6(\partial_x v)\partial_x^2 v + 2v\partial_x^3 v + \partial_x^4 v \\ &\quad = (2v + \partial_x)\left(\partial_t v - 6v^2\partial_x v + \partial_x^3 v\right) = 0 \end{aligned} \tag{1.3}$$

into the mKdV equation

$$\partial_t v - 6v^2\partial_x v + \partial_x^3 v = 0. \tag{1.4}$$

In order to derive the NLS equation we make an ansatz

$$\varepsilon\psi_v(x,t) = \varepsilon A\left(\varepsilon(x-ct), \varepsilon^2 t\right)e^{i(kx-\omega t)} + c.c. \tag{1.5}$$

for the solutions $v = v(x,t)$ of (1.4), where $0 < \varepsilon \ll 1$ is a small perturbation paramater. Equating the coefficient at $\varepsilon e^{i(kx-\omega t)}$ to zero yields the linear dispersion relation $\omega = -k^3$. At $\varepsilon^2 e^{i(kx-\omega t)}$ we find the linear group velocity $c = -3k^2$ and at $\varepsilon^3 e^{i(kx-\omega t)}$ we find that the complex-valued amplitude A satisfies the NLS equation

$$\partial_2 A = -3ik\partial_1^2 A - 6ikA|A|^2. \tag{1.6}$$

2. Approximation of the mKdV Equation via the NLS Equation

Our first approximation result is as follows.

Theorem 2.1. *Fix $s \geq 2$ and let $A \in C([0,T_0], H^{s+3})$ be a solution of the NLS equation* (1.6). *Then there exist $\varepsilon_0 > 0$ and $C > 0$ such that for all $\varepsilon \in (0,\varepsilon_0)$ there are solutions of the mKdV equation* (1.4) *such that*

$$\sup_{t\in[0,T_0/\varepsilon^2]} \left\| v(\cdot,t) - \varepsilon\psi_v(\cdot,t)\right\|_{H^s} \leq C\varepsilon^{3/2}. \tag{2.1}$$

Proof. The error function R defined by $v(x,t) = \varepsilon\psi_v(x,t) + \varepsilon^{3/2}R(x,t)$ satisfies

$$\partial_t R + \partial_x^3 R - 6\varepsilon^2\partial_x\left(\psi_v^2 R\right) - 6\varepsilon^{5/2}\partial_x\left(\psi_v R^2\right) - 2\varepsilon^3\partial_x\left(R^3\right) + \varepsilon^{-3/2}\mathrm{Res}(\varepsilon\psi_v) = 0, \tag{2.2}$$

with

$$\mathrm{Res}\left(\varepsilon\psi_v\right) = \mathrm{E}\left(\varepsilon^4\partial_1^3 A - 6\varepsilon^4\partial_1\left(A|A|^2\right)\right) - \mathrm{E}^3\left(2\varepsilon^3(ik+\varepsilon\partial_1)\left(A^3\right)\right) + c.c., \tag{2.3}$$

where $\mathrm{E} = e^{i(kx-\omega t)}$. In order to eliminate the $\mathcal{O}(\varepsilon^3)$ terms we modify the previous ansatz (1.5) by adding

$$\varepsilon^3 \frac{6ik}{3i\omega - (3ik)^3}\left(A^3\right)\mathrm{E}^3 + c.c.. \tag{2.4}$$

After this modification the residual $\mathrm{Res}(\varepsilon\psi_v)$ is of formal order $\mathcal{O}(\varepsilon^4)$. When evaluated in H^s there is a loss of $\varepsilon^{-1/2}$ due to the scaling properties of the L^2-norm. Hence there exist $\varepsilon_0 > 0$ and $C_{\mathrm{res}} > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ we have

$$\sup_{t\in[0,T_0/\varepsilon^2]} \left\| \varepsilon^{-3/2}\mathrm{Res}\left(\varepsilon\psi_v(\cdot,t)\right)\right\|_{H^s} \le C_{\mathrm{res}}\varepsilon^2. \tag{2.5}$$

By partial integration we find for $s \ge 2$ and all $m \in \{0,\ldots,s\}$ that

$$\begin{aligned}\partial_t \int (\partial_x^m R(x,t))^2 dx \le{}& C_1\varepsilon^2 \|\psi_v(\cdot,t)\|^2_{C_b^{s+1}} \|R(\cdot,t)\|^2_{H^s} \\ &+ C_2\varepsilon^{5/2}\|\psi_v(\cdot,t)\|_{C_b^{s+1}}\|R(\cdot,t)\|^3_{H^s} \\ &+ C_3\varepsilon^3\|R(\cdot,t)\|^4_{H^s} + C_4\varepsilon^2\|R(\cdot,t)\|_{H^s},\end{aligned} \tag{2.6}$$

with ε-independent constants C_j. Hence using $a \le 1 + a^2$ shows that the energy $y(t) = \|R(\cdot,t)\|^2_{H^s}$ satisfies

$$\partial_t y(t) = C_5\varepsilon^2 y(t) + C_6\varepsilon^{5/2} y(t)^{3/2} + C_7\varepsilon^3 y(t)^2 + C_8\varepsilon^2. \tag{2.7}$$

Rescaling time $T = \varepsilon^2 t$ and using Gronwall's inequality immediately shows the $\mathcal{O}(1)$ boundedness of y for all $T \in [0, T_0]$, respectively all $t \in [0, T_0/\varepsilon^2]$. Therefore, we are done. □

3. Transfer to the KdV Equation

Applying the Miura transformation (1.2) to the approximation $\varepsilon\psi_v$ defines an approximation

$$\varepsilon\psi_u = \varepsilon^2\psi_v^2 + \varepsilon\partial_x\psi_v = \varepsilon ikA\left(\varepsilon(x-ct), \varepsilon^2 t\right)e^{i(kx-\omega t)} + c.c. + \mathcal{O}\left(\varepsilon^2\right) \tag{3.1}$$

of the solution u of the KdV equation (1.1). Since

$$\|u - \varepsilon\psi_u\|_{H^{s-1}} = \left\|v^2 + \partial_x v - \varepsilon^2\psi_v^2 - \varepsilon\partial_x\psi_v\right\|_{H^{s-1}} \le \|v - \varepsilon\psi_v\|_{H^s} + \mathcal{O}\left(\varepsilon^2\right) = \mathcal{O}\left(\varepsilon^{3/2}\right) \tag{3.2}$$

the approximation theorem in the original variables follows.

Theorem 3.1. *Fix $s \geq 1$ and let $A \in C([0, T_0], H^{s+4})$ be a solution of the NLS equation* (1.6)*. Then there exist $\varepsilon_0 > 0$ and $C > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ there are solutions of the KdV equation* (1.1) *such that*

$$\sup_{t \in [0, T_0/\varepsilon^2]} \left\| u(\cdot, t) - \varepsilon \psi_u(\cdot, t) \right\|_{H^s} \leq C \varepsilon^{3/2}. \tag{3.3}$$

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