Hindawi Publishing Corporation
Advances in Mathematical Physics
Volume 2012, Article ID 831012, 17 pages
doi:10.1155/2012/831012

Research Article

Mixed Initial-Boundary Value Problem for Telegraph Equation in Domain with Variable Borders

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Received 13 March 2012; Revised 6 April 2012; Accepted 11 April 2012

Academic Editor: Burak Polat

Mixed initial-boundary value problem for telegraph equation in domain with variable borders is considered. On one part of domain's border are the boundary conditions of the first type, on other part of the boundary are set boundary conditions of the second type. Besides, the sizes of area are variable. The solution of such problem demands development of special methods. With the help of consecutive application of procedure of construction waves reflected from borders of domain, it is possible to obtain the solution of this problem in quadratures. In addition, for construction of the waves reflected from mobile border, it is necessary to apply the procedure specially developed for these purposes.

1. Introduction

Mixed initial-boundary value problems for telegraph equation in domain with variable borders arise in many applications. In particular, such problem arises in a problem about calculation of a field of stress in ropes of elevating devices. At lifting of load the rope reels up on a drum or reels off a drum at lowering a load. Therefore, the length of that part of a rope, which is reeled up on a drum, changes. If to take into account friction of a rope on a drum, elastic displacements in a rope can be described by the telegraph equation [1]. In [1] it is shown, that it occurs at dry friction. However, it is possible to show, that the telegraph equation describes elastic displacements and at viscous friction as well. Thus, there is a initial-boundary value problem for the telegraph equation in domain with variable border. Here the telegraph equation of general view is considered. Regarding a rope which hangs down from a drum, elastic displacements there are described by the wave equation. This circumstance induces to consider separately problems about elastic displacements to various parts of a rope.

Initial-boundary value problem on elastic displacements to that part of a rope which is reeled on a drum is considered in the present paper. The rope is considered as a flexible string. One end of a rope is attached to a drum and goes together with a drum. The extreme point of contact of a rope with a drum is accepted as the second end of a rope. Elastic stresses are set in this point. Change of length of a rope, which is reeled up on a drum, is taken into account as follows. Portable movement of system is understood as rotation of a drum and a rope as perfectly rigid body. Then the relative movement of a rope will be submitted as motionless in all points of a rope, except for an extreme point of contact of a rope with a drum. This last point in relative movement will make the moving equal $v(t)$, where $v(t)$ is movment of the central axis of a rope together with a drum. In relative movement all points of a rope make only elastic displacements. The axis x is directed lengthways toward conditionally straightened rope and its beginning is located in a point of attaching of a rope to a drum. The initial length of a rope, which is reeled up on a drum, is equal to l.

At the solution of initial-boundary value problems for the telegraph equation the various exact and approached methods were used. It is necessary to notice that exact solutions were obtained only for the limited number of boundary problems.

Recently for the solution of boundary problems even for fractional telegraph equations it is actively used differential transformation method [2–4]. This method is an improved version of power series method or its modifications. In this case the same is; and in a power series method, representation of the solution of a problem in the form of type Taylor's series in a neighborhood of some point or a curve is used. Thanks to the developed procedure of use at an early stage of calculations of boundary conditions, calculation of expansion coefficients becomes essentially simpler. However representation of the solution in the form of Taylor's series type demands fulfillment of additional conditions. Following conditions concern them. It is necessary that all factors in the differential equation were analytical functions of the arguments. It is necessary also that the series obtained as the solution of the problem and especially series of derivatives converged uniformly. But the most important feature consists as such expansion of solutions provides good approach only in some neighborhood of an index point or a curve. On the essential distance from such neighborhood the values of higher degree terms become dominating.

Therefore preservation in expansion of final number of the terms can lead to considerable errors. Besides, at a great distance from area of initial or boundary values series can appear divergent. Therefore differential transformation method yields satisfactory results only on small intervals of change of spatial variables and time. Besides, for application of this method it is necessary that all functions included in the equation and boundary conditions supposed the same expansion that is used for solution representation. This condition cannot be executed in general case. Necessity to be limited to calculation of final number of terms of a series leads to that the solution appears approximate. At the same time, if all given problems can be presented in the form of finite series, the solution of such problem can be obtained exactly.

For the telegraph equations, including fractional ones, it is possible to solve some initial-boundary value problems by means of a method of separation variables [5]. For application of this method it is necessary that the domain of search of the solution possessed special symmetry, and initial and boundary functions as well as the right parts of the equations, supposed expansion on eigen functions of a boundary problem. It is clear that these conditions can be executed not always.

Existing methods of the solution of initial-boundary value problems cannot be applied to the telegraph equation in cases when the domain in which the solution is found is a variable. In particular, use for this purpose of a method separation of variables is impossible, as in case of mobile borders the corresponding problem of Sturm-Liouville has only trivial eigenfunctions.

In present paper the method especially developed in [6–12] for solution of problems about movement of waves in domain with mobile borders is used.

2. Statement of the Problem

The following initial-boundary value problem is considered: in the domain $0 < x < l + v(t)$, $t > 0$ to obtain the solution of the telegraph equation

$$\frac{\partial^2 u(x,t)}{\partial x^2} - \frac{1}{a^2}\frac{\partial^2 u(x,t)}{\partial t^2} + D\frac{\partial u(x,t)}{\partial t} + B\frac{\partial u(x,t)}{\partial x} + Cu(x,t) = 0, \tag{2.1}$$

which satisfies initial conditions

$$u(x,0) = 0; \qquad u_t(x,0) = 0 \tag{2.2}$$

and mixed boundary conditions

$$u_x(l + v(t), t) = \gamma(t); \quad u(0,t) = 0, \quad t > 0. \tag{2.3}$$

Concerning function $v(t)$, describing displacement of the bottom end of a rope, it is supposed $v(0) = 0$ and from a condition of preservation of integration area of an initial-boundary value problem follows that $v(t) > -l$ at $t > 0$.

The solution of the problem put here cannot be carried out by existing methods because domain, in which the solution is found, is a variable. Therefore for the solution of such initial-boundary problems connected with the equations of hyperbolic type in [6–12] special method developed. This method has three prominent features. The first of them consists in integrated representation of solutions of the telegraph equation in the form of extending waves for an extensive class of boundary conditions [6]. Such representation is obtained by use of Riemann's method.

Use of the given integrated representation of solutions demands performance of continuation of initial and regional functions in the domain of any values of their arguments. This continuation should be carried out taking into account all conditions of statement of a problem. In it the second feature of a method consists.

At last, the third feature consists in working out of a method of construction of the waves reflected from mobile border. The given method reduces a reflexing problem to the solution of an auxiliary initial-boundary value problem with the initial conditions set at the moment of arrival of forward front of the falling wave on mobile border.

This method is used for the solution of stated problem.

3. The Solution of the Problem

For the solution of this problem the continuation of function $\gamma(t)$ on all axis t is introduced as

$$\Gamma(t) = \begin{cases} \gamma(t), & t > 0; \\ 0, & t < 0, \end{cases} \tag{3.1}$$

and the first boundary condition (2.3) is considered as continued on all axis t:

$$u_x(l + v(t), t) = \Gamma(t). \tag{3.2}$$

At the first stage the solution of the given problem is searched as

$$u_0(x,t) = 2e^{-(B/2)x} \int_0^{t+(x/a)} J_0(z) e^{(Da^2/2)(t-\eta)} \Gamma_0(\eta) d\eta, \tag{3.3}$$

with unknown function Γ_0. Here J_0, J_1-Bessel's functions as the zero and first order, respectively,

$$\begin{gathered} z = \sqrt{c_1 \left[x^2 - a^2 (t-\eta)^2\right]}; \\ c_1 = C + \frac{D^2 a^2}{4} - \frac{B^2}{4}. \end{gathered} \tag{3.4}$$

In [10] it is shown that function (3.3) satisfies (2.1) at arbitrary function Γ_0. In the same place it is shown that for solution of boundary problems with boundary conditions of the second type it is the most expedient to apply the form of the solution of a kind (3.3). Having substituted the form of the solution (3.3) in a boundary condition (3.2) we obtain

$$\begin{aligned} &\frac{2}{a} e^{-((B+Da)/2)(l+v(t))} \Gamma_0\left(t + \frac{l+v(t)}{a}\right) \\ &\quad - 2e^{-(B/2)(l+v(t))} \int_0^{t+((l+v(t))/a)} \left[\frac{B}{2} J_0(z) + c_1 (l + v(t)) \frac{J_1(z)}{z}\right] e^{(Da^2/2)(t-\eta)} \Gamma_0(\eta) d\eta = \Gamma(t). \end{aligned} \tag{3.5}$$

In (3.5)

$$z = \sqrt{c_1 \left[(v(t) + l)^2 - a^2 (t-\eta)^2\right]}. \tag{3.6}$$

Thus, if function Γ_0 is the solution of the integral equation (3.5), function (3.3) will satisfy the first boundary condition (2.3).

With the purpose of obtaining an opportunity to represent a function Γ_0 with various arguments we shall introduce into (3.5) such transformation of a variable t:

$$\tau = t + \frac{v(t) + l}{a}. \tag{3.7}$$

In order that execution of transformation (3.7) in the integral equation (3.5) was possible, it is necessary that the function t_0 inverse to τ exists. Here the case is considered when the mobile end moves with subsonic speed. That means the next condition is satisfied

$$\left|v'(t)\right| < a. \tag{3.8}$$

Then from (3.7) follows

$$\frac{d\tau}{dt} = 1 + \frac{v'(t)}{a} > 0. \tag{3.9}$$

It means that (3.7) will be strictly monotonously growing and consequently there will be an inverse to them, function t_0, also strictly monotonously growing. Thus as $\tau(0) = l/a$, we obtain that $t_0(l/a) = 0$. From (3.9) follows that $\tau > l/a$ at $t > 0$. As function $v(t)$ is determined only at $t > 0$, function $\tau(t)$ is determined also only at $t > 0$. Accordingly function $t_0(\tau)$ will be determined only at $\tau > l/a$. At the same time during construction of the solution of a considered initial-boundary value problem there is a necessity of knowledge of function $\Gamma_0(\tau)$ behavior as well at values of argument $\Gamma_0(\tau)$, smaller than l/a.

With this purpose it is necessary to execute continuation of function $v(t)$ on all axis t. It appears that continuation of function $v(t)$ on all axes t can be executed by arbitrary way, having demanded only existence of a derivative of this continuation on all axes t and performance on all axes t condition (3.8). We shall designate this continuation of function $v(t)$ through $v_1(t)$. Then on all axes t such function will be determined:

$$N(t) = \begin{cases} v(t), & t > 0; \\ v_1(t), & t < 0. \end{cases} \tag{3.10}$$

Continued on all axes t of function $\tau(t)$, we shall designate $T(t)$ and we shall determine it by expression

$$T(t) = t + \frac{N(t) + l}{a}. \tag{3.11}$$

From this expression and (3.10) it is clear that at $t > 0$, $T(t) = \tau(t)$. As function $N(t)$ satisfies to an inequality

$$\left|N'(t)\right| < a \tag{3.12}$$

at all t, function $T(t)$ will be strictly monotonously growing and as $\tau(0) = l/a$, at $t < 0$ it will be valid such inequality: $T(t) < l/a$.

As function $T(t)$ is strictly monotonous at all t, there exists inverse to this the function $T_0(T)$, and at $T \geq l/a$, $T_0(T) = T_0(\tau)$ and $T_0(T)$ will be strictly monotonously growing function. Thus, function $T_0(T)$ satisfies a condition

$$T_0(T) = \begin{cases} t_0(\tau) > 0, & T > \dfrac{l}{a}; \\ 0, & T = \dfrac{l}{a}; \\ < 0, & T < \dfrac{l}{a}. \end{cases} \tag{3.13}$$

Now after transformation (3.11) integral equation (3.5) will become

$$\begin{aligned} &\frac{1}{a} e^{-((B+Da)/2)(l+N(T_0(T)))} \Gamma_0(T) \\ &\quad - e^{-(B/2)(l+N(T_0(T)))} \int_0^T \left[\frac{B}{2} J(z) + c_1 (l + N(T_0(T))) \frac{J_1(z)}{z} \right] e^{(Da^2/2)(t-\eta)} \Gamma_0(\eta) d\eta = \frac{1}{2} \Gamma(t). \end{aligned} \tag{3.14}$$

In integral equation (3.14) becomes

$$z = \sqrt{c_1 \left[(l + N(T_0(T)))^2 - a^2 (T_0(T) - \eta)^2 \right]}. \tag{3.15}$$

From the integral equation (3.14), the quality (3.1) of function $\Gamma(t)$ and an equality (3.13) follow, that is

$$\Gamma_0(T) = 0, \quad T < \frac{l}{a}. \tag{3.16}$$

In turn, from the quality (3.16) of function $\Gamma_0(T)$ follows (3.3) satisfing initial conditions (2.2). Really, from the formula (3.3) directly follows that at $t = 0$ upper limit of integration becomes equal to x/a. But at $t = 0$,$x < l$ and on the basis of the quality (3.16) of function $\Gamma_0(T)$ follows that at $t = 0$ function (3.3) will be equal to zero. That means it satisfies the first initial condition (2.2).

Having differentiated function (3.3) on t, we shall obtain

$$\begin{aligned} \frac{\partial u_0(x,t)}{\partial t} &= \frac{2}{a} e^{-((B+Da^2)/2)x} \Gamma_0\left(t + \frac{x}{a}\right) \\ &\quad + 2a^2 e^{-(B/2)x} \int_0^{t+(x/a)} \left[\frac{D}{2} J_0(z) + c_1 (t - \eta) \frac{J_1(z)}{z} \right] e^{(Da^2/2)(t-\eta)} \Gamma_0(\eta) d\eta. \end{aligned} \tag{3.17}$$

If in the formula (3.17) we set $t = 0$, then argument of function Γ_0 and also the upper limit of integration at $x < l$ become smaller than l/a. Therefore on the basis of function's Γ_0

quality (3.16) the derivative (3.17) at $t = 0$ will equal to zero. And it means that function (3.3) will satisfy also the second initial condition (2.2).

Thus, function (3.3) satisfies all conditions of statement of an initial-boundary value problem, except for the second boundary condition (2.3). With the purpose to check this condition we shall calculate from (3.3)

$$u_0(0,t) = 2\int_0^t J_0(z)e^{(Da^2/2)(t-\eta)}\Gamma_0(\eta)d\eta. \tag{3.18}$$

From the formula (3.18) on the basis of function's Γ_0 quality (3.16) follows the function (3.3) at $t < l/a$ satisfing the second boundary condition (2.3) as well. With the purpose of satisfaction of the second boundary condition (2.3) at $t > l/a$, the solution of an initial-boundary value problem we have to search as the sum of two functions is

$$u(x,t) = u_0(x,t) + u_1(x,t), \tag{3.19}$$

where

$$u_1(x,t) = -2e^{-(B/2)x}\int_0^{t-(x/a)} J_0(z)e^{(Da^2/2)(t-\eta)}\Gamma_0(\eta)d\eta. \tag{3.20}$$

Function (3.20) satisfies (2.1) with arbitrary function Γ_0. It is obvious the function (3.19) satisfies the second boundary condition (2.3) at all t. In the same way as for function $u_0(x,t)$ it is possible to check up that the function (3.19) satisfies initial conditions (2.2).

In order that (3.19) satisfies the first boundary condition (2.3) it is necessary the function $u_1(x,t)$ satisfies a boundary condition

$$u_{1,x}(l+v(t),t) = 0, \quad t > 0. \tag{3.21}$$

Having calculated value of function $u_{1,x}(x,t)$ at point $x = l + v(t)$, we obtain

$$\begin{aligned} &-\frac{2}{a}e^{-((B+Da)/2)(l+v(t))}\Gamma_0\left(t-\frac{l+v(t)}{a}\right) \\ &+2e^{-(B/2)(l+v(t))}\int_0^{t-((l+v(t))/a)}\left[\frac{B}{2}J_0(z)+c_1(l+v(t))\frac{J_1(z)}{z}\right]e^{(Da^2/2)(t-\eta)}\Gamma_0(\eta)d\eta = 0. \end{aligned} \tag{3.22}$$

In formula (3.22)

$$z = \sqrt{c_1\left[(v(t)+l)^2 - a^2(t-\eta)^2\right]}. \tag{3.23}$$

In order that equality (3.22) was valid, on the basis of function's Γ_0 quality (3.16), it is necessary that the argument of this function in equality (3.22) satisfies condition

$$t - \frac{l+v(t)}{a} < \frac{l}{a}. \tag{3.24}$$

Thus, function (3.19) will be the solution of a problem at $t < (2l + v(t))/a$. At $t > (2l + v(t))/a$ for satisfaction of first boundary condition (2.3) in the solution (3.19) it is necessary to enter the amendment. To enter such amendment by the methods used for domains with motionless borders, as shown in [7–12] for a case of the wave equation, is impossible. Therefore for corrective action the approach developed in [7–12] is used. With this purpose we shall notice that during an interval of time determined by a condition $t = (v(t) + 2l)/a$, forward front of the wave, radiated on the mobile end, starting from the moment of time $t = 0$, will reach the end $x = 0$, will be reflected from it, and will meet the mobile end. The length of this interval of time will be determined as the less positive root τ_1 of equation

$$at = v(t) + 2l. \tag{3.25}$$

The left part of (3.25) at $t = 0$ is less than the right part. At the same time on the basis of a condition (3.8) at $t > 0$ left part of this equation grows faster than right part. Hence, the positive root of (3.25) exists. From the carried out reasoning it is clear also that the condition (3.22) will be valid at $t < \tau_1$. Really, at $t = 0$ inequality $t < (v(t) + 2l)/a$ is valid, as $v(0) = 0$. At the same time τ_1 is the less positive number at which this inequality turns into equality (3.25). Therefore correction function $u_2(x, t)$, being in essence of a wave reflected from the mobile end, is under construction as the solution of such an auxiliary initial-boundary value problem: in the domain $0 < x > l + v(t)$, $t > \tau_1$ to obtain the solution of the telegraph equation (2.1) satisfying initial conditions

$$u(x, \tau_1) = 0; \quad u_t(x, \tau_1) = 0; \quad 0 < x < l + v(\tau_1), \tag{3.26}$$

and boundary conditions

$$u_x(l + v(t), t) = -u_{1,x}(l + v(t), t); \quad u(0, t) = 0, \quad t > \tau_1. \tag{3.27}$$

The solution of this auxiliary initial-boundary value problem is constructed as function satisfying (2.1) at arbitrary function Γ_2:

$$u_2(x, t) = 2e^{-(B/2)x} \int_0^{t+(x/a)} J_0(z) e^{(Da^2/2)(t-\eta)} \Gamma_2(\eta) d\eta. \tag{3.28}$$

Having substituted function (3.28) in the first boundary condition (3.27), we obtain

$$\begin{aligned} &\frac{2}{a}\Gamma_2\left(t + \frac{v(t) + l}{a}\right) e^{-((Da+B)/2)(l+v(t))} \\ &\quad - 2e^{-(B/2)(l+v(t))} \int_0^{t+((v(t)+l)/a)} \left[\frac{B}{2} J_0(z) + c_1 \frac{l + v(t)}{z} J_1(z)\right] e^{(Da^2/2)(t-\eta)} \Gamma_2(\eta) d\eta \\ &\quad = -u_{1,x}(l + v(t), t). \end{aligned} \tag{3.29}$$

Thus, if function Γ_2 will be the solution of the integral equation (3.29), the function

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t), \tag{3.30}$$

will satisfy the second boundary condition (2.3) at all t. Having executed in (3.29) transformation (3.11), we shall obtain

$$\begin{aligned}
&2\Gamma_2(T)e^{-((Da+B)/2)(l+v(T_0(T)))} \\
&\quad - 2ae^{-(B/2)(l+v(T_0(T)))}\int_0^T \left[\frac{B}{2}J_0(z) + c_1\frac{l+v(T_0(T))}{z}J_1(z)\right]e^{(Da^2/2)(T_0(T)-\eta)}\Gamma_2(\eta)d\eta \\
&\quad = -au_{1,x}(l+v(T_0(T)),T_0(T)).
\end{aligned} \tag{3.31}$$

As the right part of the integral equations (3.29) and (3.31) is equal to zero at $t < \tau_1$, function $\Gamma_2(T)$ also will be equal to zero at $t < \tau_1$. We shall find out, which additional values of T function $\Gamma_2(T)$ will be equal to zero. As function $T(t)$ strictly monotonously grows, such inequality will be valid

$$T(t) < T(\tau_1), \quad \text{at } t < \tau_1. \tag{3.32}$$

Using in this inequality the definition (3.11) of function $T(t)$ and value τ_1 from (3.25), we shall obtain

$$T < \tau_1 + \frac{v(\tau_1)+l}{a} = \frac{3l+2v(\tau_1)}{a}. \tag{3.33}$$

Thus, function $\Gamma_2(T)$ possesses the following quality:

$$\Gamma_2(T) = 0, \quad T < \frac{3l+2v(\tau_1)}{a}. \tag{3.34}$$

If now to put in equality (3.28) $t = 0$, the upper limit of integration in this formula will accept value x/a. Taking into account that at $t = 0$ it is valid $x < l$, we shall have that $x/a < l/a$. But as $l > v(\tau_1)$, we shall obtain that

$$\frac{l}{a} < \frac{3l+2v(\tau_1)}{a}. \tag{3.35}$$

It means that at $t = 0$ function (3.28) will equal to zero, that is, will satisfy the first initial condition (2.2).

Having calculated a derivative of function (3.28) on t, we shall obtain

$$\begin{aligned}
\frac{\partial u_2(x,t)}{\partial t} &= \frac{2}{a}e^{-((B+Da^2)/2)x}\Gamma_2\left(t+\frac{x}{a}\right) \\
&\quad + 2a^2e^{-(B/2)x}\int_0^{t+(x/a)}\left[\frac{D}{2}J_0(z) + c_1(t-\eta)\frac{J_1(z)}{z}\right]e^{(Da^2/2)(t-\eta)}\Gamma_2(\eta)d\eta.
\end{aligned} \tag{3.36}$$

At $t = 0$ argument of function Γ_2 and also the upper limit of integration in the formula (3.36) will accept value x/a. Hence, as shown above, in the domain of search of the solution at such values of argument, the function Γ_2 will be equal to zero. Therefore, the derivative of function $u_2(x,t)$ on t at $t = 0$ will be equal to zero. And it means that function $u_2(x,t)$ satisfies as well the second initial condition (2.2).

Thus, function (3.30) satisfies all conditions of statement of the basic initial-boundary value problem, except for the second boundary condition (2.3). In order that this boundary condition was carried out, it is necessary that there was valid an equality

$$u_{2,x}(0,t) = 0, \quad t > 0. \tag{3.37}$$

Having substituted function (3.28) in the left part of a boundary condition (3.37), we shall obtain

$$u_{2,x}(0,t) = \frac{2}{a}\Gamma_2(t) - 2\int_0^t \frac{B}{2} J_0(z) e^{(Da^2/2)(t-\eta)}\Gamma_2(\eta)\,d\eta. \tag{3.38}$$

As follows from function's Γ_2 quality (3.34), expression (3.38) will equal zero, that is, will satisfy a boundary condition (3.37) only at validity of an inequality $t < (3l + 2v(\tau_1))/a$. For satisfaction of the second boundary condition at big t in the solution (3.30), the amendment $u_3(x,t)$ is entered that is, the solution is represented as

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t), \tag{3.39}$$

where

$$u_3(x,t) = -2e^{-(B/2)x}\int_0^{t-(x/a)} J_0(z) e^{(Da^2/2)(t-\eta)}\Gamma_2(\eta)\,d\eta. \tag{3.40}$$

Function $u_3(x,t)$ satisfies (2.1) at arbitrary function Γ_2 and should provide performance of a boundary condition

$$u_2(0,t) + u_3(0,t) = 0, \quad t > 0. \tag{3.41}$$

Fact that functions (3.28) and (3.40) satisfy a boundary condition (3.41) is practically obvious.

With the same way, as it is made for function $u_2(x,t)$, it is possible to show that function $u_3(x,t)$ will satisfy initial conditions (2.2).

Thus, function (3.39) satisfies all conditions of statement of the basic initial-boundary value problem, except for the first boundary condition (2.3). In order that this boundary condition was carried out, it is necessary that there was valid an equality

$$u_{3,x}(l + v(t), t) = 0, \quad t > 0. \tag{3.42}$$

Having calculated value of derivative function (3.40) on x in a point $x = l + v(t)$, we shall obtain

$$u_{3,x}(l+v(t),t) = \frac{2}{a}e^{((Da-B)/2)(l+v(t))}\Gamma_2\left(t - \frac{v(t)+l}{a}\right) + 2e^{-(B/2)(l+v(t))}\int_0^{t-((v(t)+l)/a)}\left[\frac{B}{2}J_0(z) + c_1\frac{l+v(t)}{z}J_1(z)\right]e^{(Da^2/2)(t-\eta)}\Gamma_2(\eta)\,d\eta. \tag{3.43}$$

On the basis of function's Γ_2 quality (3.34) one can conclude that the right part of equality (3.43) will be equal to zero if argument of function Γ_2 and the upper limit of integration in the formula (3.43) will satisfy an inequality

$$t - \frac{l+v(t)}{a} < \frac{3l+2v(\tau_1)}{a}, \tag{3.44}$$

whence follows

$$t < \frac{4l+2v(\tau_1)+v(t)}{a}. \tag{3.45}$$

Hence, as t satisfies an inequality (3.45), the boundary condition (3.42) will be carried out. The inequality (3.45) is inconvenient for use as its left and right parts depend on t. With the purpose of more convenient use of this inequality we shall consider the equation

$$t = \frac{4l+2v(\tau_1)+v(t)}{a}. \tag{3.46}$$

Also we shall designate as τ_2 the less positive root of this equation. At $t = 0$ right part of (3.46) is more than the left part. At the same time by virtue of a condition (3.8) right part of (3.46) grows faster than its left part; therefore, the positive root of (3.46) exists. Hence, number τ_2 is the less positive number at which the inequality (3.45) terns into equality. Therefore the inequality (3.45) is equivalent to the inequality

$$t < \frac{4l+2v(\tau_1)+v(\tau_2)}{a} = \tau_2. \tag{3.47}$$

Let us notice that on physical sense of an initial-boundary value problem at the moment of time $t = \tau_2$ forward front of the wave radiated with the mobile end, having twice reflected from the motionless end and having once reflected from the mobile end, it will meet again the mobile end. At $t > \tau_2$ function $u_3(x,t)$ (3.40) will not satisfy any more to a boundary condition (3.42). Therefore at $t > \tau_2$ solution of the basic initial-boundary value problem is searched as

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + u_4(x,t), \tag{3.48}$$

where function $u_4(x,t)$ is under construction as the solution of the following auxiliary initial-boundary value problem. In the domain $0 < x < l + v(t)$, $t > \tau_2$ to obtain the solution of the telegraph equation (2.1) satisfying initial conditions

$$u(x,\tau_2) = 0; \quad u_t(x,\tau_2) = 0; \quad 0 < x < l + v(\tau_2), \tag{3.49}$$

and boundary conditions

$$u_x(l + v(t), t) = -u_{3,x}(l + v(t), t); \quad u_x(0,t) = 0, \qquad t > \tau_2. \tag{3.50}$$

The solution of this auxiliary initial-boundary value problem is under construction as the function being the solution of (2.1) at arbitrary function Γ_4 is

$$u_4(x,t) = 2e^{-(B/2)x} \int_0^{t+(x/a)} J_0(z) e^{(Da^2/2)(t-\eta)} \Gamma_4(\eta) d\eta. \tag{3.51}$$

Having substituted function (3.51) in the first boundary condition (3.50), we shall obtain

$$\begin{aligned}
&\frac{2}{a}\Gamma_4\left(t + \frac{v(t)+l}{a}\right) e^{-((Da+B)/2)(l+v(t))} \\
&\quad - 2e^{-(B/2)(l+v(t))} \int_0^{t+((v(t)+l)/a)} \left[\frac{B}{2}J_0(z) + c_1 \frac{l+v(t)}{z} J_1(z)\right] e^{(Da^2/2)(t-\eta)} \Gamma_4(\eta) d\eta \\
&\quad = -u_{3,x}(l + v(t), t).
\end{aligned} \tag{3.52}$$

Thus, if function Γ_4 will be the solution of the integral equation (3.52) the function (3.48) will satisfy the second boundary condition (2.3) at all t. Having executed in (3.52) transformation (3.11), we shall obtain

$$\begin{aligned}
&2\Gamma_4(T) e^{-((Da+B)/2)(l+v(T_0(T)))} \\
&\quad - 2ae^{-(B/2)(l+v(T_0(T)))} \int_0^T \left[\frac{B}{2}J_0(z) + c_1 \frac{l+v(T_0(T))}{z} J_1(z)\right] e^{(Da^2/2)(T_0(T)-\eta)} \Gamma_4(\eta) d\eta \\
&\quad = -au_{3,x}(l + v(T_0(T)), T_0(T)).
\end{aligned} \tag{3.53}$$

As the right part of the integral equations (3.52) and (3.53) is equal to zero at $t < \tau_2$, function $\Gamma_4(T)$ also will be equal to zero at $t < \tau_2$. We shall find out at what additional values T function $\Gamma_4(T)$ will be equal to zero. As function $T(t)$ strictly monotonously grows, the next inequality will be valid

$$T(t) < T(\tau_2), \quad \text{at } t < \tau_2. \tag{3.54}$$

Using in this inequality definition (3.11) of function $T(t)$ and value τ_2 from (3.52), we shall obtain

$$T < \tau_2 + \frac{v(\tau_2) + l}{a} = \frac{5l + 2v(\tau_1) + 2v(\tau_2)}{a}. \tag{3.55}$$

Thus, function $\Gamma_4(T)$ possesses the following quality:

$$\Gamma_4(T) = 0, \quad T < \frac{5l + 2v(\tau_1) + 2v(\tau_1)}{a}. \tag{3.56}$$

Precisely the same as it is made for function $u_2(x,t)$, it is possible to show that function $u_4(x,t)$ will satisfy initial conditions (2.2). Thus, function (3.48) satisfies all conditions of statement of the basic initial-boundary value problem, except for the second boundary condition (2.3). This boundary condition function $u_4(x,t)$ will satisfy only at the some of values of $t > \tau_2$. To obtain the solution of the basic initial-boundary value problem at all $t > \tau_2$, (3.48) it is necessary to introduce the additional amendment into function (3.48).

Having continued process of corrective actions in the solution, we shall obtain that function

$$\begin{aligned} u(x,t) &= \sum_{n=0}^{\infty} 2e^{-(B/2)x} \int_0^{t+(x/a)} J_0(z) e^{(Da^2/2)(t-\eta)} \Gamma_{2n}(\eta)\, d\eta \\ &\quad - \sum_{n=0}^{\infty} 2e^{-(B/2)x} \int_0^{t-(x/a)} J_0(z) e^{(Da^2/2)(t-\eta)} \Gamma_{2n}(\eta)\, d\eta \end{aligned} \tag{3.57}$$

will be the solution of a considered initial-boundary value problem. Here function Γ_0 is the solution of the integral equation (3.14), and other functions Γ_{2n} are solutions of the following integral equations:

$$\begin{aligned} &2\Gamma_{2n}(T) e^{-((Da+B)/2)(l+v(T_0(T)))} \\ &\quad - 2ae^{-(B/2)(l+v(T_0(T)))} \int_0^T \left[\frac{B}{2} J_0(z) + c_1 \frac{l + v(T_0(T))}{z} J_1(z) \right] e^{(Da^2/2)(T_0(T)-\eta)} \Gamma_{2n}(\eta)\, d\eta \\ &\quad = -au_{2n-1,x}(l + v(T_0(T)), T_0(T)). \end{aligned} \tag{3.58}$$

Here

$$\begin{aligned} &u_{2n-1,x}(l + v(t), t) \\ &\quad = \frac{2}{a} e^{((Da-B)/2)(l+v(t))} \Gamma_{2n-2}\left(t - \frac{v(t) + l}{a} \right) \\ &\qquad + 2e^{-(B/2)(l+v(t))} \int_0^{t-((v(t)+l)/a)} \left[\frac{B}{2} J_0(z) + c_1 \frac{l + v(t)}{z} J_1(z) \right] e^{(Da^2/2)(t-\eta)} \Gamma_{2n-2}(\eta)\, d\eta. \end{aligned} \tag{3.59}$$

Thus functions Γ_{2n} possess the following qualities:

$$\Gamma_{2n}(t) = 0, \quad t < \frac{(2n+1)l + 2\sum_{i=1}^{n} v(\tau_i)}{a}, \ n = 0, 1, \ldots, \tag{3.60}$$

where τ_n is less positive root of equation

$$t = \frac{2nl + 2\sum_{i=1}^{n-1} v(\tau_i) + v(t)}{a}. \tag{3.61}$$

By virtue of these qualities, at everyone fixed $t = H$ in the formula (3.57) will be only final number of terms distinct from zero. Really, in the sums of the formula (3.57) each term under conditions (3.56) becomes equal to zero, if the upper limit of integration is less than the right part of an inequality (3.60). For the first sum of the formula (3.57) such condition at $t = H$ looks like

$$H + \frac{x}{a} < \frac{(2n+1)l + 2\sum_{i=1}^{n} v(\tau_i)}{a}, \tag{3.62}$$

whence follows

$$n > \frac{1}{2l}\left(Ha + x - l - 2\sum_{i-1}^{n} v(\tau_i)\right). \tag{3.63}$$

And as in the domain of obtaining the solution, next inequality is valid:

$$0 < x < l + v(H), \tag{3.64}$$

and we obtain that at all n, satisfying a condition

$$n > \frac{1}{2l}\left(Ha + v(H) - 2\sum_{i-1}^{n} v(\tau_i)\right), \tag{3.65}$$

all terms in the first sum of formula (3.57) will be equal to zero. Differently, summation in the first sum of the formula (3.57) needs to be made in this case not up to infinity, but up to $N-1$, where N is the less natural number satisfying an inequality (3.65).

For the second sum of the formula (3.57), condition that the upper limit of integration is less than right part of inequality (3.60) at $t = H$ looks like

$$H - \frac{x}{a} < \frac{(2n+1)l + 2\sum_{i=1}^{n} v(\tau_i)}{a}, \tag{3.66}$$

from which follows

$$n > \frac{1}{2l}\left(Ha - x - l - 2\sum_{i-1}^{n} v(\tau_i)\right). \tag{3.67}$$

Therefore on the basis of an inequality (3.64) it is obtained that at all n, satisfying a condition

$$n > \frac{1}{2l}\left(Ha - l - 2\sum_{i-1}^{n} v(\tau_i)\right), \tag{3.68}$$

all terms in the second sum of formula (3.57) will be equal to zero. Differently, summation in the second sum of the formula (3.57) needs to be made in this case not up to infinity, but up to $N_1 - 1$, where N_1 is the less natural number satisfying inequality (3.68).

All terms in the formula (3.57) are solutions of (2.1). And as for everyone fixed t number of terms in the formula (3.57) is finite, differentiation in the formula (3.57) is possible to carry out term by term. Therefore function (3.57) is the solution of (2.1).

From the formula (3.57) directly follows that at $t = 0$ and $0 < x < l$ the upper limits of integration of all integrals become smaller, than l/a. It means that on the basis of function's Γ_k qualities, (3.60) from (3.57) follows $u(x, 0) = 0$. Thus, function (3.57) satisfies the first initial condition (2.2). Having differentiated function (3.57) on t we obtain

$$\begin{aligned}
\frac{\partial u(x,t)}{\partial t} = \sum_{n=0}^{\infty} \Bigg\{ & \frac{2}{a} e^{-((B+Da^2)/2)x} \Gamma_{2n}\left(t + \frac{x}{a}\right) \\
& +2a^2 e^{-(B/2)x} \int_0^{t+(x/a)} \left[\frac{D}{2} J_0(z) + c_1 (t-\eta) \frac{J_1(z)}{z}\right] e^{(Da^2/2)(t-\eta)} \Gamma_{2n}(\eta) d\eta \Bigg\} \\
+ \sum_{n=0}^{\infty} \Bigg\{ & \frac{2}{a} e^{((Da-B)/2)x} \Gamma_{2n}\left(t - \frac{x}{a}\right) \\
& -2a^2 e^{-(B/2)x} \int_0^{t-(x/a)} \left[\frac{D}{2} J_0(z) + c_1 \frac{t-\eta}{z} J_1(z)\right] e^{(Da^2/2)(t-\eta)} \Gamma_{2n}(\eta) d\eta \Bigg\}.
\end{aligned} \tag{3.69}$$

From the formula (3.69) it is obtained that at $t = 0$ and $0 < x < l$ the upper limits of integration of all integrals and as well arguments of functions Γ_k become smaller, than l/a. It means, on the basis of function's Γ_k qualities (3.60), that $u_t(x, 0) = 0$. Thus, function (3.57) satisfies also the second initial condition (2.2).

Having calculated value of derivative of function (3.57) on x in point $x = l + v(t)$, we obtain

$$\begin{aligned}
& u_x(l + v(t), t) \\
& = \sum_{n=0}^{\infty} \Bigg\{ \frac{2}{a} \Gamma_{2n}\left(t + \frac{l + v(t)}{a}\right) e^{-((Da+B)/2)(l+v(t))} \\
& \qquad -2e^{-(B/2)(l+v(t))} \int_0^{t+((l+v(t))/a)} \left[\frac{B}{2} J_0(z) + c_1 \frac{l + v(t)}{z} J_1(z)\right] e^{(Da^2/2)(t-\eta)} \Gamma_{2n}(\eta) d\eta \Bigg\}
\end{aligned}$$

$$+\sum_{n=0}^{\infty}\left\{\frac{2}{a}e^{((Da-B)/2)(l+v(t))}\Gamma_{2n}\left(t-\frac{l+v(t)}{a}\right)\right.$$
$$\left.-2e^{-(B/2)(l+v(t))}\int_0^{t-((l+v(t))/a)}\left[\frac{B}{2}J_0(z)+c_1\frac{l+v(t)}{z}J_1(z)\right]e^{(Da^2/2)(t-\eta)}\Gamma_{2n}(\eta)d\eta\right\}. \tag{3.70}$$

In the formula (3.70) we shall write down the first term of the first sum (at $n = 0$) separately, and in the second sum we shall replace an index of summation n on $s = n+1$. We shall obtain

$$\begin{aligned}
&u_x(l+v(t),t)\\
&\quad=\frac{2}{a}\Gamma_0\left(t+\frac{l+v(t)}{a}\right)e^{-((Da+B)/2)(l+v(t))}\\
&\qquad-2e^{-(B/2)(l+v(t))}\int_0^{t+((l+v(t))/a)}\left[\frac{B}{2}J_0(z)+c_1\frac{l+v(t)}{z}J_1(z)\right]e^{(Da^2/2)(t-\eta)}\Gamma_0(\eta)d\eta\\
&\qquad+\sum_{n=1}^{\infty}\left\{\frac{2}{a}\Gamma_{2n}\left(t+\frac{l+v(t)}{a}\right)e^{-((Da+B)/2)(l+v(t))}\right.\\
&\qquad\qquad\left.-2e^{-(B/2)(l+v(t))}\int_0^{t+((l+v(t))/a)}\left[\frac{B}{2}J_0(z)+c_1\frac{l+v(t)}{z}J_1(z)\right]e^{(Da^2/2)(t-\eta)}\Gamma_{2n}(\eta)d\eta\right\}\\
&\qquad+\sum_{s=1}^{\infty}\left\{\frac{2}{a}e^{((Da-B)/2)(l+v(t))}\Gamma_{2s-2}\left(t-\frac{l+v(t)}{a}\right)\right.\\
&\qquad\qquad\left.-2e^{-(B/2)(l+v(t))}\int_0^{t-((l+v(t))/a)}\left[\frac{B}{2}J_0(z)+c_1\frac{l+v(t)}{z}J_1(z)\right]e^{(Da^2/2)(t-\eta)}\Gamma_{2s-2}(\eta)d\eta\right\}.
\end{aligned} \tag{3.71}$$

First two summands in this formula represent the left part of the integral equation (3.5) and consequently are equal to $\Gamma(t)$. Summands in the first sum of last formula represent the left parts of the integral equations (3.58) divided on a. It means they are equal to $-u_{(2n-1)x}(l+v(t),t)$. Summands in the second sum represent functions $u_{(2n-1)x}(l+v(t),t)$. Therefore all terms under signs Σ in last formula will equal zero. Hence

$$u_x(l+v(t),t)=\Gamma(t)=\gamma(t),\quad t>0. \tag{3.72}$$

And it means that function (3.57) at $t > 0$ satisfies the first boundary condition (2.3). The fact that function (3.57) satisfies the second boundary condition (2.3) is obvious.

Thus, it is shown that function (3.57) satisfing all conditions of statement of the basic initial-boundary value problem consequently is its solution.

4. Conclusion

The exact solution of the mixed initial-boundary value problem for the telegraph equation in domain with mobile borders is obtained. The solution is obtained as superposition of an initial wave and waves of reflection from the borders of a domain. It is necessary to note that the form of the solution in accuracy corresponds to those natural phenomena which, in particular, occur in a rope during its loading. The solution of a problem represents value of a field of elastic displacements in a rope. To obtain a field of pressure in a rope, it is enough to differentiate a field of displacements on x and to increase this result on the module of elasticity of a rope.

Here, the developed method of the solution of evolutionary initial-boundary value problems with mobile borders of domain is suitable for search of solutions of a wide class of similar problems.

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