Hindawi Publishing Corporation
Advances in Mathematical Physics
Volume 2012, Article ID 183503, 9 pages
doi:10.1155/2012/183503

Research Article

Combination Mode of Internal Waves Generated by Surface Wave Propagating over Two Muddy Sea Beds

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Received 6 March 2012; Accepted 9 May 2012

Academic Editor: Sanith Wijesinghe

When surface wave propagating over the two layer system usually induces internal wave in three different modes: they are external, internal and combination. In the present study, the nonlinear response of an initially flat sea bed, with two muddy sections, to a monochromatic surface progressive wave was investigated. From this theoretical result, it shows that a surface water wave progressing over two different muddy sections, the surface wave will excite two opposite-traveling short interfacial waves, forming a nearly standing wave at the interface of the fresh water and the muddy layer. Meanwhile, two opposite-outgoing "mud" waves each with very long wavelength will be simultaneously induced at the interface of two muddy sections. As a result, the amplitudes of the two short internal waves are found to grow exponentially in time. Furthermore, it will be much difficult to excite the internal waves when surface water wave progressing over two muddy sections with the large density gap.

1. Introduction

Generally, wave propagating over the two layer system usually induces internal wave in three different modes: they are external, internal, and combination, as shown in Figure 1. These internal waves dissipate rapidly with imaginary and real wave numbers of a similar magnitude and remove energy from the surface waves. Also, nonlinear damping mechanisms have been proposed based on wave-wave instabilities and interactions.

The present work is motivated by recent studies on the interaction between a progressive surface wave and the nearly standing subharmonic internal waves in a two-layer system. It is well known that the loading of progressive surface waves, a silty sediment

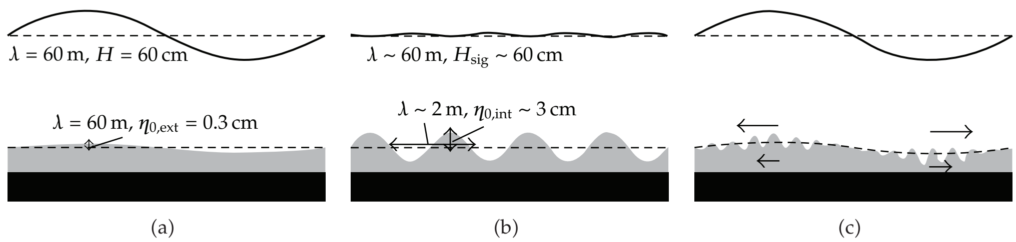

Figure 1: Schematic diagram of the two layer system showing (a) the external wave mode with the same frequency and wavelength for both the air-water and water-mud interface waves, (b) the internal wave mode with the same frequency, but much shorter wavelength for the water-mud interface waves, and (c) the combination of the two. The arrows in panel (c) indicate shear across the mud water interface, which could lead to a shear instability mechanism to generate the internal mode waves in each half cycle of the surface wave.

bed was repeatedly and extensively fluidized. The broad-based interest in understanding this phenomenon induces from the application to studies in sediment transport, wave attenuation, and the design of marine structures. The nonlinear response of an initially flat sea bed to a monochromatic surface progressive wave was studied by Wen [1] for using the multiple scale perturbation method. She found that two opposite-traveling subliminal internal "mud" waves are triggered and form a resonant triad with the surface wave. The resonant generation of internal waves on sediment bed was presented as a new mechanism of sediment suspension. Nonlinear wave interactions are considered to be as an important aspect of the dynamics of the oceans and the atmosphere. Of particular interest are resonant interactions, which are important in the redistribution of energy among wave modes with different spatial and temporal scales. Ball [2] used a second-order nonlinear resonance theory to analyze that linear growth of an internal wave could result from the interactions between two finite-amplitude surface waves. Watson et al. [3] extended the investigation by analyzing how a spectrum of surface waves could generate a corresponding spectrum of internal waves. Wen's [1] work was followed by Hill and Foda [4], who treated the problem in two dimensions for both an inviscid and a viscous lower layer. Hill and Foda [5] and Jamali [6] have presented theoretical and experimental studies of the resonant interaction between a surface wave and two oblique interfacial waves. Despite many similarities between the findings, there is one seemingly major difference. Hill and Foda's [5] analysis predicts only narrow bands of frequency, density ratio, and direction angle within which growth is possible. However, Jamali [6] predicted and observed wave growth over wide ranges of frequency and direction angle, and for all the density ratios that he investigated. Therefore, Jamali et al. [7] presented the study to investigate the contradictory results between the findings of Hill and Foda [5] and Jamali [6]. From their result, it is showed that the crucial difference between the two studies is in the dynamic interfacial boundary condition. The boundary condition used by Hill and Foda is missing a term proportional to the time derivative of the square of the velocity shear across the interface. When this missing term is included in the analysis, the theoretical predictions are consistent with the results of Jamali's [6] laboratory experiments. Both of the Hill and Foda's [5] and Jamali's [6] study found that the interfacial waves are short, have a frequency of nearly half that of the surface wave, and propagate in nearly opposite directions. The nonlinear response of an initially flat sea bed, with two muddy sections, to a monochromatic surface progressive wave was investigated in the present study. Based on an analysis similar to that of Hill and Foda's paper [5], the multiple-scale perturbation method was adopted, and the boundary value problem was

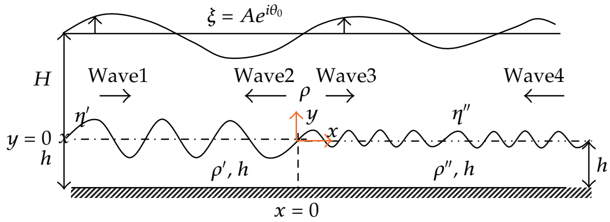

Figure 2: Configuration of the problem in the two-layer system with two muddy sections.

expanded in a power series of the surface-wave steepness. The linear harmonics and the conditions for resonance were obtained by the leading order, while the temporal evolution equations for the internal-wave amplitudes were investigated by a second-order analysis. It was found that result for equal density of two muddy sections is similar to that of Hill and Foda's paper [5]. Two opposite-traveling internal "mud" waves are selectively excited and form a resonant triad with the progressive surface wave. However, for a surface water wave progressing over two different muddy sections, the surface wave will also excite only two opposite-traveling short interfacial waves, forming a nearly standing wave at the interface of the fresh water and the muddy layer. Meanwhile, two opposite-outgoing "mud" waves each with very long wavelength will be simultaneously induced at the interface of two muddy sections. As a result, the amplitudes of the two short internal waves are found to grow exponentially in time.

2. Formulation

As shown in Figure 2, the origin of a two-dimensional Cartesian coordinate system is placed on the undisturbed interface between a surface layer of depth H and density ρ and a lower depth h at the interface between two different density sections, ρ and ρ''. The y-coordinate is defined as pointing vertically upward, and the density rations, $\gamma = \rho/\rho$ and $\gamma = \rho/\rho^{''}$, are assumed to be less than unity. To the leading order, the wave field is assumed to be made up of a linear progressive surface wave of amplitude A, wave number k, and frequency ω, propagating in the positive x-direction. Firstly, we assumed that the perturbation internal waves at left muddy section have amplitudes a_1 and a_2, wave numbers λ_1 and λ_2, frequencies σ_1 and σ_2, and propagate in the positive and negative x-directions, respectively. Then for the right muddy section, there are also the perturbation internal waves with amplitudes a_3 and a_4, wave numbers λ_3 and λ_4, frequencies σ_3 and σ_4, and move in the positive and negative x-directions, respectively. It is noted that a_1, a_2, a_3, a_4, and A are taken to be complex and $\lambda_1 \sim \lambda_4$ and $\sigma_1 \sim \sigma_4$ are all defined to have positive real values. For resonant interactions to occur in two muddy sections, the following resonance conditions are imposed on the four-handed wave numbers and frequencies:

$$\begin{aligned}
\lambda_1 - \lambda_2 - \lambda_4 &= k, \\
\lambda_1 + \lambda_3 - \lambda_4 &= k, \\
\sigma_1 - \sigma_2 - \sigma_4 &= \omega, \\
\sigma_1 + \sigma_3 - \sigma_4 &= \omega.
\end{aligned} \tag{2.1}$$

Expressing the flow-field in the two-layer inviscid system in terms of a velocity potential Φ, we assume that Φ satisfies Laplace's equation throughout the depth of the fluid

$$\nabla^2\Phi = 0, \qquad -h \leq y \leq H + \xi. \tag{2.2}$$

The displacement ξ of the free surface from its static elevation $y = H$ is given by

$$\xi = Ae^{i\theta_0} \tag{2.3}$$

in which the phase function $\theta_0 \equiv (kx - \omega t)$. The free surface will oscillate synchronously with ξ, with an amplitude A in response to the passage of the surface wave.

At the free surface, the usual kinematic and dynamic conditions are given by

$$\begin{gathered} \frac{D\xi}{Dt} = \Phi y, \qquad y = H + \xi, \\ \Phi_t + g\xi + \frac{1}{2}\nabla\Phi \cdot \nabla\Phi = 0, \qquad y = H + \xi. \end{gathered} \tag{2.4}$$

At the interface between the water and the slurry, there are similar conditions of continuity of pressure and vertical velocity. Therefore, on the left muddy section we have

$$\begin{gathered} \rho\left(\Phi_t^l + g\eta' + \frac{1}{2}\nabla\Phi^l \cdot \nabla\Phi^l\right)^+ = \rho'\left(\Phi_t^l + g\eta' + \frac{1}{2}\nabla\Phi^l \cdot \nabla\Phi^l\right)^-, \quad y = \eta', \\ \frac{D\eta'}{Dt} = \Phi_y^{l+} = \Phi_y^{l-}, \quad y = \eta'. \end{gathered} \tag{2.5}$$

Similar conditions of continuity of normal velocities and traction stresses are imposed at the disturbed interface on the right muddy section

$$\begin{gathered} \rho\left(\Phi_t^r + g\eta'' + \frac{1}{2}\nabla\Phi^r \cdot \nabla\Phi^r\right)^+ = \rho''\left(\Phi_t^l + g\eta'' + \frac{1}{2}\nabla\Phi^r \cdot \nabla\Phi^r\right)^-, \quad y = \eta'', \\ \frac{D\eta''}{Dt} = \Phi_y^{r+} = \Phi_y^{r-}, \quad y = \eta''. \end{gathered} \tag{2.6}$$

Meanwhile at the interface between two muddy mass of sections, conditions of continuity of pressure and continuity of water and mud are shown as followed

$$\begin{gathered} \left[\rho(\Phi_l)_x\right]_{y=\eta'}^{y=H+\xi} = \left[\rho(\Phi_r)_x\right]_{y=\eta''}^{y=H+\xi}, \quad \text{at } x = 0, \text{ (forwater)}, \\ \left[\rho'(\Phi_l)_x\right]_{y=0}^{y=\eta'} = \left[\rho''(\Phi_r)_x\right]_{y=0}^{y=\eta''}, \quad \text{at } x = 0, \text{ (formud)}. \end{gathered} \tag{2.7}$$

Note that the l and r superscripts denote the tiny interface between left muddy section and right muddy section, and + and − superscripts show evaluation just above and just below the interface between the water and the fluid-mud, respectively. And finally, at the bottom of mud layer, we have the no flux condition

$$\Phi_y = 0, \qquad y = -h. \tag{2.8}$$

With the system for the interface and free surface being assumed to be only weakly nonlinear interaction, Taylor's expansion at the interface and free surface is used to eliminate ξ, η', and η''. And by using the method of successive approximations, we have three free wave harmonics in the velocity potential Φ^l for left side ($x < 0$), that is we assume the following expansion for Φ^l:

$$\Phi^l = \varepsilon\varphi(y)e^{i(kx+\omega t)} + \varepsilon^2\left\{\psi(y)e^{i(\lambda_1 x+\sigma_1 t)} + \chi(y)e^{i(\lambda_2 x-\sigma_2 t)}\right\} + \Phi^l_{n.l.} + \text{c.c.} \tag{2.9}$$

The first three terms of the above expansion represent the three free harmonics; first term is the surface wave, and the following two terms are for the internal waves, $\Phi^l_{n.l.}$ represents the harmonics, and c.c. denotes complex conjugate. The expansion parameter is taken to be the steepness of the surface wave, $\varepsilon = kA$. Our analysis is restricted to the case of small internal waves, only so the internal wave harmonics appear at $O(\varepsilon^2)$ and not at $O(\varepsilon)$.

Similarly, the velocity potential Φ^r for fight side ($x > 0$) is expanded as follows:

$$\Phi^r = \varepsilon\varphi(y)e^{i(kx-\omega t)} + \varepsilon^2\left\{\psi(y)e^{i(\lambda_3 x-\sigma_3 t)} + \chi(y)e^{i(\lambda_4 x+\sigma_4 t)}\right\} + \Phi^r_{n.l.} + \text{c.c.} \tag{2.10}$$

3. Perturbation Solution

For the solution procedure, a standard perturbation analysis for a weakly nonlinear wave-field system is used. The solution procedure involves solving the above boundary value problem in an ordered sequence, by separating terms in the governing equations and the boundary conditions according to their order in ε and their phase. Substituting (2.9) and (2.10) in the governing equations and collecting the leading $O(\varepsilon)$ order, the approximate linear solutions for the interacting harmonics are obtained. For the dispersion relationship of surface wave,

$$\begin{aligned} &\omega^4\left\{(\gamma'\gamma'')^{1/2}\coth(kh)\coth(kH) + 1\right\} - \omega^2(\gamma'\gamma'')^{1/2} \\ &\quad\times\{\coth(kH) + \coth(kh)\}gk + \left\{(\gamma'\gamma'')^{1/2} - 1\right\}g^2k^2 = 0. \end{aligned} \tag{3.1}$$

For the dispersion relationship for internal waves of two muddy sections, the following is obtained

$$\begin{aligned}
\sigma_1^2 &= \frac{g\left\{1-(\gamma'\gamma'')^{1/2}\right\}}{(\gamma'\gamma'')^{1/2}+\coth(\lambda_1 h)}, \quad x<0,\\
\sigma_2^2 &= \frac{g\left\{1-(\gamma'\gamma'')^{1/2}\right\}}{(\gamma'\gamma'')^{1/2}+\coth|\lambda_2 h+\lambda_4 h|}, \quad x<0,\\
\sigma_3^2 &= \frac{g\left\{1-(\gamma'\gamma'')^{1/2}\right\}}{(\gamma'\gamma'')^{1/2}+\coth(\lambda_1 h+\lambda_3 h)}, \quad x>0,\\
\sigma_4^2 &= \frac{g\left\{1-(\gamma'\gamma'')^{1/2}\right\}}{(\gamma'\gamma'')^{1/2}+\coth|(\lambda_4 h)|}, \quad x>0.
\end{aligned} \tag{3.2}$$

Quadratic interactions between the above linear harmonics are analyzed at the second order, $O(\varepsilon^2)$. Since the homogeneous version of the boundary value problem had a nontrivial solution, the inhomogeneous problem has a solution only if the forcing terms are orthogonal to the homogeneous solution. Invoking solvability, through the use of Green's theorem, the desired evolution equations via the internal wave amplitudes are obtained. For simplicity, the amplitude equations—via solvability—may be straightforwardly shown as follows:

$$\begin{aligned}
\frac{da_l}{dt} &= i\alpha_l a_r A,\\
\frac{da_r}{dt} &= i\alpha_r a_l A^*,
\end{aligned} \tag{3.3}$$

in which $a_l = a_1 + a_2 + a_4$ and $a_r = a_1 + a_3 + a_4$. α_l and α_r are the interaction coefficients. Taking cross differentiation of (3.3), the growth for amplitudes of the internal waves is governed by exponentials:

$$\begin{aligned}
&a_l, a_r \propto \exp\{\pm\sqrt{-\alpha_l\alpha_r}At\},\\
\alpha &= \sqrt{-\alpha_l\alpha_r} = \frac{1}{2}|A|\left\{\frac{-g}{H}\left(1-2(\gamma'\gamma'')^{1/2}\frac{\sigma_2+\sigma_4}{\sigma_1+\sigma_3}+\frac{k}{\lambda_1+\lambda_3}\right)\right.\\
&\quad\left.\times\left(1-2(\gamma'\gamma'')^{1/2}\frac{\sigma_1+\sigma_3}{\sigma_2+\sigma_4}-\frac{k}{\lambda_2+\lambda_4}\right)(\lambda_1+\lambda_3)(\lambda_2+\lambda_4)\right\}^{1/2}.
\end{aligned} \tag{3.4}$$

Under the condition of α being purely imaginary, then the amplitudes a_l and a_r are not able to grow with time. Equivalently, the surface water wave will be stable to these internal wave perturbations. Only when α is real, the growth of the internal wave amplitudes will occur.

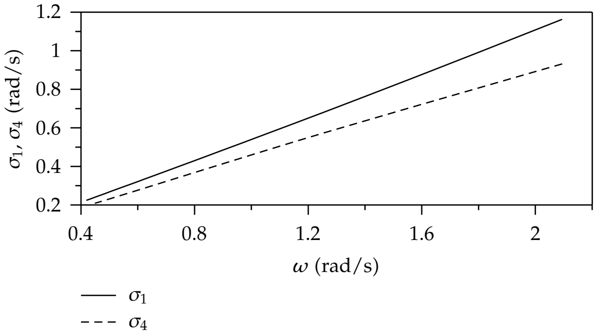

Figure 3: Two internal wave frequencies as functions of surface wave frequency ω, $H = 4\,\mathrm{m}$, $\gamma' = \gamma'' = 0.83$, and $h = 0.1\,\mathrm{m}$.

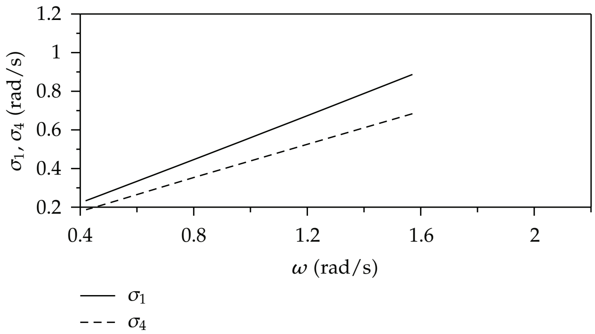

Figure 4: Two internal wave frequencies as functions of surface wave frequency ω, $H = 4\,\mathrm{m}$, $\gamma' = \gamma'' = 0.83$, and $h = 0.3\,\mathrm{m}$.

4. Results and Discussion

In an effort to develop a mechanism of the generation for internal waves of two muddy sections by surface wave, an inviscid second-order nonlinear resonant interaction has been analyzed. An example of the same density for two muddy sections, the internal-wave parameters as functions of ω for the cases of $\gamma' = \gamma'' = 0.83$, $H = 4\,\mathrm{m}$, $h = 0.1\,\mathrm{m}$ and $\gamma' = \gamma'' = 0.83$, $H = 4\,\mathrm{m}$, $h = 0.3\,\mathrm{m}$ is shown in Figures 3 and 4. The result of the present study is similar to that of Hill and Foda's paper [5]. It is obvious that there is a surface-wave frequency, a cut-off frequency, below which resonant triads do not exist. The cut-off frequency is a function of H, h, γ', and γ''. Furthermore, the critical frequency corresponds to these cases in which the internal waves in muddy layer are nearly subharmonic to the surface wave, that is, $\sigma_1 \approx \sigma_4 \approx \omega/2$, and are propagating in the same and opposite directions to the surface wave. Figure 5 shows the growth rate of internal waves as a function of the surface wave frequency for $\gamma'' = 0.83$, $H = 4\,\mathrm{m}$, and $h = 0.1\,\mathrm{m}$ and $0.3\,\mathrm{m}$. The growth rate of internal waves will be suppressed when the depth of mud layer increases. That is because if the mud layer depth being too thick, the interaction among surface and interfacial internal waves weakens. Internal waves are also subdued. However, for a surface water wave progressing over two different muddy sections ($\gamma' \neq \gamma''$), the surface wave will also excite only two opposite-traveling interfacial short waves, forming a nearly standing wave at the interface of the fresh water and the muddy layer. Meanwhile, two opposite-outgoing

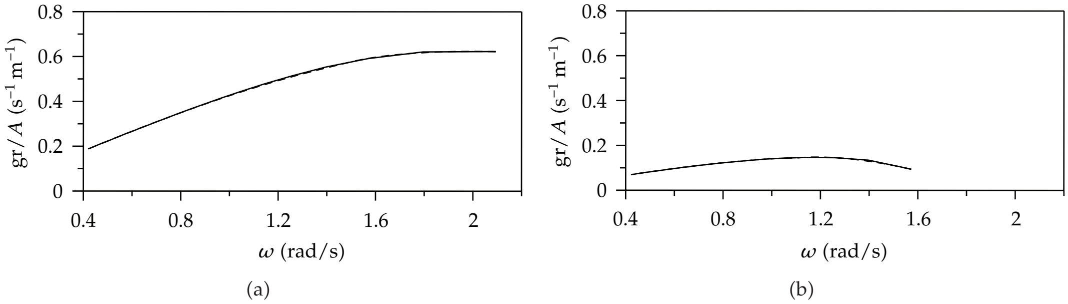

Figure 5: Theoretical determined internal wave growth rates as functions of surface wave frequency ω, $H = 4\,\text{m}$, $\gamma' = \gamma'' = 0.83$, (a) $h' = 0.1\,\text{m}$, and (b) $h = 0.3\,\text{m}$.

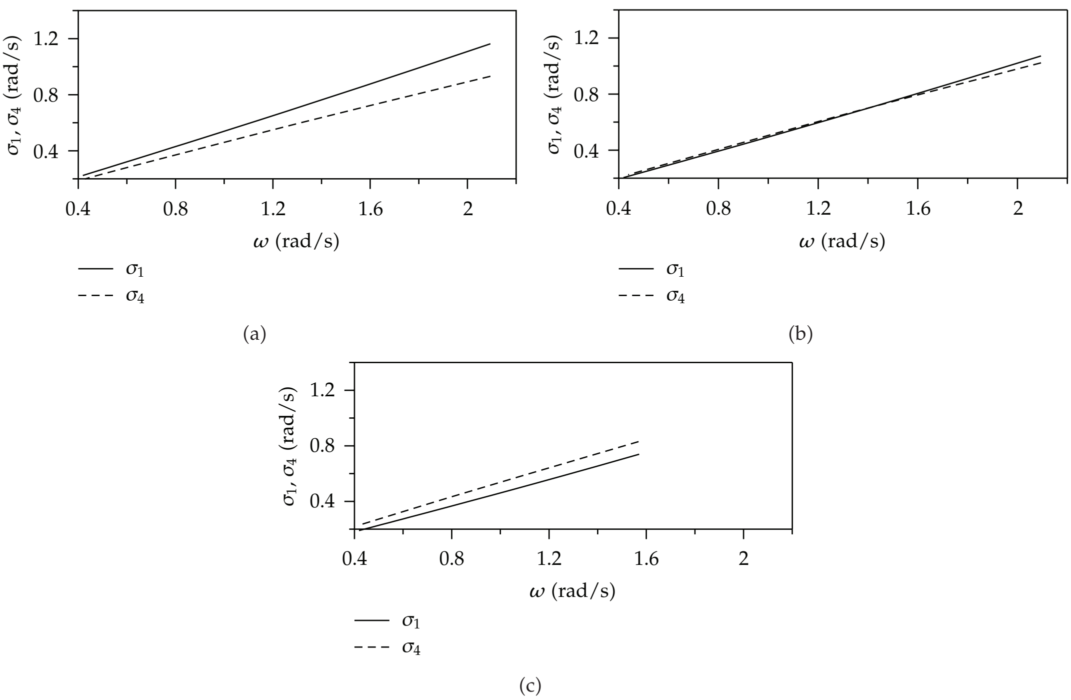

Figure 6: Two internal wave frequencies as functions of surface wave frequency ω, $H = 4\,\text{m}$, $h = 0.1\,\text{m}$ (a) $\gamma' = \gamma'' = 0.83$, (b) $\gamma' = 0.83$, $\gamma'' = 0.77$, and (c) $\gamma' = 0.83$, $\gamma'' = 0.71$.

"mud" waves each with very long wavelength will be simultaneously induced at the interface between two muddy sections. The wave numbers of these two long internal waves will be of the order $10^{-5} \sim 10^{-4}$ and have opposite value; $\lambda_2 = -\lambda_3 \approx 10^{-5} \sim 10^{-4}$ for the cases of different mud layer depth. Figure 6 shows that if the density of the right muddy section becomes increasingly larger than that of left side, then the right resonant internal wave frequency will also have increasingly higher value than the left. Meanwhile if the density difference between two muddy sections increases, the growth rate of the two resonant short internal waves will be suppressed. This result can be seen in Figure 7. The result means that it will

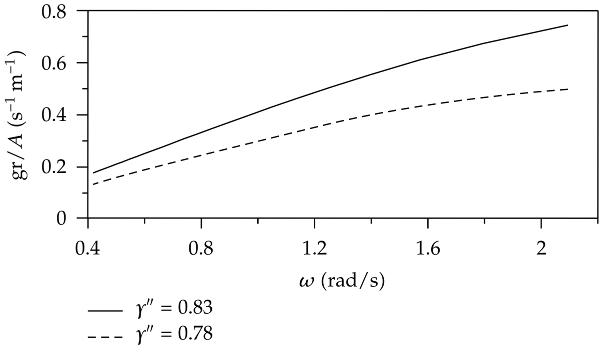

Figure 7: Growth rate of internal short waves for different density ratios of the right muddy section ($\gamma'' = 0.83, 0.78$), water depth $H = 4\,\mathrm{m}$, mud layer depth $h = 0.1\,\mathrm{m}$, and density ratio of the left muddy section $\gamma' = 0.83$.

be much difficult to excite the internal waves when surface water wave progressing over two muddy sections with the large density gap.

5. Conclusions

The resonant generation of internal waves on a sediment bed was presented as a new mechanism of sediment suspension. In this study, the corresponding theoretical analysis for equal density of two muddy sections is similar to that of the previous studies. A progressive surface wave simultaneously generates two opposite-traveling short internal waves. However, for a surface water wave progressing over two different muddy sections, the surface wave will also excite only two opposite-traveling interfacial short waves, forming the triad resonance with the surface wave. Meanwhile, two opposite-outgoing long internal waves will be also triggered at the interface between two muddy sections. Furthermore, it will be much difficult to excite the internal waves when surface water wave is progressing over two muddy sections with the large density gap.

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