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Research Article

Werner State Structure and Entanglement Classification

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We present applications of the representation theory of Lie groups to the analysis of structure and local unitary classification of Werner states, sometimes called the *decoherence-free* states, which are states of n quantum bits left unchanged by local transformations that are the same on each particle. We introduce a multiqubit generalization of the singlet state and a construction that assembles these qubits into Werner states.

1. Introduction

Quantum entanglement, a feature of quantum theory named by Schrödinger [1] and employed by Bell [2, 3] in the rejection of local realism, has come to be seen as a resource for quantum information processing tasks including measurement-based quantum computation, teleportation, and some forms of quantum cryptography. Driven by applications to computation and communication, entanglement of composite systems of n quantum bits, or qubits, is of particular interest.

The problem of entanglement is to understand nonlocal properties of states and to answer operational questions such as when two given states can be interconverted by local operations on individual subsystems. This inspires the mathematical problem of classifying orbits of the local unitary group action on the space of states.

The goal of this paper is to address these questions for the Werner states, which are defined to be those states invariant under the action of any particular single-qubit unitary operator acting on all n qubits. Werner states have found a multitude of uses in quantum information science. Originally introduced in 1989 for two particles [4] to distinguish between classical correlation and the Bell inequality satisfaction, Werner states have found use in the description of noisy quantum channels [5], as examples in nonadditivity claims [6] and in the study of deterministic purification [7]. In what may prove to be a practical application to computing in noisy environments, Werner states lie in the decoherence-free subspace for collective decoherence [8–10]. A recent example of how analysis of state structure can be useful is the work of Migdał and Banaszek [11] on protecting information against the loss of a qubit using Werner states.

We apply the representation theory of Lie groups, in particular the Clebsch-Gordan decomposition of representations of $SU(2)$ on tensor products and the representation theory of $SO(3)$ on polynomials in three variables, to obtain structural theorems and local unitary classification for Werner states. We summarize recent results for the special cases of *pure* Werner states [12] and *symmetric* Werner states [13] in Section 3. We present new results for the general case of mixed Werner states in Section 4. We introduce a generalization of the singlet state and use these states to construct Werner states.

2. Local Unitary Group Action

Let $G = (SU(2))^n$ denote the local unitary (LU) group for n-qubit states. An LU operator $g = (g_1, g_2, \ldots, g_n)$ acts on an n-qubit density matrix ρ (i.e., a $2^n \times 2^n$ positive semidefinite matrix with trace 1) by

$$\rho \longmapsto g\rho g^\dagger := (g_1 \otimes g_2 \otimes \cdots \otimes g_n)\rho\left(g_1^\dagger \otimes g_2^\dagger \otimes \cdots \otimes g_n^\dagger\right). \tag{2.1}$$

In this notation, the Werner states are defined to be the set of density matrices ρ such that $\rho = g^{\otimes n}\rho(g^\dagger)^{\otimes n}$ for all g in $SU(2)$. We will write Δ to denote the subgroup

$$\Delta = \{(g, g, \ldots, g) : g \in SU(2)\} \tag{2.2}$$

of the LU group G.

The set of n-qubit density matrices is a convex set inside of the vector space $\mathcal{V}^{\otimes n}$, where $\mathcal{V}$ is the 4-dimensional real vector space of 2×2 Hermitian matrices. A convenient basis for $\mathcal{V}$ is $\{\sigma_0, \sigma_1, \sigma_2, \sigma_3\}$, where σ_0 is the 2×2 identity matrix, and $\sigma_1 = \sigma_x$, $\sigma_2 = \sigma_y$, and $\sigma_3 = \sigma_z$ are the Pauli matrices. Every element ρ (whether or not ρ is positive or has trace 1) of $\mathcal{V}^{\otimes n}$ can be uniquely written in the form $\rho = \sum_I s_I \sigma_I$, where $I = i_1, i_2 \ldots, i_n$ is a multi-index with $i_k = 0, 1, 2, 3$ for $1 \leq k \leq n$, and σ_I denotes

$$\sigma_I = \sigma_{i_1} \otimes \sigma_{i_2} \otimes \cdots \otimes \sigma_{i_n}, \tag{2.3}$$

with real coefficients s_I.

Sitting inside $\mathcal{V}^{\otimes n}$ is the space of pure states, which are the rank 1 density matrices of the form $|\psi\rangle\langle\psi|$, where $|\psi\rangle$ is a vector in the Hilbert space $\mathcal{H} = (\mathbf{C}^2)^{\otimes n}$ of pure n-qubit states. We will use the computational basis vectors $|I\rangle$ for $\mathcal{H}$, where $I = i_1, i_2 \ldots, i_n$ is a multi-index with $i_k = 0, 1$ for $1 \leq k \leq n$. The expansion of a pure state vector $|\psi\rangle$ in the computational basis has the form $|\psi\rangle = \sum_I c_I |I\rangle$, where the coefficients c_I are complex. Note that we use the same multi-index notation I for "mod 4" multi-indices for tensors of the Pauli matrices in $\mathcal{V}^{\otimes n}$, and for "mod 2" multi-indices for computational basis vectors in $\mathcal{H}$. The distinction will be clear from context.

3. Pure and Symmetric Werner States

In [12], we prove a structure theorem for the pure Werner states based on the following geometric construction. Begin with a circle with an even number $n = 2m$ of marked points, labeled 1 through n in order around the circle, say clockwise. Let $\mathcal{P}$ be a partition of $\{1, 2, \ldots, n\}$ into two-element subsets. Each two-element subset $\{a, b\}$ determines a chord connecting a and b. We impose the condition that no two chords coming from $\mathcal{P}$ may intersect. For each chord C in $\mathcal{P}$, let $|s_C\rangle$ be a singlet state $(1/\sqrt{2})(|01\rangle - |10\rangle)$ in the two qubits at the ends of C, and define the state $|s_{\mathcal{P}}\rangle$ to be the product

$$|s_{\mathcal{P}}\rangle = \bigotimes_{C \in \mathcal{P}} |s_C\rangle \tag{3.1}$$

of singlet states $|s_C\rangle$, over all C in $\mathcal{P}$. We call states of the form $|s_{\mathcal{P}}\rangle$ "nonintersecting chord diagram states." Figure 1 illustrates the two possibilities for 4 qubits.

We show that any linear combination of chord diagram states is a Werner state, and conversely, any pure Werner state can be written uniquely as a linear combination of nonintersecting chord diagram states. Further, these linear combinations are unique representatives of their LU equivalence class, up to a phase factor. Representation theory and combinatorics enter the story in the proof that the nonintersecting chord diagram states span the space of pure Werner states. The Werner states are the trivial summand in the decomposition into irreducible submodules of the $SU(2)$-space $\mathcal{H} = (\mathbf{C}^2)^{\otimes n}$. The dimension of the trivial summand is equal to the Catalan number

$$\frac{1}{m+1}\binom{2m}{m} \tag{3.2}$$

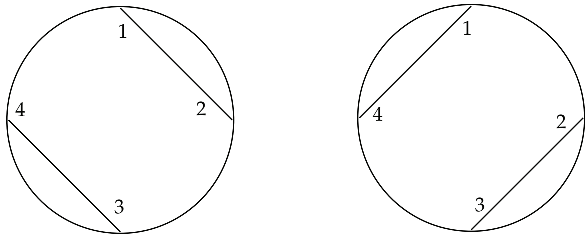
when the number of qubits $n = 2m$ is even, and the dimension of this space is zero when n is odd. The nonintersecting chord diagrams with $n = 2m$ nodes are one of the well-known sets enumerated by the Catalan numbers [14]. Together with an argument that the nonintersecting chord diagram states are linearly independent, the fact that these two numbers agree establishes that the chord diagram states form a linear basis for the space of pure Werner states.

In [13], we consider the case of pure and mixed Werner states that are invariant under permutations of qubits, also called *symmetric* states. Given nonnegative integers n_1, n_1, n_3 with $n_1 + n_2 + n_3 \leq n$, we identify the monomial $x^{n_1} y^{n_2} z^{n_3}$ in three variables with the matrix

$$\rho = \alpha \operatorname{Sym}\left(\sigma_0^{\otimes n_0} \otimes \sigma_1^{\otimes n_1} \otimes \sigma_2^{\otimes n_2} \otimes \sigma_3^{\otimes n_3}\right), \tag{3.3}$$

where $n_0 = n - n_1 - n_2 - n_3$, the symmetrizing operator Sym sums all the permutations of the products of n_k copies of σ_k for $k = 0, 1, 2, 3$, and α is a normalization factor. This establishes a correspondence between mixed symmetric states (not necessarily Werner states) and real polynomials in three variables. Using the representation theory of $SO(3)$, we show that the symmetric Werner states correspond to polynomials that are linear combinations of $(x^2 + y^2 + z^2)^m$ for some $m \leq \lfloor n/2 \rfloor$. Further, any two such states are local unitarily inequivalent.

Now we turn to the general case of mixed Werner states.



$P = \{\{1,2\},\{3,4\}\}$

$s_P = |0101\rangle + |1010\rangle - |0110\rangle - |1001\rangle$

(a)

$Q = \{\{1,4\},\{2,3\}\}$

$s_Q = |0011\rangle + |1100\rangle - |0101\rangle - |1010\rangle$

(b)

Figure 1: The two non-intersecting 4-qubit chord diagrams and their associated singlet product states.

4. The General Case of Mixed Werner States

We begin with the construction of a family of density matrices C_n that generalize the singlet state.

Given an n-qubit binary string I, let $C(I)$ denote the pure state

$$C(I) = \alpha \sum_{k=0}^{n-1} \omega^k \left| \pi^k I \right\rangle, \tag{4.1}$$

where $\omega = e^{2\pi i/n}$ and π is the cyclic permutation of $\{1,2,\ldots,n\}$ given by $1 \mapsto n$, $k \mapsto k-1$ for $2 \leq k \leq n$, and α is a normalizing factor so that $|C(I)| = 1$, whenever $C(I) \neq 0$ (notice that $C(00) = 0$, so is not a state). For example,

$$C(001) = \frac{1}{\sqrt{3}} \Big(|001\rangle + e^{2\pi i/3}|010\rangle + e^{4\pi i/3}|100\rangle \Big). \tag{4.2}$$

Let C_n denote the density matrix

$$C_n = \beta \sum_I C(I) C(I)^\dagger, \tag{4.3}$$

where β is a normalizing factor so that $tr(C_n) = 1$. Observe that C_2 is the density matrix $|s\rangle\langle s|$ of the singlet state $|s\rangle = (1/\sqrt{2})(|01\rangle - |10\rangle)$, so that the C_n states are n-qubit generalizations of the singlet.

Next we form products of C_k states to make Werner states. (It is perhaps nontrivial to show that the C_n and the diagram states constructed from them below are indeed Werner states. This can be done with straightforward calculations, but at the expense of technical overhead. We refer the interested reader to our paper [15] which gives details on the action of the Lie algebra of the local unitary group on density matrices. One can show that the generators of the Lie algebra of the Werner stabilizer group $\Delta = \{(g, g, \ldots, g) : g \in SU(2)\}$ annihilate C_n.) As with the case of pure Werner states, we utilize diagrams. This time we consider diagrams consisting of n points labeled $1, 2, \ldots, n$ on a circle, with nonintersecting polygons that have vertices in the given set of n points. Again, there is a Catalan number $(1/(n+1))\binom{2n}{n}$ of such n-vertex diagrams [14]. (There is a one-to-one correspondence between these "nonintersecting polygon" diagrams and the "nonintersecting chord" diagrams on $2n$ points in our pure Werner states analysis of the previous section. Given a nonintersecting chord diagram with $2n$ vertices, rename the vertices $1, 1', 2, 2', \ldots, n, n'$ and then glue each pair jj' for $1 \le j \le n$.)

Given an n-vertex nonintersecting polygon diagram $\mathfrak{D}$, we construct a state $\rho_{\mathfrak{D}}$,

$$\rho_{\mathfrak{D}} = \bigotimes_{U \in \mathfrak{D}} C_U, \tag{4.4}$$

where the tensor has positions specified by elements of the partition $\mathfrak{D}$ and C_U denotes the state $C_{|U|}$ in qubit positions in U. Figure 2 shows an example.

Here is our main conjecture.

Conjecture. The states $\rho_{\mathfrak{D}}$ form a basis for the space of Werner states (in the larger space of real linear combinations of Pauli tensors).

Again, representation theory says that we have the right dimension: the 1-qubit density matrix representation space $\mathcal{V}$ decomposes into irreducible $SU(2)$-submodules as follows:

$$\mathcal{V} = \{\text{span of } \sigma_0\} \oplus \{\text{span of } \sigma_1, \sigma_2, \sigma_3\} = \mathbf{R}^1 \oplus \mathbf{R}^3, \tag{4.5}$$

where the $\mathbf{R}^3$ summand is isomorphic to the adjoint representation. The complexification $\mathcal{V}_C$ is isomorphic to $\mathbf{C}^1 \oplus \mathbf{C}^3 \approx \mathbf{C}^2 \otimes \mathbf{C}^2$, which is 2-qubit pure state space. In general, the complexification $(\mathcal{V}^{\otimes n})_C$ of n-qubit density matrix space $\mathcal{V}^{\otimes n}$ is isomorphic to $2n$-qubit pure state space, as $SU(2)$ spaces. Thus the real dimension of the trivial summand for n-qubit density matrices is equal to the complex dimension of the trivial summand for $2n$-qubit pure states, which is the Catalan number $(1/(n+1))\binom{2n}{n}$. This establishes that we only need to show that the diagram states are independent in order to prove the conjecture.

We conclude with a conjecture regarding a precise statement about the stabilizer subgroup of the local unitary group for our constructed Werner states. The full stabilizer of a Werner state ρ, that is, the set

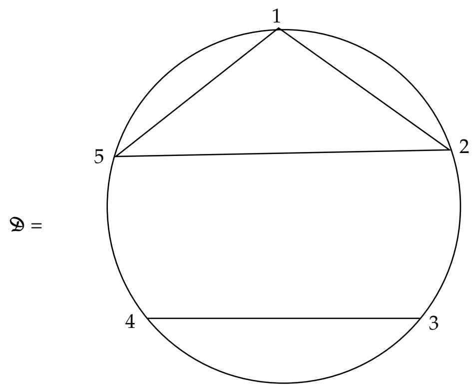
$$\text{Stab}_\rho = \left\{ g \in G : \rho = g\rho g^\dagger \right\} \tag{4.6}$$

of all local unitary transformations that fix ρ, could be larger than the subgroup Δ of the unitary group. For example, a diagram state $\rho_{\mathfrak{D}}$ is stabilized by the subgroup

$$\Delta_{\mathfrak{D}} := \prod_{U \in \mathfrak{D}} \Delta_U, \tag{4.7}$$

where Δ_U denotes the subgroup that consists of elements $(g, g, \ldots, g)$ in qubits in U, and all other coordinates are the identity.

In [12], we give a criterion on the diagrams that appear in the expansion of a pure Werner state for when the stabilizer subgroup of a pure Werner state is precisely the subgroup Δ of the local unitary group, and not larger. The criterion is that for any bipartition of the set



$\rho_{\mathfrak{D}} = (C_3)_{\{1,2,5\}} \bigotimes (C_2)_{\{3,4\}}$

Figure 2: A non-intersection polygon diagram state.

of qubits, there must be a diagram (with nonzero coefficient) in the expansion of the given state that has a chord with one end in each of the sets of the partition. Our final conjecture is a generalization of this idea to the general mixed Werner case.

Consider a poset lattice of partitions of $\{1, 2, \ldots, n\}$ (we consider all partitions, with and without crossing polygons), where $\mathfrak{D} \leq \mathfrak{D}'$ if $\mathfrak{D}'$ is a subdivision of $\mathfrak{D}$. The n-gon is the least element at the bottom, and the all-singleton diagram is the greatest element at the top of this lattice. The noncrossing polygon diagram lattice is a sublattice. There is a corresponding lattice of subgroups of local unitary group G, where H is less than or equal to K in the partial order if H is a subgroup of K. The subgroup Δ is the least element at the bottom and LG at the top. A diagram $\mathfrak{D}$ corresponds to the subgroup $\Delta_{\mathfrak{D}}$ defined above. We conjecture that

$$\mathrm{Stab}_{\sum a_{\mathfrak{D}}\rho_{\mathfrak{D}}} = \mathrm{glb}\{\Delta_{\mathfrak{D}} : a_{\mathfrak{D}} \neq 0\} = \bigcap_{\mathfrak{D}: a_{\mathfrak{D}} \neq 0} \Delta_D, \tag{4.8}$$

where "glb" denotes the greatest lower bound in the lattice. This would give a picture criterion for when a Werner state has the Werner stabilizer (and not a larger one).

5. Summary and Outlook

We have surveyed known results on the structure and local unitary equivalence classification of Werner states for the special cases of pure states and symmetric states. We have presented a diagram-based construction for the general case of the mixed Werner states that generalizes the "sums of products of singlets" construction known for pure states. Finally, we conjecture that the general construction will prove to be a basis for the Werner states and that this basis will lead to local unitary classification and a precise analysis of stabilizer subgroups.

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