

Research Article

Approximate Analytical Solution for Nonlinear System of Fractional Differential Equations by BPs Operational Matrices

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Received 20 February 2013; Revised 6 March 2013; Accepted 7 March 2013

Academic Editor: José Tenreiro Machado

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We present two methods for solving a nonlinear system of fractional differential equations within Caputo derivative. Firstly, we derive operational matrices for Caputo fractional derivative and for Riemann-Liouville fractional integral by using the Bernstein polynomials (BPs). In the first method, we use the operational matrix of Caputo fractional derivative (OMCFD), and in the second one, we apply the operational matrix of Riemann-Liouville fractional integral (OMRLF). The obtained results are in good agreement with each other as well as with the analytical solutions. We show that the solutions approach to classical solutions as the order of the fractional derivatives approaches 1.

1. Introduction

Differential equations of fractional order have been subjected to many studies due to their frequent appearance in various applications in fluid mechanics, viscoelasticity, biology, physics, engineering, and so on. Recently, a large amount of literature was developed regarding the application of fractional differential equations in nonlinear dynamics (see, e.g., [1–11] and the references therein). Thus, a huge attention has been given to the solution of fractional ordinary differential equations, integral equations, and fractional partial differential equations of physical interest. As it is known, there exists no method that yields an exact solution for fractional differential equations. Various methods have been proposed in order to solve the fractional differential equations. These methods include the homotopy perturbation method [12–15], Adomian's decomposition method [16–20], variation iteration method [12–14, 21–23], homotopy analysis method [24], differential transform method [25], operational matrices [26–28], and nonstandard finite difference scheme [29].

In this paper, we investigate the nonlinear system of fractional differential equations as

$$\begin{aligned} D^{\alpha_i} x_i(t) &= g_i(t, X(t)), \\ i &= 1, \dots, n, \quad 0 < t \leq 1, \quad 0 < \alpha_i \leq 1, \end{aligned} \quad (1)$$

and the initial condition

$$X(0) = X_0, \quad (2)$$

where $X(t) = [x_1(t), \dots, x_n(t)]^T$ and $X_0 = [x_{0,1}, \dots, x_{0,n}]^T$. Also, $g_i : [0, 1] \times R^n \rightarrow R$ are multivariable polynomial functions.

The structure of the paper is given later. In Section 2, we present some preliminaries and properties in fractional calculus and Bernstein polynomials. In Section 3, we make operational matrices for product, power, Caputo fractional derivative, and Riemann-Liouville fractional integral by BPs. In Section 4, we apply two methods for solving nonlinear

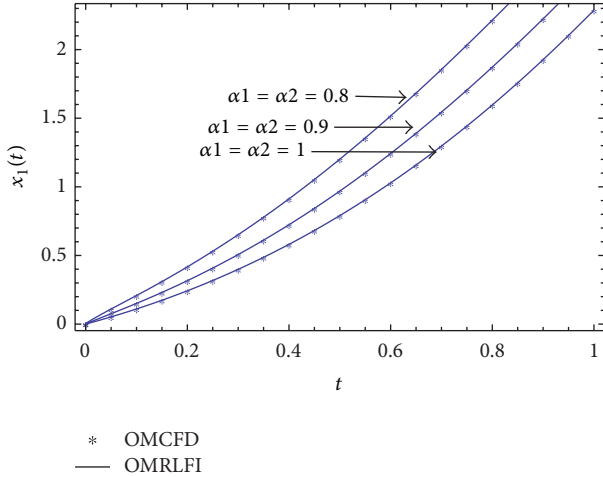


FIGURE 1: Approximate solutions of $x_1(t)$ for $m = 10$ and different values of α_1, α_2 by OMCDF and OMRLFI in Example 12.

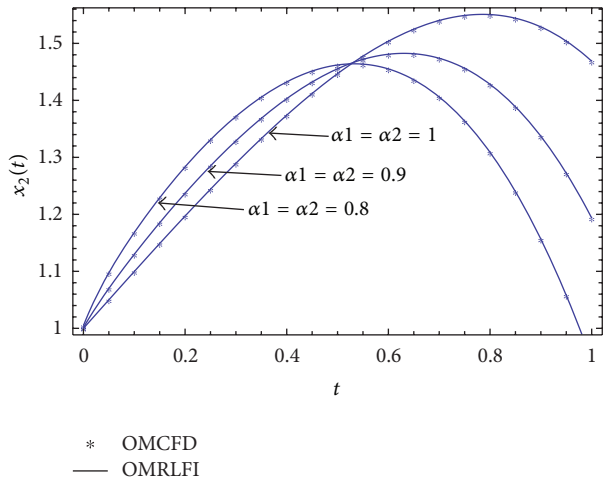


FIGURE 2: Approximate solutions of $x_2(t)$ for $m = 10$ and different values of α_1, α_2 by OMCDF and OMRLFI in Example 12.

system of fractional differential equations by BPs. In Section 5, numerical examples are simulated to demonstrate the high performance of the proposed method. Conclusions are presented in Section 6.

2. Basic Tools

In this section, we recall some basic definitions and properties of the fractional calculus and Bernstein polynomials.

Definition 1 (see [2, 7, 10]). The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, of a function $f \in C_\mu$, $\mu \geq -1$, is defined as

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-x)^{\alpha-1} f(x) dx, \quad \alpha > 0, t > 0, \quad (3)$$

$$I^0 f(t) = f(t),$$

and for $n-1 < \alpha \leq n$, $n \in \mathbb{N}$, $t > 0$, $f \in C_{-1}^n$, the fractional derivative of $f(t)$ in the Caputo sense is defined as

$$D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-x)^{n-\alpha-1} f^{(n)}(x) dx, \quad (4)$$

where for $n \in \mathbb{N}$, $\mu \in \mathbb{R}$ we have

$$C_\mu = \{f(t) \mid f(t) > 0 \text{ for } t > 0, \\ f(t) = t^p f_1(t) \text{ where } p > \mu, f_1(t) \in C[0, \infty)\}, \\ C_\mu^n = \{f(t) \mid f^{(n)}(t) \in C_\mu\}.$$

(5)

Also, if $n-1 < \alpha \leq n$, $n \in \mathbb{N}$, and $f \in C_\mu^n$, $\mu \geq -1$, then

$$(1) \quad D^\alpha I^\alpha f(t) = f(t), \quad (6)$$

$$(2) \quad I^\alpha D^\alpha f(t) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(0^+) \frac{x^k}{k!}, \quad t > 0. \quad (7)$$

Definition 2 (see [30]). The Bernstein polynomials (BPs) of m th degree are defined on the interval $[0, 1]$ as follows:

$$B_{i,m}(x) = \binom{m}{i} x^i (1-x)^{m-i}, \quad i = 0, 1, \dots, m. \quad (8)$$

Lemma 3. One can write $\Phi_m(x) = A T_m(x)$, where A is a matrix upper triangular, $T_m(x) = [1, x, \dots, x^m]^T$, and $\Phi_m(x) = [B_{0,m}(x), B_{1,m}(x), \dots, B_{m,m}(x)]^T$.

Proof. (see [26]). \square

Definition 4. We can define the dual matrix $Q_{(m+1) \times (m+1)}$ on the basis of Bernstein polynomials of m th degree as follows:

$$Q = \int_0^1 \Phi(x) \Phi(x)^T dx, \quad (9)$$

where

$$(Q)_{i+1,j+1} = \int_0^1 B_{i,m}(x) B_{j,m}(x) dx \\ = \frac{\binom{m}{i} \binom{m}{j}}{(2m+1) \binom{2m}{i+j}}, \quad i, j = 0, 1, \dots, m. \quad (10)$$

Lemma 5. Let $L^2[0, 1]$ be a Hilbert space with the inner product $\langle f, g \rangle = \int_0^1 f(x)g(x)dx$ and $y \in L^2[0, 1]$. Then, we can find the unique vector $c = [c_0, c_1, \dots, c_m]^T$ such that $c^T \Phi_m(x)$ is the best approximation of $y(x)$ from space $S_m = \text{Span}\{B_{0,m}(x), B_{1,m}(x), \dots, B_{m,m}(x)\}$. Moreover, one can get $c = Q^{-1} \langle y, \Phi_m \rangle$, such that $\langle y, \Phi_m \rangle = [\langle y, B_{0,m} \rangle, \langle y, B_{1,m} \rangle, \dots, \langle y, B_{m,m} \rangle]^T$.

Proof. (see [31]). \square

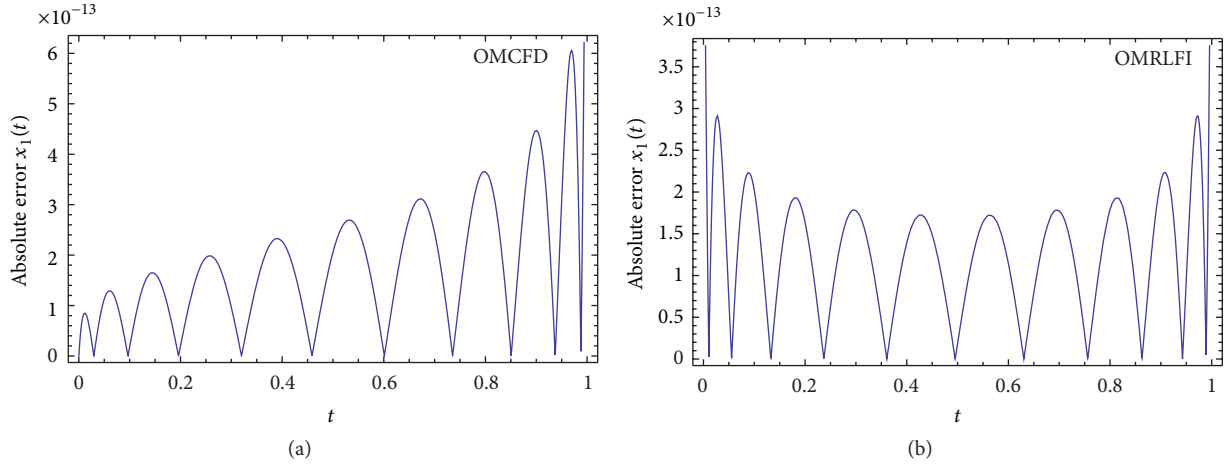


FIGURE 3: Plot of absolute error function $x_1(t)$ for $\alpha_1 = \alpha_2 = 1$ and $m = 10$ by OMCDF and OMRLFI in Example 12.

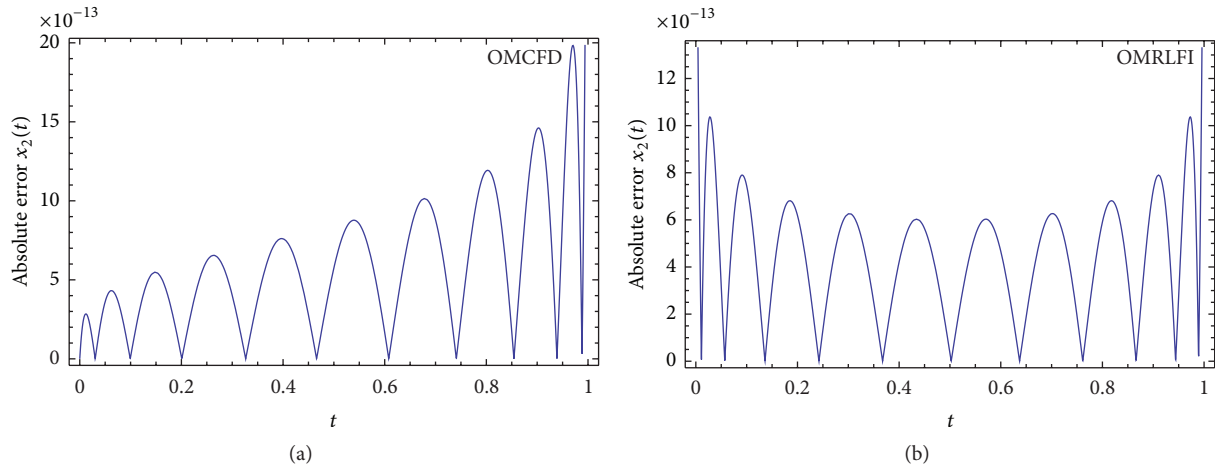


FIGURE 4: Plot of absolute error function $x_2(t)$ for $\alpha_1 = \alpha_2 = 1$ and $m = 10$ by OMCDF and OMRLFI in Example 12.

Lemma 6. Suppose that the function $y : [0, 1] \rightarrow R$ is $m + 1$ times continuously differentiable (i.e., $y \in C^{m+1}([0, 1])$). If $c^T B$ is the best approximation y out of S_m , then

$$\|y - c^T B\|_{L^2[0,1]} \leq \frac{\widehat{K}}{(m + 1)! \sqrt{2m + 3}}, \quad (11)$$

where $\widehat{K} = \max_{x \in [0,1]} |y^{(m+1)}(x)|$. Also, if $y \in C^\infty([0, 1])$, then the error bound vanishes.

Proof. (see [32]). □

3. Operational Matrices of Bernstein Polynomials

In Section 3, we recall the operational matrices for product, power, Caputo fractional derivative and Riemann-Liouville fractional integral by BPs.

Lemma 7. Suppose that $c_{(m+1) \times 1}$ is an arbitrary vector. The operational matrix of product $\widehat{C}_{(m+1) \times (m+1)}$ using BPs can be given as follows:

$$c^T \Phi_m(x) \Phi_m(x)^T \approx \Phi_m(x)^T \widehat{C}. \quad (12)$$

Proof. (see [27]). □

Corollary 8. Suppose that $y(t) \approx c^T \Phi_m(t)$, $x(t) \approx d^T \Phi_m(t)$, and $\widehat{C}_{(m+1) \times (m+1)}$ is the operational matrix of product using BPs for vector c . One can get the approximate function for $x(t)y(t)$ using BPs as follows:

$$y(t)x(t) \approx \Phi_m(t)^T \widehat{C}d. \quad (13)$$

Proof. By using Lemma 7, it is clear. □

Corollary 9. Suppose that $y(t) \approx c^T \Phi_m(t)$ and $\widehat{C}_{(m+1) \times (m+1)}$ is the operational matrix of product using BPs for vector c . One

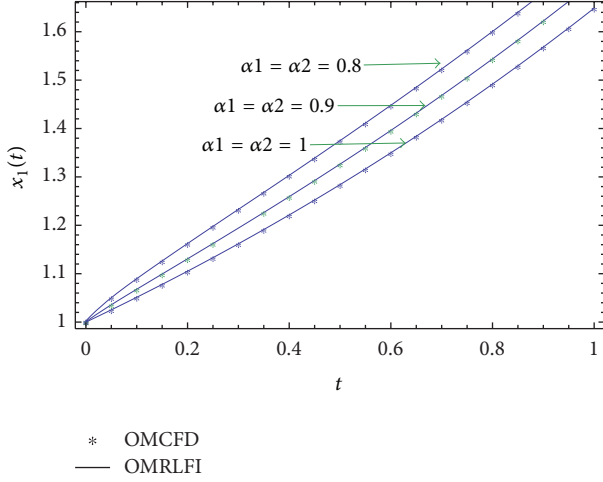


FIGURE 5: Approximate solutions of $x_1(t)$ for $m = 10$ and different values of α_1, α_2 by OMCFD and OMRLFI in Example 13.

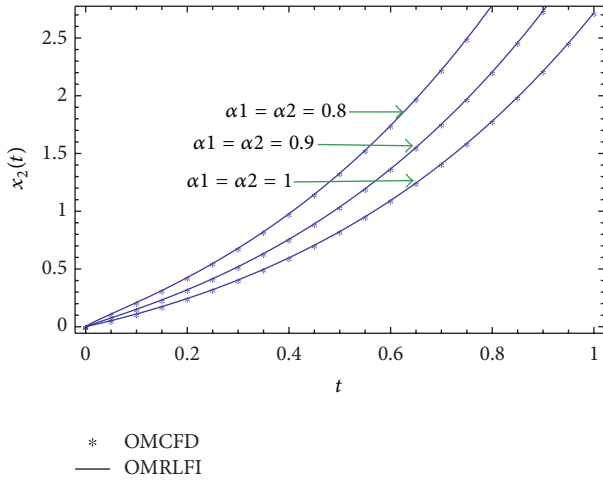


FIGURE 6: Approximate solutions of $x_2(t)$ for $m = 10$ and different values of α_1, α_2 by OMCFD and OMRLFI in Example 13.

can get the approximate function for $y^k(t)$ ($k \in \mathbb{N}$), using BPs as follows:

$$y^k(t) \approx \Phi_m(t)^T \widetilde{C}_k, \quad (14)$$

where $\widetilde{C}_k = \widehat{C}^{k-1} c$.

Proof. (see [26]). \square

Theorem 10. One can get BPs operational matrix D_α from order $(m+1) \times (m+1)$ for the Caputo fractional derivative as follows:

$$\begin{aligned} D^\alpha \Phi_m(t) &= \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-x)^{n-\alpha-1} \Phi_m^{(n)}(x) dx \\ &\approx D_\alpha \Phi_m(t). \end{aligned} \quad (15)$$

Proof. See [26] for details. \square

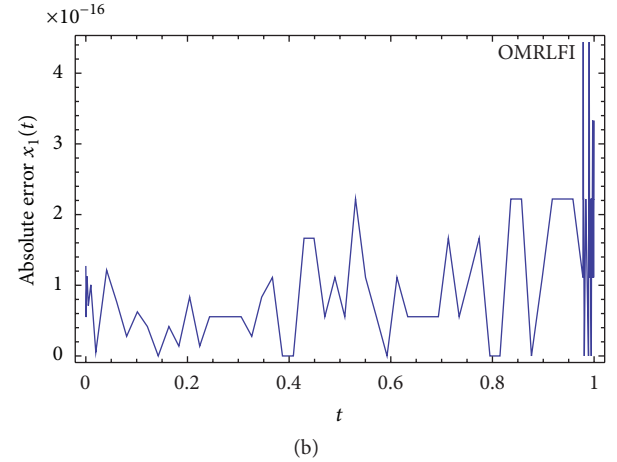
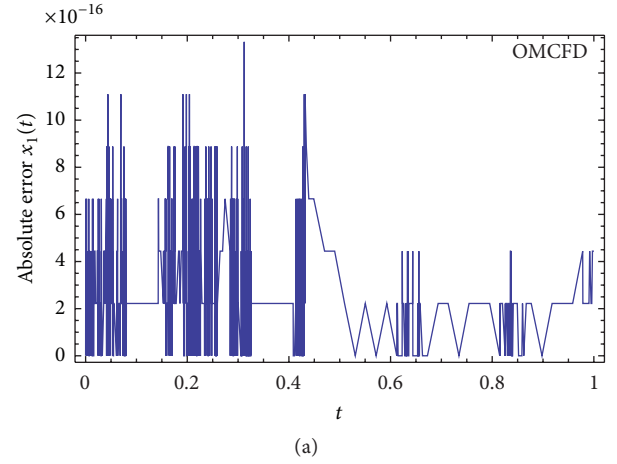


FIGURE 7: Plot of absolute error function $x_1(t)$ for $\alpha_1 = \alpha_2 = 1$ and $m = 10$ by OMCFD and OMRLFI in Example 13.

Theorem 11. One can obtain the operational matrix F_α from order $(m+1) \times (m+1)$ for the Riemann-Liouville fractional integral on the basis of BPs from order m as

$$I^\alpha \Phi_m(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-x)^{\alpha-1} \Phi_m(x) dx \approx F_\alpha \Phi_m(t). \quad (16)$$

Proof. See [28] for details. \square

4. Solving System of Fractional Differential Equations

In this section, we use two methods for solving system of fractional differential equations. In the first method, we use the operational matrix for Caputo fractional derivative (OMCFD), and in the second method, we apply the operational matrix for Riemann-Liouville fractional integral (OMRLFI).

4.1. Solving the Problem by OMCFD. Using Lemma 5, we can approximate the functions $x_i(t)$ as follows:

$$x_i(t) \approx C_i^T \Phi_m(t), \quad i = 1, \dots, n, \quad (17)$$

where $C_i \in \mathbb{R}^{(m+1) \times 1}$.

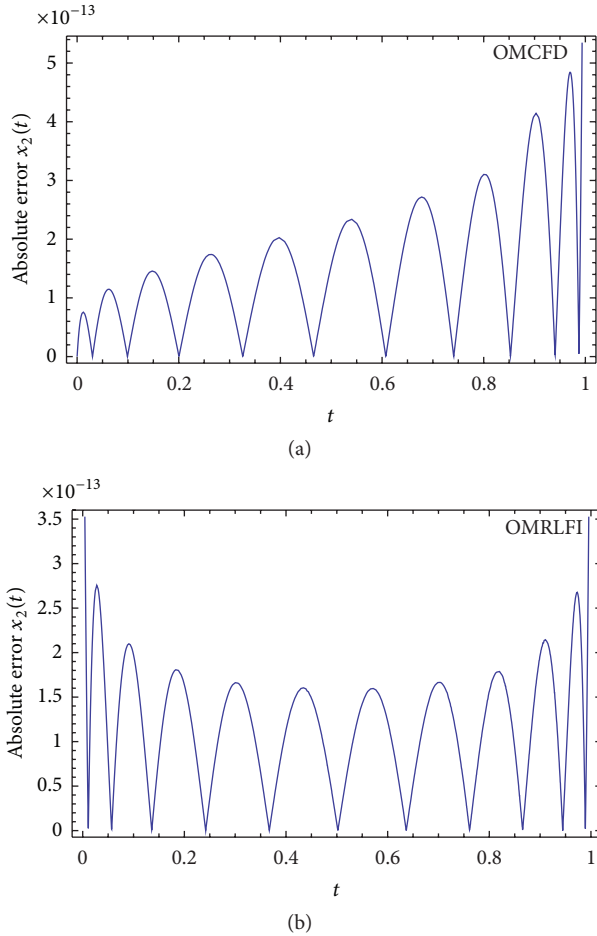


FIGURE 8: Plot of absolute error function $x_2(t)$ for $\alpha_1 = \alpha_2 = 1$ and $m = 10$ by OMCFD and OMRLFI in Example 13.

From (17) and (15) we can write

$$D^{\alpha_i} x_i(t) \approx C_i^T D_{\alpha_i} \Phi_m(t), \quad i = 0, \dots, n. \quad (18)$$

Therefore, problem (1) and (2) reduces to the following problem:

$$C_i^T D_{\alpha_i} \Phi_m(t) = g_i(t, C_1^T \Phi_m(t), \dots, C_n^T \Phi_m(t)), \quad (19)$$

$$i = 1, \dots, n,$$

and the initial condition

$$C_i^T \Phi_m(0) = x_{0,i}, \quad i = 1, \dots, n. \quad (20)$$

Now, using Lemma 5 we can approximate all of the known functions in the system (19). Then, by using Lemma 7 and Corollaries 8 and 9, since functions g_i are polynomial, we obtain the following approximations:

$$g_i(t, X(t)) \approx G_i(C_1, \dots, C_n) \Phi_m(t), \quad i = 1, \dots, n, \quad (21)$$

where $G_i : R^{(m+1) \times n} \rightarrow R^{1 \times (m+1)}$.

Also, for each i ($i = 1, \dots, n$), by using tau method [33] we can generate algebraic equations from (19) and (21) as follows

$$\begin{aligned} \tilde{G}_{i,j} = \int_0^1 (C_i^T D_{\alpha_i} - G_i(C_1, \dots, C_n)) \\ \times \Phi_m(t) B_{j,m}(t) dt = 0, \quad j = 0, \dots, m-1, \end{aligned} \quad (22)$$

and from (23) we set $\tilde{G}_{i,m} = C_i^T \Phi_m(0) - x_{0,i}$.

Finally, problem (1) and (2) has been reduced to the system of algebraic equations

$$\tilde{G}_{i,j}(C_1, \dots, C_n) = 0, \quad i = 1, \dots, n, \quad j = 0, \dots, m. \quad (23)$$

The aforementioned system can be solved for C_i by Newton's iterative method. Then, we get the approximate value of the functions $x_i(t)$ from (17).

4.2. Solving the Problem by OMRLFI. This method consists of two steps.

Step 1. Initial conditions are used to reduce a given initial-value problem to a problem with zero initial conditions. Therefore we have a modified system, incorporating the initial values.

Step 2. The BPs operational matrix of Riemann-Liouville fractional integral is used to transform the problem into a system of algebraic equations.

Now, from (2) we define

$$x_i(t) = x_{0,i} + z_i(t), \quad i = 1, 2, \dots, n, \quad (24)$$

where $z_i(t)$, $i = 1, 2, \dots, n$, are the new unknown functions.

Substituting (24) in (1) and (2), we have the following system:

$$\begin{aligned} D^{\alpha_i} z_i(t) = f_i(t, Z(t)), \\ i = 1, \dots, n, \quad 0 < t \leq 1, \quad 0 < \alpha_i \leq 1, \end{aligned} \quad (25)$$

and the initial condition

$$z_i(0) = 0, \quad i = 1, \dots, n, \quad (26)$$

where $Z(t) = [z_1(t), \dots, z_n(t)]^T$ and $f_i : [0, 1] \times R^n \rightarrow R$ are multivariable polynomial functions. We use the following approximation:

$$D^{\alpha_i} z_i(t) \approx \tilde{C}_i^T \Phi_m(t), \quad i = 1, \dots, n, \quad (27)$$

where $\tilde{C}_i \in R^{(m+1) \times 1}$ are unknown vectors. From (7), (27), and Theorem 11, we can write

$$\begin{aligned} z_i(t) = I^{\alpha_i} D^{\alpha_i} z_i(t) \approx I^{\alpha_i} (\tilde{C}_i^T \Phi_m(t)) \\ = \tilde{C}_i^T I^{\alpha_i} \Phi_m(t) \approx \tilde{C}_i^T F_{\alpha_i} \Phi_m(t). \end{aligned} \quad (28)$$

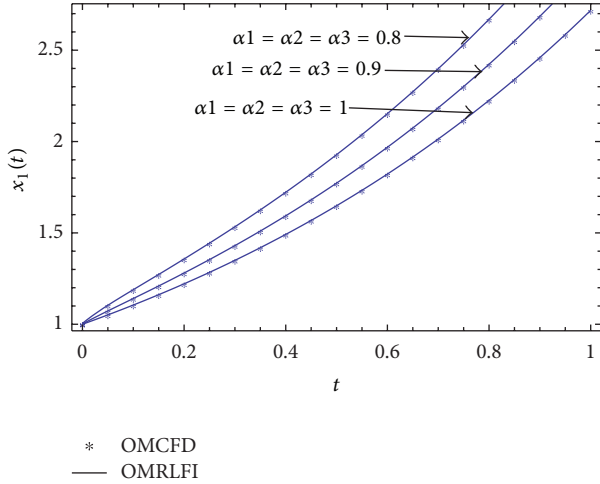


FIGURE 9: Approximate solutions of $x_1(t)$ for $m = 10$ and different values of $\alpha_1, \alpha_2, \alpha_3$ by OMCDF and OMRLFI in Example 14.

So, by (27) and (28), problem (25) and (26) reduces to the following problem:

$$\tilde{C}_i^T \Phi_m(t) = f_i(t, \tilde{C}_1^T F_{\alpha_1} \Phi_m(t), \dots, \tilde{C}_n^T F_{\alpha_n} \Phi_m(t)), \quad (29)$$

$$i = 1, \dots, n.$$

As we saw in the previous section, we can obtain the following approximations:

$$f_i(t, X(t)) \approx F_i(\tilde{C}_1, \dots, \tilde{C}_n) \Phi_m(t), \quad i = 1, \dots, n, \quad (30)$$

where $F_i : R^{(m+1) \times n} \rightarrow R^{1 \times (m+1)}$. So, from (29) and (30) we have

$$(\tilde{C}_i^T - F_i(\tilde{C}_1, \dots, \tilde{C}_n)) \Phi_m(t) = 0, \quad i = 1, \dots, n. \quad (31)$$

Therefore, we have reduced problem (1) and (2) to the system of algebraic equations as follows:

$$\tilde{C}_i^T - F_i(\tilde{C}_1, \dots, \tilde{C}_n) = 0, \quad (32)$$

where this system can be solved for \tilde{C}_i by Newton's iterative method. Finally we obtain the approximate of the functions $x_i(t)$ by

$$x_i(t) \approx x_{0,i} + \tilde{C}_i^T F_{\alpha_i} \Phi_m(t), \quad i = 1, 2, \dots, n. \quad (33)$$

5. Examples

To demonstrate the applicability and to validate the numerical scheme, we apply the present method for the following examples.

Example 12. Consider the following linear system of fractional differential equations [24, 25]:

$$\begin{aligned} D^{\alpha_1} x_1(t) &= x_1(t) + x_2(t), \\ D^{\alpha_2} x_2(t) &= -x_1(t) + x_2(t), \quad 0 < \alpha_1, \alpha_2 \leq 1, \end{aligned} \quad (34)$$

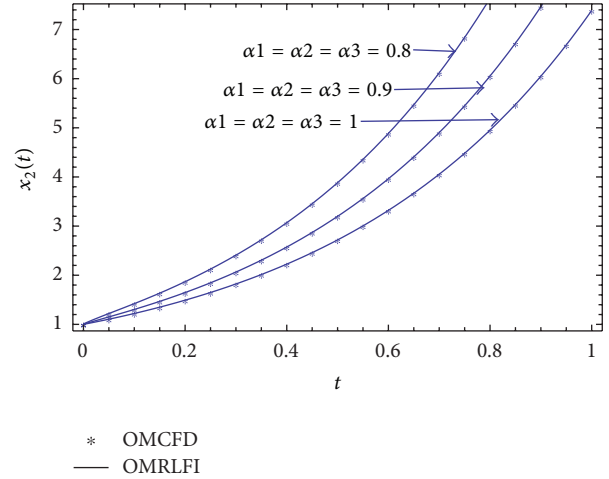


FIGURE 10: Approximate solutions of $x_2(t)$ for $m = 10$ and different values of $\alpha_1, \alpha_2, \alpha_3$ by OMCDF and OMRLFI in Example 14.

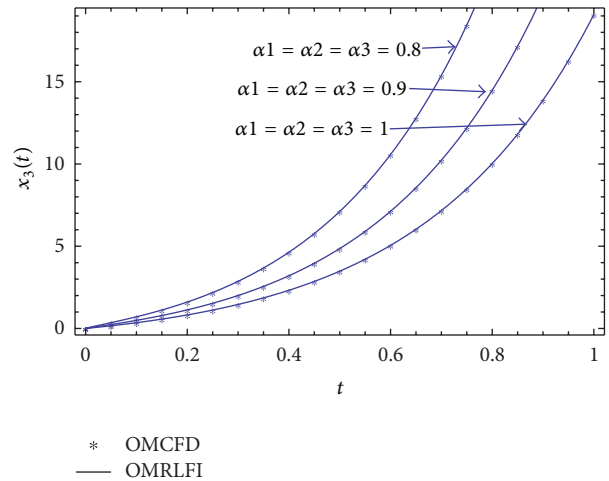


FIGURE 11: Approximate solutions of $x_3(t)$ for $m = 10$ and different values of $\alpha_1, \alpha_2, \alpha_3$ by OMCDF and OMRLFI in Example 14.

with initial condition

$$x_1(0) = 0, \quad x_2(0) = 1. \quad (35)$$

For this problem we have the exact solution in the case of $\alpha_1 = \alpha_2 = 1$ as

$$\begin{aligned} x_1(t) &= e^t \sin(t), \\ x_2(t) &= e^t \cos(t). \end{aligned} \quad (36)$$

We solved this problem by OMCDF and OMRLFI. Figures 1 and 2 show the approximate solutions of $x_1(t)$ and $x_2(t)$, respectively, as a function of time for $m = 10$, for different values of α_1, α_2 . The results show that numerical solutions are in good agreement with each other, in both methods. Also, these figures show that as α_1, α_2 approach close to 1, the numerical solutions approach to the solutions for $\alpha_1 = \alpha_2 = 1$ as expected. In Figures 3 and 4, we see the absolute error of

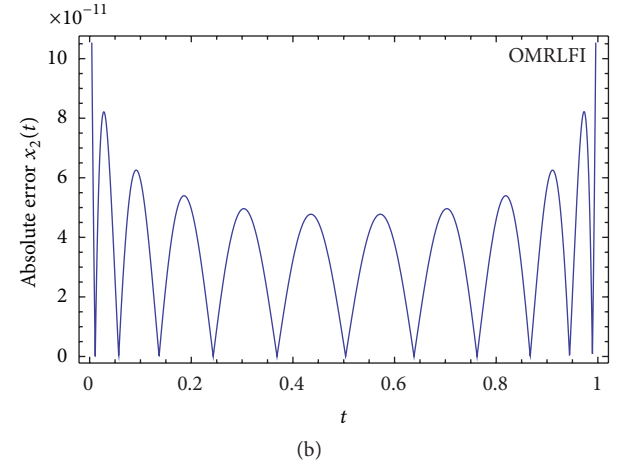
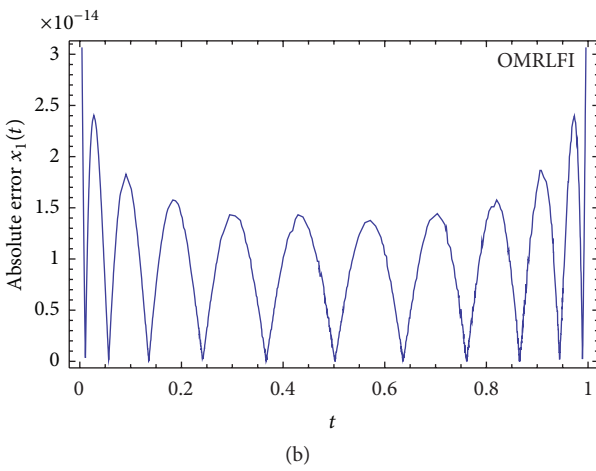
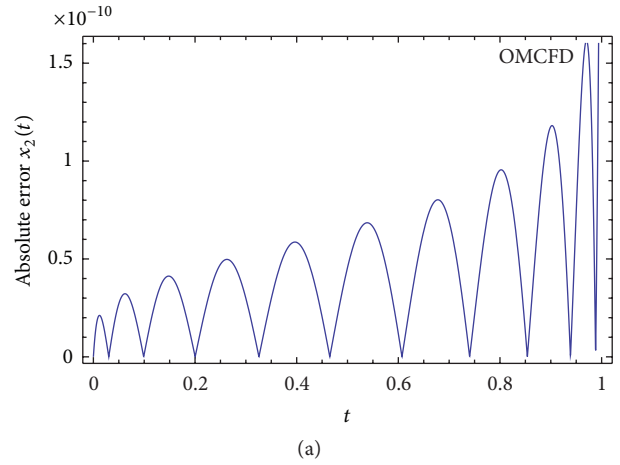
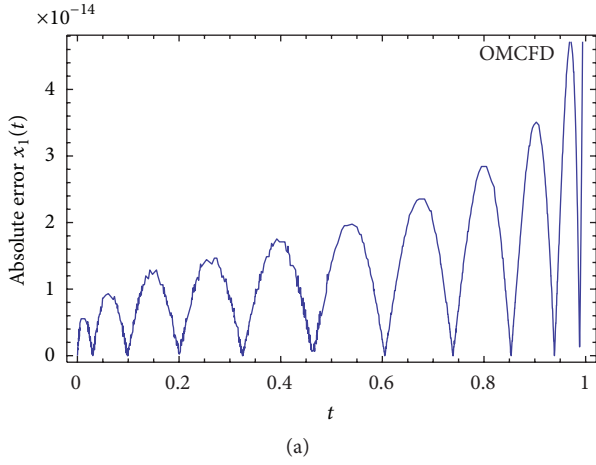


FIGURE 12: Plot of absolute error function $x_1(t)$ for $\alpha_1 = \alpha_2 = \alpha_3 = 1$ and $m = 10$ by OMCDFD and OMRLFI in Example 14.

FIGURE 13: Plot of absolute error function $x_2(t)$ for $\alpha_1 = \alpha_2 = \alpha_3 = 1$ and $m = 10$ by OMCDFD and OMRLFI in Example 14.

both methods, for $m = 10$, $\alpha_1 = \alpha_2 = 1$. In these figures, we can see that obtained results using the presented methods agree well with the analytical solutions for $\alpha_1 = \alpha_2 = 1$.

Example 13. Let us consider the following nonlinear fractional system [24] as follows:

$$D^{\alpha_1} x_1(t) = \frac{x_1(t)}{2}, \tag{37}$$

$$D^{\alpha_2} x_2(t) = x_1^2(t) + x_2(t), \quad 0 < \alpha_1, \alpha_2 \leq 1,$$

such that

$$x_1(0) = 1, \quad x_2(0) = 0. \tag{38}$$

The exact solution of this system, when $\alpha_1 = \alpha_2 = 1$, is

$$\begin{aligned} x_1(t) &= e^{t/2}, \\ x_2(t) &= te^t. \end{aligned} \tag{39}$$

Figures 5 and 6 show the approximate solutions of $x_1(t)$ and $x_2(t)$, respectively, for different values of α_1, α_2 by OMCDFD

and OMRLFI. We conclude that as α_1, α_2 approach close to 1, the numerical solutions approach solutions for $\alpha_1 = \alpha_2 = 1$ as expected. Furthermore, in both methods, the results agree well with each other. Figures 7 and 8 show that, the absolute error of obtained results for $m = 10$ and $\alpha_1 = \alpha_2 = 1$ using OMCDFD and OMRLFI is in good agreement with the exact solution.

Example 14. Consider the nonlinear system of fractional differential equations [24]:

$$\begin{aligned} D^{\alpha_1} x_1(t) &= x_1(t), \\ D^{\alpha_2} x_2(t) &= 2x_1^2(t), \end{aligned} \tag{40}$$

$$D^{\alpha_3} x_3(t) = 3x_1(t)x_2(t), \quad 0 < \alpha_1, \alpha_2, \alpha_3 \leq 1,$$

with the initial conditions given by

$$x_1(0) = 1, \quad x_2(0) = 1, \quad x_3(0) = 0. \tag{41}$$

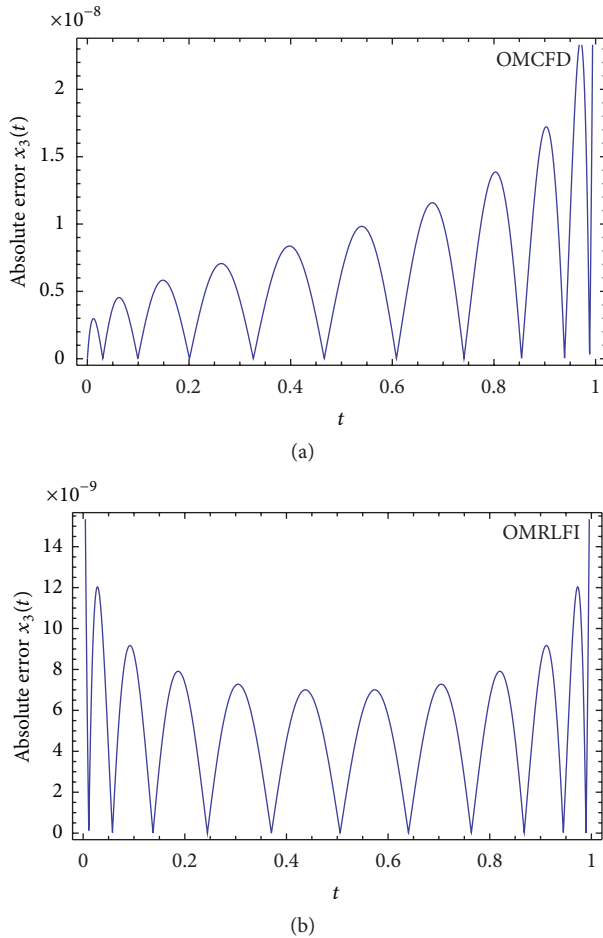


FIGURE 14: Plot of absolute error function $x_3(t)$ for $\alpha_1 = \alpha_2 = \alpha_3 = 1$ and $m = 10$ by OMCDF and OMRLFI in Example 14.

The exact solution of this system, when $\alpha_1 = \alpha_2 = \alpha_3 = 1$, becomes

$$\begin{aligned} x_1(t) &= e^t, \\ x_2(t) &= e^{2t}, \\ x_3(t) &= e^{3t} - 1. \end{aligned} \quad (42)$$

We can see the approximate solutions of $x_1(t)$, $x_2(t)$ and $x_3(t)$, by OMCDF and OMRLFI for $m = 10$ and different values of α_1 , α_2 and α_3 , in Figures 9, 10, and 11. These figures show that, when α_1 , α_2 , and α_3 approach close to 1, the numerical solutions approach the solutions for $\alpha_1 = \alpha_2 = \alpha_3 = 1$ as expected. In Figures 9–11, we observe that results of OMCDF and OMRLFI overlap. In Figures 12, 13, and 14, we see the absolute error of the obtained results for $m = 10$ and $\alpha_1 = \alpha_2 = \alpha_3 = 1$ in both methods.

6. Conclusion

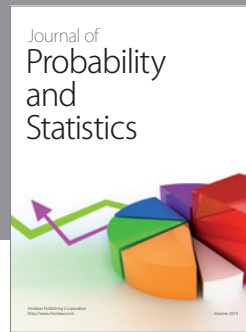
In this paper, we get operational matrices of the product, Caputo fractional derivative, and Riemann-Liouville fractional integral by Bernstein polynomials. Then by using

these matrices, we proposed two methods that reduced the nonlinear systems of fractional differential equations to the two system of algebraic equations that can be solved easily. Finally, numerical examples are simulated to demonstrate the high performance of the proposed method. We saw that the results of both methods were in good agreement with each other, and the classical solutions were recovered when the order of the fractional derivative goes to 1.

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