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Research Article

The Asymptotic Synchronization Analysis for Two Kinds of Complex Dynamical Networks

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We consider a class of complex networks with both delayed and nondelayed coupling. In particular, we consider the situation for both time delay-independent and time delay-dependent complex dynamical networks and obtain sufficient conditions for their asymptotic synchronization by using the Lyapunov-Krasovskii stability theorem and the linear matrix inequality (LMI). We also present some simulation results to support the validity of the theories.

1. Introduction

A complex dynamical network is a large set of interconnected nodes that represent the individual elements of the system and their mutual relationships. Owing to their immense potential for applications to various fields, complex networks have been intensively investigated in the past decade in areas as diverse as mathematics, physics, biology, engineering, and even the social sciences [1–3]. The synchronization problem for complex networks was first posed by Saber and Murray [4, 5] who also introduced a theoretical framework for their investigation by viewing them as the adjustments of the rhythms of their interaction states [6] and many different kinds of synchronization phenomena and models have also been discovered such as complete synchronization, phase synchronization, lag synchronization, antisynchronization, impulsive synchronization, and projective synchronization.

Time delays are an important consideration for complex networks although these were usually ignored in early investigations of synchronization and control problems [6–11]. To make up for this deficiency, uniformly distributed time delays have recently been incorporated into network models [12–25] and Wang et al. [18] even considered networks with both delayed and nondelayed couplings and obtained sufficient conditions for asymptotic stability. Similarly, Wu and Lu [19] investigated the exponential synchronization of general weighted delay and nondelay coupled complex dynamical networks with different topological structures. There remains, however, much room for improvement in both the scope of the systems considered by Wang and Xu as well as in their methods of proofs.

The main contributions of this paper are two-fold. Firstly, we present a more general model for networks with both delayed and nondelayed couplings and derive criteria for their asymptotical synchronization. Secondly, we apply the Lyapunov-Krasovskii theorem and the LMIs to ensure the inevitable attainment of the required synchronization.

The rest of the paper is organized as follows. In Section 2, we present the general complex dynamical network model under consideration and state some preliminary definitions and results. In Section 3, we present the main results of this paper. In particular, we consider the situation for both time delay-independent and time delay-dependent complex dynamical networks and derive sufficient conditions for their asymptotic synchronization by using the Lyapunov-Krasovskii stability theorem and the linear matrix inequality (LMI). In Section 4, we present some numerical simulation results that verify our theoretical results. The paper concludes in Section 5.

2. Preliminaries and Model Description

In general, a linearly coupled ordinary differential equation system (LCODES) can be described as follows:

$$\frac{dx_i(t)}{dt} = f(x_i(t)) + c_1 \sum_{j\neq i, j=1}^{N} b_{ij} A x_j(t) + c_2 \sum_{j\neq i, j=1}^{N} b'_{ij} A' x_j(t-\tau). \tag{2.1}$$

Since $x_i - x_i = 0$ for all $i = 1, \ldots, N$, we can choose any values for a_{ii} in the above equations. Hence, letting $b_{ii} = -\sum_{j\neq i, j=1}^{N} b_{ij}$ and $b'_{ii} = -\sum_{j\neq i, j=1}^{N} b'_{ij}$, the above equations can be rewritten as follows:

$$\frac{dx_i(t)}{dt} = f(x_i(t)) + c_1 \sum_{j=1}^{N} b_{ij} A x_j(t) + c_2 \sum_{j=1}^{N} b'_{ij} A' x_j(t-\tau), \tag{2.2}$$

where N is the number of nodes, $x^i(t) = (x_{i1}, x_{i2}, \ldots, x_{iN})^T \in R^n$ are the state variables of the ith node, $t \in [0, +\infty)$ and $f : R^n \to R^n$ is a continuously differentiable function. The constants c_1 and c_2 (possibly distinct) are the coupling strengths, $b_{ij} \geq 0$, $b'_{ij} > 0$ (for $i, j = 1, \ldots, N$), $A, A' \in R^{n\times n}$ are inner-coupled matrices, $B, B' \in R^{n\times n}$ are coupled matrices with zero-sum rows with $b_{ij}, b'_{ij} \geq 0$ for $i \neq j$ that determines the topological structure of the network. We assume that B and B' are symmetric and irreducible matrices so that there are no isolated nodes in the system.

If all the eigenvalues of a matrix $A \in R^{n\times n}$ are real, then we denote its ith eigenvalue by $\lambda_i(A)$ and sort them by $\lambda_1(A) \leq \lambda_2(A) \leq \cdots \leq \lambda_n(A)$. A symmetric real matrix A is positive definite (semidefinite) if $x^T A x > 0$ (≥ 0) for all nonzero x and denoted by $A > 0$ $(A \geq 0)$. Finally, I stands for the identity matrix and the dimensions of all vectors and matrices should be clear in the context.

Definition 2.1. A complex network with delayed and nondelayed coupling (2.2) is said to achieve asymptotic synchronization if

$$x_1(t) = x_2(t) = \cdots = x_N(t) = s(t), \quad t \longrightarrow +\infty, \tag{2.3}$$

where $s(t)$ is a solution of the local dynamics of an isolated node satisfying $s(\dot{t}) = f(s(t))$.

Definition 2.2. A matrix $L = (l_{ij})_{i,j=1}^{N}$ is said to belong to the class $A1$, denoted by $L \in A1$ if

(1) $l_{ij} \leq 0, i \neq j, l_{ii} = -\sum_{j=1, j\neq i}^{N} l_{ij}, i = 1, 2, \ldots, N,$
(2) *L is irreducible.*

If $L \in A1$ is symmetrical, then we say that L belongs to the class $A2$, denoted by $L \in A2$.

Lemma 2.3 (see [26]). *If $L \in A1$, then* $\operatorname{rank}(L) = N - 1$, *that is, 0 is an eigenvalue of L with multiplicity 1, and all the nonzero eigenvalues of L have positive real part.*

Lemma 2.4 (Wang and Chen [11]). *If $G = (g_{ij})_{N\times N}$ satisfies the above conditions, then there exists a unitary matrix $\Phi = (\phi_1, \ldots, \phi_N)$ such that*

$$G^T \phi_k = \lambda_k \phi_k, \quad k = 1, 2, \ldots, N, \tag{2.4}$$

where λ_i, $i = 1, 2, \ldots, N$, are the eigenvalues of G.

Lemma 2.5 (Schur complement [22]). *The linear matrix inequality (LMI)*

$$\begin{pmatrix} Q(x) & S(x) \\ S(x)^T & R(x) \end{pmatrix} > 0, \tag{2.5}$$

where $Q(x)$ and $R(x)$ are symmetric matrices and $S(x)$ is a matrix with suitable dimensions is equivalent to one of the following conditions:

(i) $Q(x) > 0, R(x) - S(x)^T Q(x)^{-1} S(x) > 0;$
(ii) $R(x) > 0, Q(x) - S(x) R(x)^{-1} S(x) > 0.$

Lemma 2.6 ((the Lyapunov-Krasovskii stability theorem). (Kolmanovskii and Myshkis, Hale and Verduyn Lunel [16])). *Consider the delayed differential equation*

$$x(t) = \dot{f}(t, x(t)), \tag{2.6}$$

where $f : R\times C \to R^n$ is continuous and takes $R\times$ (bounded subsets of C) into bounded subsets of R^n, and let $u, v, w : R^+ \to R^+$ be continuous and strictly monotonically nondecreasing functions with $u(s)$, $v(s)$, $w(s)$ being positive for $s > 0$ and $u(0) = v(0) = 0$. If there exists a continuous functional $V : R \times C \to R$ such that

$$\begin{gathered} u(\|x\|) \leq V(t, x) \leq v(\|x\|), \\ \dot{V}(t, x(t, x(t))) \leq -w(\|x(t)\|), \end{gathered} \tag{2.7}$$

where $\dot{V}$ is the derivative of V along the solutions of the above delayed differential equation, then the solution $x = 0$ of this equation is uniformly asymptotically stable.

Remark 2.7. The functional V is called a Lyapunov-Krasovskii functional.

Lemma 2.8 (Moon et al. [22]). *Let $a(\cdot) \in R^{n_a}$, $b(\cdot) \in R^{n_b}$ and $M(\cdot) \in R^{n_a \times n_b}$ be defined on an interval Ω. Then, for any matrices $X \in R^{n_a \times n_a}$, $Y \in R^{n_a \times n_b}$, and $Z \in R^{n_b \times n_b}$, one has*

$$-2\int_{\Omega} a(x)^T M b(x) dx \leq \int_{\Omega} \begin{bmatrix} a(x) \\ b(x) \end{bmatrix}^T \begin{bmatrix} X & Y - M \\ Y^T - M^T & Z \end{bmatrix} \begin{bmatrix} a(x) \\ b(x) \end{bmatrix} dx, \tag{2.8}$$

where

$$\begin{bmatrix} X & Y \\ Y^T & Z \end{bmatrix} \geq 0. \tag{2.9}$$

Lemma 2.9. *For all positive-definite matrices P and vectors x and y, one has*

$$-2x^T y \leq \inf_{P>0} \left\{ x^T P x + y^T P^{-1} y \right\}. \tag{2.10}$$

Lemma 2.10 (see [16]). *Consider the delayed dynamical network* (2.2). *Let*

$$\begin{aligned} 0 &= \lambda_1 > \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_N, \\ 0 &= \mu_1 > \mu_2 \geq \mu_3 \geq \cdots \geq \mu_N \end{aligned} \tag{2.11}$$

be the eigenvalues of the outer-coupling matrices B and B', respectively. If the n-dimensional linear time-delayed and nontime delayed system

$$\dot{w}_i(t) = J(t) w_i(t) + c_1 \lambda_i A w_i(t) + c_2 \mu_i A' w_i(t - \tau), \quad k = 2, 3, \ldots, N, \tag{2.12}$$

of $N - 1$ differential equations is asymptotically stable about their zero solutions for some Jacobian matrix $J(t) \in R^{n \times n}$ of $f(x(t))$ at $s(t)$, then the synchronized states (2.3) *are asymptotically stable.*

3. The Criteria for Asymptotic Synchronization

In this section, we derive the conditions for the asymptotic synchronization of time-delayed coupled dynamical networks when they are either time-dependent or time-independent.

3.1. Case 1: The Time Delay-Independent Stability Criterion

Theorem 3.1. *Consider the general time delayed and non-time delayed complex dynamical network* (2.2)*. If there exist two positive definite matrices P and $Q > 0$ such that*

$$\begin{bmatrix} J(t)^T P + PJ(t) + 2c_1\lambda_i A + Q & c_2\mu_N PA' \\ c_2\mu_N A'P & -Q \end{bmatrix} > 0, \tag{3.1}$$

then the synchronization manifold (2.3) *of network* (2.2) *can be asymptotically synchronized for all fixed time delay $\tau > 0$.*

Proof. For each fixed $i = 1, 2, \ldots, N$, choose the Lyapunov-Krasovskii functional

$$V_i(t) = w_i(t)^T P w_i(t) + \int_{t-\tau}^{t} w_i(s)^T Q w_i(s) ds \tag{3.2}$$

for some matrices $P > 0$ and $Q > 0$ to be determined. Then the derivative of $V_i(t)$ along the trajectories of (3.2) is

$$\frac{dV_i(t)}{dt} = \dot{w}_i(t)^T P w_i(t) + w_i(t)^T P \dot{w}_i(t) + w_i(t)^T Q w_i(t) - w_i(t-\tau)^T Q w_i(t-\tau) \tag{3.3}$$

which, upon substitution of (2.12), gives

$$\begin{aligned}
\dot{V}_i(t) = {} & \left[J(t)w_i(t) + c_1\lambda_i A w_i(t) + c_2\mu_i A' w_i(t-\tau)\right]^T P w_i(t) + w_i(t)^T P \\
& \times \left[J(t)w_i(t) + c_1\lambda_i A w_i(t) + c_2\mu_i A' w_i(t-\tau)\right] + w_i(t)^T Q w_i(t) - w_i(t-\tau)^T Q w_i(t-\tau) \\
= {} & w_i(t)^T J(t)^T w_i(t) + w_i(t)^T \lambda_i A^T P w_i(t) + w_i(t-\tau)^T \mu_i A' P w_i(t) \\
& + w_i(t)^T P J(t) w_i(t) + w_i(t)^T c_1 \lambda_i A w_i(t) + w_i(t)^T P c_2 \mu_i A' w_i(t-\tau) \\
& + w_i(t)^T Q w_i(t) - w_i(t-\tau)^T Q w_i(t-\tau) \\
= {} & w_i(t)^T \left[J(t)^T P + PJ(t) + 2c_1\lambda_i A + Q\right] w_i(t) + 2w_i(t)^T c_2\mu_i P A' w_i(t-\tau) \\
& - w_i(t-\tau)^T Q w_i(t-\tau).
\end{aligned} \tag{3.4}$$

Now, by using the inequality in Lemma 2.9, we have

$$2w_i(t)^T c_2\mu_i P A'^{w_i}(t-\tau) \leq w_i(t-\tau)^T Q w_i(t-\tau) + w_i(t)^T {c_2}^2 {\mu_i}^2 P A' Q A' P w_i(t), \tag{3.5}$$

which, upon substituting (3.5) into (3.2), gives

$$\dot{V}_i(t) \leq w_i(t)^T \left[J(t)^T P + PJ(t) + 2c_1\lambda_i A + c_2{\mu_i}^2 w_i(t)^T P A' Q A' A w(t) + Q\right] w_i(t). \tag{3.6}$$

It therefore follows from the Schur complement (Lemma 2.5) and the linear matrix inequality (3.1) that $\dot{V}_i(t) < 0$ for all the $N-1$ equations in the general time delayed and non-time delayed system (2.12) and hence the system (2.12) is asymptotically synchronized by the Lyapunov-Krasovskii stability theorem. So, by Theorem 3.1, the synchronization manifold (2.3) of the network (2.2) is asymptotically synchronized. This completes the proof of the theorem. □

The following corollaries follow immediately from the above theorem.

Corollary 3.2. *Consider the general non-time delayed complex dynamical network*

$$\dot{x}_i(t) = f(x_i(t)) + c_1\sum_{j=1}^{N} b_{ij}Ax_j(t). \tag{3.7}$$

If there exists a positive definite matrix $P > 0$ such that

$$J(t)^T P + c_1\lambda_i AP < 0, \tag{3.8}$$

then the synchronization manifold (2.3) *of network* (3.7) *can be asymptotically synchronized.*

Proof. From Lemma 2.10, we have

$$\dot{w}_i(t) = J(t)w_i(t) + c_1\lambda_i Aw_i(t) \tag{3.9}$$

and the result follows by choosing the Lyapunov functional $V_i(t) = (1/2)w_i(t)^T Pw_i(t)$. □

Corollary 3.3. *Consider the general time delayed complex dynamical network*

$$\dot{x}_i(t) = f(x_i(t)) + c_2\sum_{j=1}^{N} b'_{ij}A'x_j(t-\tau). \tag{3.10}$$

If there exist two positive definite matrices $P > 0$ and $Q > 0$ such that

$$\begin{bmatrix} J(t)^T P + PJ(t) + Q & c_2\mu_N PA' \\ c_2\mu_N A'P & -Q \end{bmatrix} < 0, \tag{3.11}$$

then the synchronization manifold (2.3) *of network* (3.10) *can be asymptotically synchronized.*

Remark 3.4. The results of [16] are obtainable as particular cases of Theorem 3.1.

Remark 3.5. The above analysis is applicable to a general system with arbitrary time delays. A simpler synchronization scheme, however, could be applied to systems with time delays that are already known and are small in value.

3.2. Case 2: The Criterion for Time Delay-Dependent Stability

Theorem 3.6. *Consider the general time delayed and non-time delayed complex dynamical network* (2.2) *with a fixed time delay $\tau \in (0, h]$ for some small h. If there exist three positive definite matrices $P, Q, Z > 0$, such that*

$$\begin{bmatrix} (1,1) & Pc_2\mu_i A' - Y + h(J(t) + c_1\lambda_i A)^T Z c_2\mu_i A' \\ c_2\mu_i A'^T P - Y^T + hc_2\mu_i A'^T Z(J(t) + c_1\lambda_i A) & hc_2\mu_i^2 A'^T Z A' - Q \end{bmatrix} < 0 \tag{3.12}$$

with

$$\begin{gathered} (1,1) = \left(J(t) + c_1\lambda_i A + c_2\mu_i A'\right)^T P + P\left(J(t) + c_1\lambda_i A + c_2\mu_i A'\right) + hX \\ + Y^T + Y + Q + h(J(t) + c_1\lambda_i A)^T (J(t) + c_1\lambda_i A) \\ \begin{bmatrix} X & Y \\ Y^T & Z \end{bmatrix} \geq 0, \end{gathered} \tag{3.13}$$

then the synchronization manifold (2.3) *of network* (2.2) *can be asymptotically synchronized.*

Proof. For each fixed $i = 1, 2, \ldots, N$, choose the Lyapunov-Krasovskii functional

$$V_i(t) = w_i(t)^T P w_i(t) + \int_{t-\tau}^{t} w_i(s)^T Q w_i(s) + \int_{-\tau}^{0}\int_{t+\beta}^{t} \dot{w}_i(s)^T Z \dot{w}_i(s) ds d\beta \tag{3.14}$$

for some matrices $P, Q, Z > 0$ to be determined and let

$$\begin{gathered} V_1 = \dot{w}_i(t)^T P w_i(t), \qquad V_2 = \int_{t-\tau}^{t} w_i(s)^T Q w_i(s), \\ V_3 = \int_{-\tau}^{0}\int_{t+\beta}^{t} \dot{w}_i(s)^T Z \dot{w}_i(s) ds d\beta. \end{gathered} \tag{3.15}$$

Then, $V_i(t) = V_1 + V_2 + V_3$ and it follows from the Newton-Leibniz equation that

$$\int_{t-\tau}^{t} \dot{w}_i(\xi) d\xi = w_i(t) - w_i(t - \tau) \tag{3.16}$$

so that (2.12) can be transformed into

$$\dot{w}_i(t) = \left(J(t) + c_1\lambda_i A + c_2\mu_i A'\right) w_i(t) - c_2\mu_i A'^{\int_{t-\tau}^{t} \dot{w}_i}(s) ds. \tag{3.17}$$

Hence

$$
\begin{aligned}
V_1 &= \dot{w}_i(t)^T P w_i(t) + w_i(t)^T P \dot{w}_i(t) \\
&= w_i(t)^T \Big[\big(J(t) + c_1\lambda_i A + c_2\mu_i A'\big)^T P + P\big(J(t) + c_1\lambda_i A + c_2\mu_i A'\big) \Big] w_i(t) \\
&\quad - 2w_i(t)^T P c_2\mu_i A' \int_{t-\tau}^{t} \dot{w}_i(s)ds
\end{aligned}
\tag{3.18}
$$

and so, by Lemma 2.9, we have

$$
\begin{aligned}
&- 2w_i(t)^T P c_2\mu_i A' \int_{t-\tau}^{t} \dot{w}_i(s)ds \\
&\quad = -2\int_{t-\tau}^{t} w_i(t)^T \left(P c_2\mu_i A'\right) \dot{w}_i(s)ds \\
&\quad \leq \int_{t-\tau}^{t} \begin{bmatrix} w_i(t) \\ \dot{w}_i(s) \end{bmatrix}^T \begin{bmatrix} X & Y - P c_2\mu_i A' \\ Y^T - A'^T c_2\mu_i P & Z \end{bmatrix} \begin{bmatrix} w_i(t) \\ \dot{w}_i(s) \end{bmatrix} dx \\
&\quad = \int_{t-\tau}^{t} w_i(t)^T X w_i(t)ds + \int_{t-\tau}^{t} \dot{w}_i(s)^T Z \dot{w}_i(s)ds + 2\int_{t-\tau}^{t} w_i(t)^T \left(Y - P c_2\mu_i A'\right)\dot{w}_i(s)ds \\
&\quad = \tau w_i(t)^T X w_i(t) + 2w_i(t)^T \left(Y - P c_2\mu_i A'\right) \int_{t-\tau}^{t} \dot{w}_i(s)ds + \int_{t-\tau}^{t} \dot{w}_i(s)^T Z \dot{w}_i(s) \\
&\quad = \tau w_i(t)^T X w_i(t) + 2w_i(t)^T \left(Y - P c_2\mu_i A'\right) w_i(t) - 2w_i(t)^T \left(Y - P c_2\mu_i A'\right) w_i(t-\tau) \\
&\qquad + \int_{t-\tau}^{t} \dot{w}_i(s) Z \dot{w}_i(s)
\end{aligned}
\tag{3.19}
$$

and so

$$
\begin{aligned}
V_1 \leq\, & w_i(t)^T \Big[\big(J(t) + c_1\lambda_i A + c_2\mu_i A'\big)^T P + P\big(J(t) + c_1\lambda_i A + c_2\mu_i A'\big) \Big] w_i(t) \\
&+ 2w_i(t)^T \left(P c_2\mu_i A' - Y\right) w_i(t-\tau) + \int_{t-\tau}^{t} \dot{w}_i(s)^T Z w_i(s)ds.
\end{aligned}
\tag{3.20}
$$

Similarly, we have

$$\begin{aligned}
V_2 &= w_i(t)^T Q w_i(t) - w_i(t-\tau)^T Q w_i(t-\tau),\\
V_3 &= \tau \dot{w}_i(t)^T Z \dot{w}_i(t) - \int_{t-\tau}^{t} \dot{w}_i(s)^T Z \dot{w}_i(s) ds \le h\left[(J(t) + c_1\lambda_i A) w_i(t) + c_2\mu_i A' w_i(t-\tau)\right]^T\\
&\quad \times Z\left[(J(t) + c_1\lambda_i A) w_i(t) + c_2\mu_i A' w_i(t-\tau)\right] - \int_{t-\tau}^{t} \dot{w}_i(s)^T Z \dot{w}_i(s) ds\\
&= h w_i(t)^T (J(t) + c_1\lambda_i A)^T Z (J(t) + c_1\lambda_i A) w_i(t) + h w_i(t)^T\\
&\quad \times (J(t)t + c_1\lambda_i A)^T Z c_2\mu_i A' w_i(t-\tau) + h w_i(t-\tau)^T c_2\mu_i A'^T Z^T\\
&\quad \times (J(t) + c_1\lambda_i A) w_i(t) + h w_i(t-\tau)^T {\mu_i}^2 A'^T Z A' w_i(t-\tau) - \int_{t-\tau}^{t} \dot{w}_i(s)^T Z \dot{w}_i(s) ds\\
&= h w_i(t)^T (J(t) + c_1\lambda_i A)^T Z (J(t) + c_1\lambda_i A) w_i(t) + 2h w_i(t)^T\\
&\quad \times (J(t) + c_1\lambda_i A)^T Z c_2\mu_i A' w_i(t-\tau) + h w_i(t-\tau)^T {\mu_i}^2 A'^T Z A' w_i(t-\tau)\\
&\quad - \int_{t-\tau}^{t} \dot{w}_i(s)^T Z \dot{w}_i(s) ds
\end{aligned} \tag{3.21}$$

and so

$$\begin{aligned}
\dot{V}_i(t) &= V_1 + V_2 + V_3\\
&\le w_i(t)^T \Big[\left(J(t) + c_1\lambda_i A + c_2\mu_i A'\right)^T P + P\left(J(t) + c_1\lambda_i A + c_2\mu_i A'\right)\\
&\qquad + hX + Y^T + Y + Q + h(J(t) + c_1\lambda_i A)^T (J(t) + c_1\lambda_i A)\Big] w_i(t)\\
&\quad + w_i(t-\tau)^T \left[h{\mu_i}^2 A'^T Z A' - Q\right] w_i(t-\tau)\\
&\quad + 2 w_i(t)^T \left[\left(P c_2\mu_i A' - Y\right) + h(J(t) + c_1\lambda_i)^T Z c_2\mu_i A'\right] w_i(t-\tau).
\end{aligned} \tag{3.22}$$

Finally, we have

$$\dot{V}_i(t) \le \begin{bmatrix} w_i(t) \\ w_i(t-\tau) \end{bmatrix}^T \begin{bmatrix} (1,1) & (1,2) \\ (2,1) & hc_2\mu^2 A'^T Z A'_Q \end{bmatrix} \begin{bmatrix} w_i(t) \\ w_i(t-\tau) \end{bmatrix}, \tag{3.23}$$

where

$$
\begin{aligned}
(1,1) &= \left(J(t) + c_1\lambda_i A + c_2\mu_i A'\right)^T P + P\left(J(t) + c_1\lambda_i A + c_2\mu_i A'\right) + hX \\
&\quad + Y^T + Y + Q + h(J(t) + c_1\lambda_i A)^T (J(t) + c_1\lambda_i A) \\
(1,2) &= Pc_2\mu_i A' - Y + h(J(t) + c_1\lambda_i A)^T Zc_2\mu_i A' \\
(2,1) &= c_2\mu_i A'^T P - Y^T + hc_2\mu_i A'^T Z(J(t) + c_1\lambda_i A) \\
&\begin{bmatrix} X & Y \\ Y^T & Z \end{bmatrix} \geq 0.
\end{aligned}
\tag{3.24}
$$

It now follows from Lemma 2.5 that the conditions of the theorem are equivalent to $\dot{V}_i(t) < 0$ and that by the Lyapunov-Krasovskii Stability Theorem all the nodes of the system (2.12) are asymptotically stable when (3.12) and (3.13) hold for $i = 1, 2, \ldots, N$. This completes the proof of Theorem 3.6. □

Corollary 3.7. *Consider the general time delayed complex dynamical network* (3.10) *with a fixed time delay* $\tau \in (0, h]$

$$
\dot{x}_i(t) = f(x_i(t)) + c_2 \sum_{j=1}^{N} b'_{ij} A' x_j(t - \tau) \tag{3.25}
$$

for some $h < +\infty$. *If there exist two positive definite matrices,* $P, Q > 0$, X, Y, *and* Z *such that*

$$
\begin{bmatrix} (1,1) & c_2\mu_i PA' - Y + hJ(t)^T Zc_2\mu_i A' \\ c_2\mu_A'^T P - Y^T + hc_2\mu_i A'^T ZJ(t) & hc_2{}^2\mu_i{}^2 A'^T ZA' - Q \end{bmatrix} < 0, \tag{3.26}
$$

where $(1,1) = PJ(t) + J(t)^T + hX + Y^T + Y + Q + hJ(t^T ZJ(t))$, *then the synchronization manifold* (2.3) *of network* (3.10) *is asymptotic synchronization.*

Remark 3.8. The proof can be found in [16]. Those are the two results of general complex dynamical network with fixed time-invariant delay $\tau \in (0, h]$ for some $h < +\infty$; the conclusions are less conservative than the time-independent delay. The delay-dependent stability is another method applying to the delayed system. And it could provide a useful and meaningful upper bound of the delay h, which could ensure the delayed system achieves asymptotic synchronization only if the time delay is less than h.

4. Numerical Simulations

The above criteracould be applied to networks with different topologies and different size. We put two examples to illustrate the validity of the theories.

Example 4.1. We use a three-dimensional stable nonlinear system as an example to illustrate the main results, Theorem 3.1, of our paper; this is the time delay-independent situation. The model could be described as follows:

$$\begin{bmatrix} \dot{x}_{i1} \\ \dot{x}_{i2} \\ \dot{x}_{i3} \end{bmatrix} = \begin{bmatrix} -x_{i1} + x_{i2}^2 \\ -2x_{i2} \\ -3x_{i3} + x_{i2}x_{i3} \end{bmatrix}, \quad i = 1, 2, 3. \tag{4.1}$$

The solution of the 3-dimensional stable nonlinear system equations can be written as

$$x_{i1} = c_1 e^{-t} - c e^{-4t}, \qquad x_{i2} = c_2 e^{-2t}, \qquad x_{i3} = \frac{c_3 e^{-3t} - c_2 e^{2t}}{2}, \tag{4.2}$$

which is asymptotically stable at the equilibrium point of the system $s(t) = 0$, where $c = -c_2^2/3$ and c_1, c_2, c_3 are all constants. It is easy to see that the Jacobian matrix is $J = \mathrm{diag}\{-1, -2, -3\}$. We assume the inner-coupling matrices A, A' are all identity matrices, namely, $A = A' = \mathrm{diag}\{1, 1, 1\}$, and the outer coupling configuration matrices

$$B = B' = \begin{bmatrix} -2 & 1 & 0 & 1 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 1 & 0 & 1 & 2 \end{bmatrix}. \tag{4.3}$$

The eigenvalues of the coupling matrices are $\lambda(B) = \lambda(B') = \{0, -1.5, -1.5\}$. We choose the coupling strength $c_1 = 0.5$, $c_2 = 1$. By using Theorem 3.1 and the LMI Toolbox in MATLAB, we obtained the following common two positive-definite matrices:

$$P = \mathrm{diag}\{1.1204, 12.3091, 10.165\}, \qquad Q = \mathrm{diag}\{2.3710, 22.5713, 26.0849\}. \tag{4.4}$$

According to the conditions in Theorem 3.1, we know the synchronized state $s(t)$ is global asymptotically stable for any fixed delay. The quantity

$$E(t) = \sqrt{\left(\frac{\sum_{i=1}^{N} |x_i(t) - s(t)|^2}{N} \right)} \tag{4.5}$$

is used to measure the quality of the synchronization process. We plot the evolution of $E(t)$ in the upper part in Figure 1. For the time delay here we choose $\tau = 0.1$. The lower subplot indicates the synchronization results of the network.

Example 4.2. We use a 4-nodes networks model as another example to illustrate the Theorem 3.6; this is the time delay-dependent situation. The model could be described as follows:

$$\begin{bmatrix} \dot{x}_{i1} \\ \dot{x}_{i2} \\ \dot{x}_{i3} \end{bmatrix} = \begin{bmatrix} -x_{i1} \\ -2x_{i2} + x_{i3}^2 \\ -3x_{i3} + x_{i2}x_{i3} \end{bmatrix}, \quad i = 1, 2, 3, 4. \tag{4.6}$$

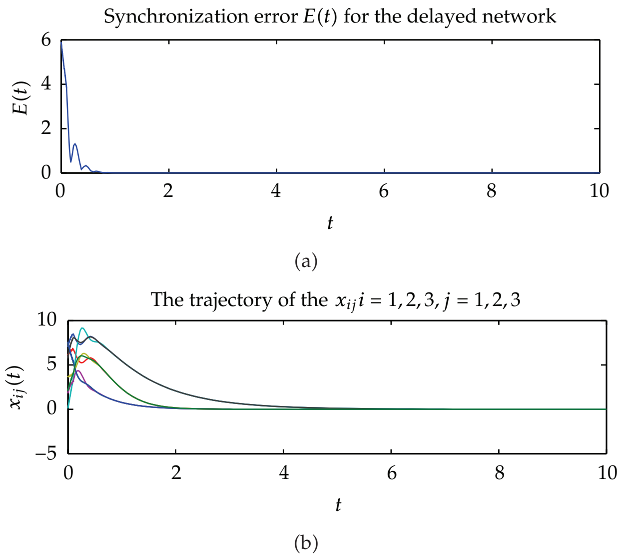

Figure 1: Synchronization evolution $E(t)$ for the delay-independent network with $c_1 = 0.5$, $c_2 = 1$, and $\tau = 0.1$.

We choose the same coupling strength $c_1 = 0.5$, $c_2 = 1$; the eigenvalues of the coupling matrices are $\lambda(B) = \lambda(B') = \{0, -2, -2, -4\}$. By using Theorem 3.6 and the LMI Toolbox in MATLAB, we obtained the following matrices:

$$\begin{aligned} &P = \operatorname{diag}\{1.4078, 1.4057, 1.4054\}, \qquad Q = \{1.4157, 1.4157, 1.4157\}, \\ &Z = \{-1.4218, -1.4303, -1.4367\}, \qquad X = \{18.3024, 19.6808, 21.0815\}, \\ &Y = \{0.0221, 0.0208, 0.0196\}. \end{aligned} \tag{4.7}$$

By using Theorem 3.6 in this paper, it is found that the maximum delay bound for the complex dynamical network to form asymptotic synchronization is $h = 1$. $E(t)$ are defined the same as in the example. We plot the evolution of $E(t)$ in upper part in Figure 2. The lower subplot indicates the synchronization results of the network. It can be seen from the figures that the network in this example can achieve asymptotic synchronization.

5. Conclusion

This paper considered a class of complex networks with both time delayed and non-time delayed coupling. We derived, respectively, a sufficient criterion for time delay-dependent and time delay-independent asymptotic synchronization which are more general than those obtained in previous works. These asymptotic synchronization results were obtained by using the Lyapunov-Krasovskii stability theorem and the linear matrix inequality. Two simple examples were also used to validate the theoretical analysis.

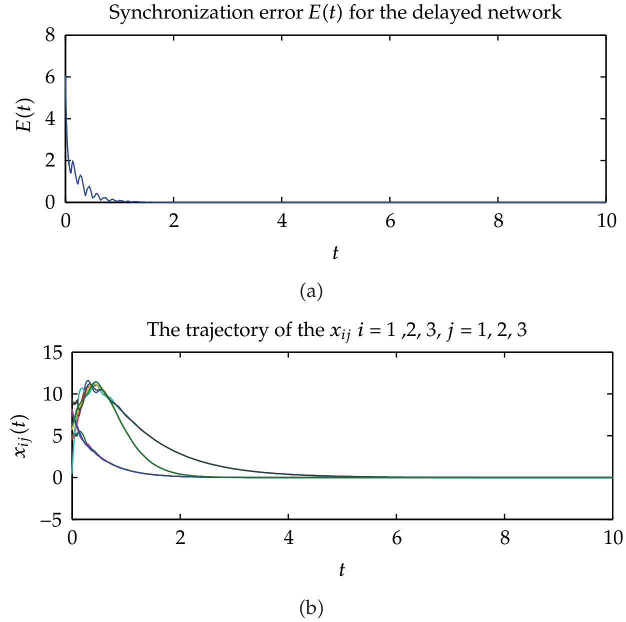

Figure 2: Synchronization evolution $E(t)$ for the delay-dependent network with $c_1 = 0.5$, $c_2 = 1$, and $\tau = 0.1$.

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References

[1] S. H. Strogatz, "Exploring complex networks," *Nature*, vol. 410, no. 6825, pp. 268–276, 2001.

[2] A.-L. Barabási and R. Albert, "Emergence of scaling in random networks," *Science*, vol. 286, no. 5439, pp. 509–512, 1999.

[3] X. F. Wang, "Complex networks: topology, dynamics and synchronization," *International Journal of Bifurcation and Chaos in Applied Sciences and Engineering*, vol. 12, no. 5, pp. 885–916, 2002, Chaos control and synchronization (Shanghai, 2001).

[4] R. O. Saber and R. M. Murray, "Flocking with Obstacle Avoidance: cooperation with Limited Communication in Mobile Networks," in *Proceedings of the 42nd IEEE Conference on Decision and Control*, pp. 2022–2028, December 2003.

[5] R. O. Saber and R. M. Murray, "Graph rigidity and distributed formation stabilization of multi-vehicle systems," in *Proceedings of the 41st IEEE Conference on Decision and Control*, pp. 2965–2971, usa, December 2002.

[6] A. Pikovsky, M. Rosenblum, and J. Kurths, *Synchronization: A Universal Concept in Nonlinear Sciences*, vol. 12 of *Cambridge Nonlinear Science Series*, Cambridge University Press, Cambridge, 2001.

[7] X. F. Wang and G. Chen, "Synchronization in scale-free dynamical networks: robustness and fragility," *IEEE Transactions on Circuits and Systems. I. Fundamental Theory and Applications*, vol. 49, no. 1, pp. 54–62, 2002.

[8] X. F. Wu, C. Xu, and J. Feng, "Mean synchronization of pinning complex networks with linearly and nonlinearly time-delay coupling," *International Journal of Digital Content Technology and its Applications*, vol. 5, no. 3, pp. 33–46, 2011.

[9] X. S. Yang, J. D. Cao, and J. Q. Lu, "Synchronization of coupled neural networks with random coupling strengths and mixed probabilistic time-varying delays," *International Journal of Robust and Nonlinear Control*. In press.

[10] J. Cao, Z. Wang, and Y. Sun, "Synchronization in an array of linearly stochastically coupled networks with time delays," *Physica A*, vol. 385, no. 2, pp. 718–728, 2007.

[11] X. F. Wang and G. Chen, "Synchronization in scale-free dynamical networks: robustness and fragility," *IEEE Transactions on Circuits and Systems. I*, vol. 49, no. 1, pp. 54–62, 2002.

[12] W. Sun, F. Austin, J. Lü, and S. Chen, "Synchronization of impulsively coupled complex systems with delay," *Chaos*, vol. 21, no. 3, Article ID 033123, 7 pages, 2011.

[13] W. He, F. Qian, J. Cao, and Q.-L. Han, "Impulsive synchronization of two nonidentical chaotic systems with time-varying delay," *Physics Letters A*, vol. 375, no. 3, pp. 498–504, 2011.

[14] J. Tang, J. Ma, M. Yi, H. Xia, and X. Yang, "Delay and diversity-induced synchronization transitions in a small-world neuronal network," *Physical Review E*, vol. 83, no. 4, Article ID 046207, 2011.

[15] C. W. Wu and L. O. Chua, "Synchronization in an array of linearly coupled dynamical systems," *IEEE Transactions on Circuits and Systems. I*, vol. 42, no. 8, pp. 430–447, 1995.

[16] C. Li and G. Chen, "Synchronization in general complex dynamical networks with coupling delays," *Physica A. Statistical Mechanics and its Applications*, vol. 343, no. 1–4, pp. 263–278, 2004.

[17] D. Xu and Z. Su, "Synchronization criterions and pinning control of general complex networks with time delay," *Applied Mathematics and Computation*, vol. 215, no. 4, pp. 1593–1608, 2009.

[18] L. Wang, H.-P. Dai, and Y.-X. Sun, "Random pseudofractal networks with competition," *Physica A*, vol. 383, no. 2, pp. 763–772, 2007.

[19] X. Wu and H. Lu, "Exponential synchronization of weighted general delay coupled and non-delay coupled dynamical networks," *Computers & Mathematics with Applications*, vol. 60, no. 8, pp. 2476–2487, 2010.

[20] S. Wen, S. Chen, and W. Guo, "Adaptive global synchronization of a general complex dynamical network with non-delayed and delayed coupling," *Physics Letters A*, vol. 372, no. 42, pp. 6340–6346, 2008.

[21] W. Lu and T. Chen, "New approach to synchronization analysis of linearly coupled ordinary differential systems," *Physica D*, vol. 213, no. 2, pp. 214–230, 2006.

[22] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, SIAM, Philadelphia, 1994.

[23] Y. Kuramoto, *Chemical Oscillations, Waves, and Turbulence*, vol. 19 of *Springer Series in Synergetics*, Springer, Berlin, Germany, 1984.

[24] A. Turing, "The chemical basis of morphogenesis," *Philosophical Transactions of the Royal Society B*, vol. 237, p. 37, 1952.

[25] F. C. Hoppensteadt and E. M. Izhikevich, "Pattern recognition via synchronization in phase-locked loop neural networks," *IEEE Transactions on Neural Networks*, vol. 11, no. 3, pp. 734–738, 2000.

[26] W. Lu and T. Chen, "Synchronization of coupled connected neural networks with delays," *IEEE Transactions on Circuits and Systems. I*, vol. 51, no. 12, pp. 2491–2503, 2004.