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Research Article

Delta Shock Waves for a Linearly Degenerate Hyperbolic System of Conservation Laws of Keyfitz-Kranzer Type

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This paper is devoted to the study of delta shock waves for a hyperbolic system of conservation laws of Keyfitz-Kranzer type with two linearly degenerate characteristics. The Riemann problem is solved constructively. The Riemann solutions include exactly two kinds. One consists of two (or just one) contact discontinuities, while the other contains a delta shock wave. Under suitable generalized Rankine-Hugoniot relation and entropy condition, the existence and uniqueness of delta shock solution are established. These analytical results match well the numerical ones. Finally, two kinds of interactions of elementary waves are discussed.

1. Introduction

Consider the following hyperbolic system of conservation laws:

$$\begin{aligned} &\rho_t + (\rho(u-P))_x = 0, \\ &(\rho u)_t + (\rho u(u-P))_x = 0, \end{aligned} \tag{1}$$

where $P = P(\rho)$ with $\rho \geq 0$. Model (1) belongs to the nonsymmetric system of Keyfitz-Kranzer type [1, 2] as

$$\begin{aligned} &\rho_t + (\rho\phi(\rho, u_1, u_2, \ldots, u_n))_x = 0, \\ &(\rho u_i)_t + (\rho u_i \phi(\rho, u_1, u_2, \ldots, u_n))_x = 0, \quad i = 1, 2, \ldots, n. \end{aligned} \tag{2}$$

System (1) was also introduced as a macroscopic model for traffic flow by Aw and Rascle [3], where ρ and u are the density and the velocity of cars on the roadway, and the function P is smooth and strictly increasing and it satisfies

$$\rho P''(\rho) + 2P'(\rho) > 0 \quad \text{for } \rho > 0. \tag{3}$$

Model (1) was also studied by Lu [2]. By using the compensated compactness method, he established the existence of global bounded weak solutions of the Cauchy problem under the following two assumptions on $P(\rho)$, respectively:

$$\begin{aligned} &P(0) = 0, \qquad \lim_{\rho\to 0} \rho P'(\rho) = 0, \\ &\rho P''(\rho) + 2P'(\rho) > 0 \quad \text{for } \rho > 0, \\ &\lim_{\rho\to 0} \rho P(\rho) = 0, \qquad \lim_{\rho\to\infty} (\rho P(\rho))' \geq A, \\ &\rho P''(\rho) + 2P'(\rho) > 0 \quad \text{for } \rho > 0. \end{aligned} \tag{4}$$

One can find that system (1) has two characteristics; one is always linearly degenerate while the other is linearly degenerate or genuinely nonlinear depending on the behaviors of $p(\rho)$. In [2, 3], the condition (3) is required, which implies that the second characteristic is genuinely nonlinear. Thus, one interesting topic is the case when both characteristics are linearly degenerate.

These motivate us to consider (1) with

$$P = P(\rho) = -\frac{1}{\rho}, \tag{5}$$

which is the prototype function satisfying

$$\rho P''(\rho) + 2P'(\rho) \equiv 0 \quad \text{for } \rho > 0. \tag{6}$$

For system (1) with (5) or (6), a distinctive feature is that both eigenvalues are linearly degenerate; that is, this is a fully linearly degenerate system. Thus, all the classical elementary waves only consist of contact discontinuities. Moreover, the overlapping of linearly degenerate characteristics may result in the formation of delta shock wave. A delta shock wave is a generalization of an ordinary shock wave. Speaking informally, it is a kind of discontinuity, on which at least one of the state variables may develop an extreme concentration in the form of a Dirac delta function with the discontinuity as its support. It is more compressive than an ordinary shock wave in the sense that more characteristics enter the discontinuity line. From the physical point of view, a delta shock wave represents the process of concentration of the mass and formation of the universe. For related results of delta shock waves, we refer the readers to the papers [4–24] and the references therein.

Putting (5) into (1), we obtain the equivalent system

$$\begin{gathered} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + u)_x = 0. \end{gathered} \tag{7}$$

Firstly, we consider the Riemann problem for (7) with initial data

$$(u, \rho)(x, t = 0) = \begin{cases} (u_-, \rho_-), & x < 0, \\ (u_+, \rho_+), & x > 0. \end{cases} \tag{8}$$

By the analysis method in phase plane, when $u_+ + (1/\rho_+) > u_-$, we can construct Riemann solution only involving two different (or just one) contact discontinuities. However, for the case $u_+ + (1/\rho_+) \leq u_-$, we find that the Riemann solution can not be constructed by these classical contact discontinuities and delta shock wave should occur. For delta shock wave, by a definition of measure solutions to (7), we derive the generalized Rankine-Hugoniot relation which describes the relation among the limit states on both sides of the discontinuity, location, propagation speed, weight, and the assignment of the component u on its discontinuity relative to the delta shock wave. The entropy condition is an overcompressive one and guarantees the uniqueness of solution. With the generalized Rankine-Hugoniot relation and entropy condition, we solve the delta shock solution uniquely. Secondly, we simulate the obtained Riemann solutions by using N-T scheme [25] and get the numerical results coinciding with the analytical ones. Thirdly, we study two kinds of interactions of delta shock waves and contact discontinuities by considering the initial-value problem with initial data as

$$(u, \rho)(x, t = 0) = \begin{cases} (u_-, \rho_-), & x < 0, \\ (u_m, \rho_m), & 0 < x < a, \\ (u_+, \rho_+), & x > a, \end{cases} \tag{9}$$

where $a > 0$ is a constant. In the first kind, the interaction of two contact discontinuities leads to a delta shock wave. In the second one, the delta shock wave vanishes after it interacts with a contact discontinuity. Such two cases are not encountered in the interactions of elementary waves for some other systems, such as the zero-pressure gas dynamics.

This paper is organized as follows. In Section 2, we present some preliminaries and construct the Riemann solution consisting of contact discontinuities by phase plane analysis method. Then, in Section 3, with suitable generalized Rankine-Hugoniot relation and entropy condition, we solve delta shock solution uniquely. Section 4 presents some numerical results of Riemann solutions obtained by using the Nessyahu-Tadmor scheme [25]. Finally, in Section 5, we study two kinds of interactions of delta shock waves and contact discontinuities.

2. Preliminaries and Solutions Involving Contact Discontinuities

In this section, we analyze some basic properties and then solve the Riemann problem (7)-(8) by the analysis in phase plane. The system has two eigenvalues

$$\lambda_1 = u, \qquad \lambda_2 = u + \frac{1}{\rho} \tag{10}$$

with associated right eigenvectors

$$\vec{r}_1 = (1, 0)^T, \qquad \vec{r}_2 = \left(1, \frac{1}{\rho^2}\right)^T \tag{11}$$

satisfying

$$\nabla \lambda_i \cdot \vec{r}_i \equiv 0, \quad i = 1, 2. \tag{12}$$

Thus, system (7) is fully linearly degenerate. The linear degeneracy also excludes the possibility of rarefaction waves and shock waves.

As usual, we seek the self-similar solution

$$(u, \rho)(t, x) = (u, \rho)(\xi), \quad \xi = \frac{x}{t}, \tag{13}$$

for which system (7) becomes

$$\begin{gathered} -\xi \rho_\xi + (\rho u)_\xi = 0, \\ -\xi (\rho u)_\xi + (\rho u^2 + u)_\xi = 0, \end{gathered} \tag{14}$$

and initial data (8) changes to the boundary condition

$$(u, \rho)(\pm\infty) = (u_\pm, \rho_\pm). \tag{15}$$

This is a two-point boundary value problem of first-order ordinary differential equations with the boundary values in the infinity.

For any smooth solution, (14) turns into

$$\begin{pmatrix} u - \xi & \rho \\ 0 & u + \dfrac{1}{\rho} - \xi \end{pmatrix} \begin{pmatrix} d\rho \\ du \end{pmatrix} = 0. \tag{16}$$

It provides either the general solutions (constant states)

$$(u, \rho) = \text{const.} \quad (\rho > 0) \tag{17}$$

or singular solutions

$$\xi = \lambda_1 = u, \qquad du = 0,$$
$$\xi = \lambda_1 = u + \frac{1}{\rho}, \qquad d\left(u + \frac{1}{\rho}\right) = 0. \tag{18}$$

Integrating (18) from (u_-, ρ_-) to (u, ρ), respectively, one can get that

$$\begin{aligned} &\xi = u, \\ &u = u_-, \\ &\xi = u + \frac{1}{\rho}, \\ &u + \frac{1}{\rho} = u_- + \frac{1}{\rho_-}. \end{aligned} \tag{19}$$

Indeed, (19) are contact discontinuities; see (21) and (22) in the following.

For a bounded discontinuous solution at $\xi = \omega$, the Rankine-Hugoniot relation

$$\begin{aligned} -\omega[\rho] + [\rho u] &= 0, \\ -\omega[\rho u] + \left[\rho u^2 + u\right] &= 0 \end{aligned} \tag{20}$$

holds, where $[q] = q_- - q$ is the jump of q across the discontinuous line and ω is the velocity of the discontinuity. A simple calculation gives

$$\begin{aligned} \omega &= u, \\ u &= u_-, \end{aligned} \tag{21}$$

$$\begin{aligned} \omega &= u + \frac{1}{\rho}, \\ u + \frac{1}{\rho} &= u_- + \frac{1}{\rho_-}. \end{aligned} \tag{22}$$

Obviously, (21) and (22) are two kinds of contact discontinuities, denoted by $\overleftarrow{J}$ and $\overrightarrow{J}$, respectively. In the phase plane, the curves expressed by the second equations in (21) and (22) are called the contact discontinuity curves.

Starting from any point (u_-, ρ_-), we draw the curve $u = u_-$ for $\rho > 0$ in the phase plane, which is parallel to the ρ-axis. We also draw the curve $u + (1/\rho) = u_- + (1/\rho_-)$ for $\rho > 0$, which is monotonically increasing and has the lines $\rho = 0$ and $u = u_- + (1/\rho_-)$ as its two asymptotic lines. Also, from the point $(u_- - (1/\rho_-), \rho_-)$, we draw the curve $u + (1/\rho) = u_-$.

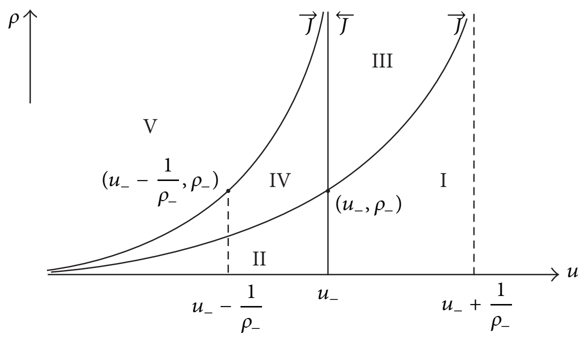


FIGURE 1: Contact discontinuity curves.

Then the phase plane is divided into five regions as follows (shown in Figure 1):

$$\begin{aligned} \mathrm{I} &= \left\{(u,\rho) \mid u_- < u < u_- + \frac{1}{\rho_-}, \right. \\ &\left. 0 < \rho < \frac{1}{u_- + \rho_-^{-1} - u}\right\} \cup \left\{u_- + \frac{1}{\rho_-} < u < +\infty, \right. \\ &\left. 0 < \rho < +\infty\right\}, \\ \mathrm{II} &= \left\{(u,\rho) \mid -\infty < u < u_-, 0 < \rho < \frac{1}{u_- + \rho_-^{-1} - u}\right\}, \\ \mathrm{III} &= \left\{(u,\rho) \mid u_- < u < u_- + \frac{1}{\rho_-}, \right. \\ &\left. \frac{1}{u_- + \rho_-^{-1} - u} < \rho < +\infty\right\}, \\ \mathrm{IV} &= \left\{(u,\rho) \mid -\infty < u < u_-, \right. \\ &\left. \frac{1}{u_- + \rho_-^{-1} - u} < \rho < \frac{1}{u_- - u}\right\}, \\ \mathrm{V} &= \left\{(u,\rho) \mid -\infty < u < u_-, \frac{1}{u_- - u} < \rho < +\infty\right\}. \end{aligned} \tag{23}$$

By the analysis method in the phase plane, when $(u_+, \rho_+) \in \mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}$, namely, $u_+ + (1/\rho_+) > u_-$, one can easily construct the solution consisting of two different (or just one) contact discontinuities (see Figure 2), in which,

$$\overleftarrow{J}: x = u_- t, \qquad \overrightarrow{J}: x = \left(u_+ + \frac{1}{\rho_+}\right)t, \tag{24}$$

the intermediate states (u^*, ρ^*) connecting two contact discontinuities satisfy

$$u^* = u_-, \qquad u^* + \frac{1}{\rho^*} = u_+ + \frac{1}{\rho_+}. \tag{25}$$

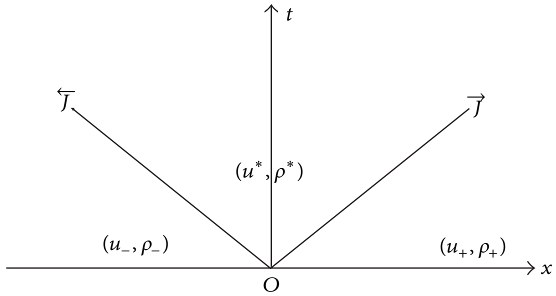


FIGURE 2: Riemann solution involving two contact discontinuities.

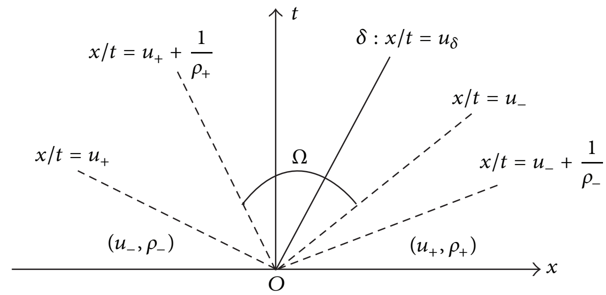


FIGURE 3: Riemann solution containing delta shock wave.

3. Delta Shock Solution

In this section, let us in detail discuss the last case when $(u_+, \rho_+) \in V\ (u_-, \rho_-)$; that is, $u_+ + (1/\rho_+) \le u_-$. The characteristic lines from initial data will overlap in a domain $\Omega = \{(x,t) : u_+ + (1/\rho_+) < x/t < u_-\}$ shown in Figure 3. So singularity must happen in Ω. It is easy to know that the singularity is impossible to be a jump with finite amplitude because the Rankine-Hugoniot relation is not satisfied on the bounded jump. In other words, there is no solution which is piecewise smooth and bounded. Motivated by [7, 8, 16, 21, 22], we seek solutions with delta distribution at the jump.

Denote by $BM(R)$ the space of bounded Borel measures on R, and then the definition of a measure solution of (7) in $BM(R)$ can be given as follows (see [16, 22, 24]).

Definition 1. A pair (u, ρ) is called a measure solution to (7) if there hold

$$\begin{gathered} \rho \in L^\infty((0,\infty), BM(R)) \cap C((0,\infty), H^{-s}(R)), \\ u \in L^\infty((0,\infty), L^\infty(R)) \cap C((0,\infty), H^{-s}(R)), \quad s > 0, \\ u \text{ is measurable with respect to } \rho \\ \text{at almost for all } t \in (0,\infty), \end{gathered} \tag{26}$$

and (7) is satisfied in measured and distributional senses; that is,

$$\begin{gathered} I_1 = \iint_{R^+\times R} (\phi_t + u\phi_x)\, d\rho\, dt = 0, \\ I_2 = \iint_{R^+\times R} u(\phi_t + u\phi_x)\, d\rho\, dt + \iint_{R^+\times R} u\phi_x dx\, dt = 0 \end{gathered} \tag{27}$$

for all test function $\phi \in C_0^\infty(R^+ \times R)$.

Definition 2. A two-dimensional weighted delta function $w(s)\delta_L$ supported on a smooth curve L parameterized as $t = t(s)$, $x = x(s)$ $(c \le s \le d)$ is defined by

$$\langle w(s)\delta_L, \phi(t,x)\rangle = \int_c^d w(s)\,\phi(t(s), x(s))\, ds \tag{28}$$

for all $\phi \in C_0^\infty(R^2)$.

Let us consider the discontinuity of (7) of the form

$$(u,\rho)(x,t) = \begin{cases} (u_l, \rho_l)(x,t), & x < x(t), \\ (u_\delta(t), w(t)\,\delta(x - x(t))), & x = x(t), \\ (u_r, \rho_r)(x,t), & x > x(t), \end{cases} \tag{29}$$

in which $(u_l, \rho_l)(x,t)$ and $(u_r, \rho_r)(x,t)$ are piecewise smooth bounded solutions of (7), for which we also set $dx(t)/dt = u_\delta(t)$ for the concentration in ρ needs to travel at the speed of discontinuity (also see [16, 22, 24]). We assert that the discontinuity (29) is a measure solution to (7) if and only if the following relation

$$\begin{aligned} \frac{dx(t)}{dt} &= u_\delta(t), \\ \frac{dw(t)}{dt} &= -u_\delta(t)[\rho] + [\rho u], \\ \frac{dw(t)\,u_\delta(t)}{dt} &= -u_\delta(t)[\rho u] + [\rho u^2 + u] \end{aligned} \tag{30}$$

is satisfied.

In fact, for any $\phi \in C_0^\infty(R^+ \times R)$, with Green's formula and taking (7) in mind, one can calculate

$$\begin{aligned} I_2 &= \iint_{R^+\times R} u(\phi_t + u\phi_x)\, d\rho\, dt + \iint_{R^+\times R} u\phi_x dx\, dt \\ &= \int_0^{+\infty}\int_{-\infty}^{x(t)} \left(\rho_l u_l \phi_t + \rho_l u_l^2 \phi_x\right) dx\, dt \\ &\quad + \int_0^{+\infty}\int_{-\infty}^{x(t)} u_l \phi_x dx\, dt \\ &\quad + \int_0^{+\infty}\int_{x(t)}^{+\infty} \left(\rho_r u_r \phi_t + \rho_r u_r^2 \phi_x\right) dx\, dt \end{aligned}$$

$$
\begin{aligned}
&+\int_0^{+\infty}\int_{x(t)}^{+\infty} u_r\phi_x dx\,dt \\
&+\int_0^{+\infty}\{w(t)\,u_\delta(t)\,(\phi_t+u_\delta(t)\,\phi_x)\}\,dt \\
&=\int_0^{+\infty}\int_{-\infty}^{x(t)}\left\{(\rho_l u_l\phi)_t+\left(\left(\rho_l u_l^2+u_l\right)\phi\right)_x\right\}dx\,dt \\
&-\int_0^{+\infty}\int_{-\infty}^{x(t)}\left\{(\rho_l u_l)_t+\left(\rho_l u_l^2+u_l\right)_x\right\}\phi\,dx\,dt \\
&+\int_0^{+\infty}\int_{x(t)}^{+\infty}\left\{(\rho_r u_r\phi)_t+\left(\left(\rho_r u_r^2+u_r\right)\phi\right)_x\right\}dx\,dt \\
&-\int_0^{+\infty}\int_{x(t)}^{+\infty}\left\{(\rho_r u_r)_t+\left(\rho_r u_r^2+u_r\right)_x\right\}\phi\,dx\,dt \\
&+\int_0^{+\infty} w(t)\,u_\delta(t)\,\frac{d\phi}{dt}dt \\
&=\int_0^{+\infty}\int_{-\infty}^{x(t)}\left\{(\rho_l u_l\phi)_t+\left(\left(\rho_l u_l^2+u_l\right)\phi\right)_x\right\}dx\,dt \\
&+\int_0^{+\infty}\int_{x(t)}^{+\infty}\left\{(\rho_r u_r\phi)_t+\left(\left(\rho_r u_r^2+u_r\right)\phi\right)_x\right\}dx\,dt \\
&+\int_0^{+\infty} w(t)\,u_\delta(t)\,\frac{d\phi}{dt}dt \\
&=\oint_{\partial\Omega_-}-\rho_l u_l\phi\,dx+\left(\rho_l u_l^2+u_l\right)\phi\,dt \\
&+\oint_{\partial\Omega_+}-\rho_r u_r\phi\,dx+\left(\rho_r u_r^2+u_r\right)\phi\,dt \\
&-\int_0^{+\infty}\frac{dw(t)\,u_\delta(t)}{dt}\phi\,dt \\
&=\int_0^{+\infty}\Bigg(-\rho_l u_l\frac{dx}{dt}+\left(\rho_l u_l^2+u_l\right) \\
&\qquad+\rho_r u_r\frac{dx}{dt}-\left(\rho_r u_r^2+u_r\right)\Bigg)\phi\,dt \\
&-\int_0^{+\infty}\frac{dw(t)\,u_\delta(t)}{dt}\phi\,dt \\
&=\int_0^{+\infty}\left\{-u_\delta(t)\,[\rho u]+\left[\rho u^2+u\right]-\frac{dw(t)\,u_\delta(t)}{dt}\right\}\phi\,dt,
\end{aligned}
\tag{31}
$$

where $\Omega_-=\{(x,t):t\in R^+,-\infty<x<x(t)\}$, $\Omega_+=\{(x,t):t\in R^+,x(t)<x<+\infty\}$, $\partial\Omega_\pm$ is the boundaries of $\Omega_\pm$, and $\oint_{\partial\Omega_\pm}$ are the line integral on boundaries $\partial\Omega_\pm$.

Similarly, we have

$$
\begin{aligned}
I_1&=\iint_{R^+\times R}(\phi_t+u\phi_x)\,d\rho\,dt \\
&=\int_0^{+\infty}\left\{-u_\delta(t)\,[\rho]+[\rho u]-\frac{dw(t)}{dt}\right\}\phi\,dt.
\end{aligned}
\tag{32}
$$

With the above results, we can reach the assertion easily.

Relation (30) reflects the exact relationship among the limit states on two sides of the discontinuity, the weight, propagation speed and the location of the discontinuity. We call it the generalized Rankine-Hugoniot relation of the discontinuity.

In addition, the admissibility (entropy) condition for the discontinuity (29) is

$$
\lambda_2(\rho_r,u_r)\le u_\delta(t)\le\lambda_1(\rho_l,u_l), \tag{33}
$$

which means that all characteristics on both sides of the discontinuity are not outcoming and guarantees uniqueness of solution.

Definition 3. A discontinuity which is presented in the form (29) and satisfies (30) and (33) will be called a delta shock wave to system (7), symbolized by δ.

In what follows, the generalized Rankine-Hugoniot relation will in particular be applied to the Riemann problem (7)-(8) for the case $(u_+,\rho_+)\in V(u_-,\rho_-)$, which is equivalent to the inequality

$$
\lambda_1(u_+,\rho_+)<\lambda_2(u_+,\rho_+)\le\lambda_1(u_-,\rho_-)<\lambda_2(u_-,\rho_-), \tag{34}
$$

that is,

$$
u_+<u_++\frac{1}{\rho_+}\le u_-<u_-+\frac{1}{\rho_-}. \tag{34'}
$$

At this moment, the Riemann solution is a delta shock wave besides two constant states with the form

$$
(u,\rho)(x,t)=\begin{cases}(u_-,\rho_-), & x<x(t),\\ (u_\delta(t),w(t)\,\delta(x-x(t))), & x=x(t),\\ (u_+,\rho_+), & x>x(t).\end{cases} \tag{35}
$$

To determine $x(t)$, $w(t)$, and $u_\delta(t)$ uniquely, we solve (30) with initial data

$$
t=0:x(0)=0,\qquad w(0)=0 \tag{36}
$$

under entropy condition (33) which is, at this moment,

$$
u_++\frac{1}{\rho_+}\le u_\delta(t)\le u_-. \tag{33'}
$$

From (30) and (36), it follows that

$$
\begin{aligned}
w(t)&=-[\rho]\,x(t)+[\rho u]\,t,\\
w(t)\,u_\delta(t)&=-[\rho u]\,x(t)+\left[\rho u^2+u\right]t.
\end{aligned}
\tag{37}
$$

Multiplying the first equation in (37) by $u_\delta(t)$ and then subtracting it from the second one, we obtain that

$$
[\rho]\,x(t)\,u_\delta(t)-[\rho u]\,u_\delta(t)\,t-[\rho u]\,x(t)+\left[\rho u^2+u\right]t=0, \tag{38}
$$

or

$$\frac{d}{dt}\left(\frac{[\rho]}{2}x^2(t)-[\rho u]x(t)t+\frac{[\rho u^2+u]}{2}t^2\right)=0. \tag{39}$$

It gives

$$[\rho]x^2(t)-2[\rho u]x(t)t+[\rho u^2+u]t^2=0. \tag{40}$$

From (40), we can find that $x'(t)=u_\delta(t):=u_\delta$ is a constant and $x(t)=u_\delta t$. Then (40) can be rewritten into

$$H(u_\delta):=[\rho]u_\delta^2-2[\rho u]u_\delta+[\rho u^2+u]=0. \tag{41}$$

When $\rho_-=\rho_+$, (41) is just a linear equation of u_δ, and then we have

$$\begin{aligned}u_\delta&=\frac{(u_-+u_++(1/\rho_-))}{2},\\ x(t)&=\frac{(u_-+u_++(1/\rho_-))t}{2},\\ w(t)&=\rho_-(u_--u_+)t,\end{aligned} \tag{42}$$

which satisfies the entropy condition $(33')$ obviously.

When $\rho_-\neq\rho_+$, (41) is just a quadratic equation with respect to u_δ, and the discriminant is

$$\begin{aligned}\Delta&=4[\rho u]^2-4[\rho][\rho u^2+u]\\ &=4\rho_-\rho_+(u_--u_+)\left(u_-+\frac{1}{\rho_-}-u_+-\frac{1}{\rho_+}\right)>0\end{aligned} \tag{43}$$

by virtue of (34), and then we can find

$$\begin{aligned}u_\delta&=\frac{[\rho u]-\sqrt{[\rho u]^2-[\rho][\rho u^2+u]}}{[\rho]},\\ w(t)&=\sqrt{[\rho u]^2-[\rho][\rho u^2+u]}\,t,\\ x(t)&=\frac{[\rho u]-\sqrt{[\rho u]^2-[\rho][\rho u^2+u]}}{[\rho]}t,\end{aligned} \tag{44}$$

$$\begin{aligned}u_\delta&=\frac{[\rho u]+\sqrt{[\rho u]^2-[\rho][\rho u^2+u]}}{[\rho]},\\ w(t)&=-\sqrt{[\rho u]^2-[\rho][\rho u^2+u]}\,t,\\ x(t)&=\frac{[\rho u]+\sqrt{[\rho u]^2-[\rho][\rho u^2+u]}}{[\rho]}t.\end{aligned} \tag{45}$$

Next, with the help of the entropy condition $(33')$, we will choose the admissible solution to Riemann problem (7)-(8) from (44) and (45). Taking into account $(34')$, one can calculate that

$$\begin{aligned}&-[\rho]\lambda_1(u_-,\rho_-)+[\rho u]=\rho_+(u_--u_+)>0,\\ &-[\rho]\lambda_2(u_+,\rho_+)+[\rho u]=\rho_-\left(u_-+\frac{1}{\rho_-}-u_+-\frac{1}{\rho_+}\right)>0,\\ &[\rho](\lambda_1(u_-,\rho_-))^2-2[\rho u]\lambda_1(u_-,\rho_-)+[\rho u^2+u]\\ &\quad=-\rho_+(u_--u_+)\left(u_--u_+-\frac{1}{\rho_+}\right)\leq 0,\\ &[\rho](\lambda_2(u_+,\rho_+))^2-2[\rho u]\lambda_2(u_+,\rho_+)+[\rho u^2+u]\\ &\quad=\rho_-\left(u_--u_+-\frac{1}{\rho_+}\right)\left(u_-+\frac{1}{\rho_-}-u_+-\frac{1}{\rho_+}\right)\geq 0.\end{aligned} \tag{46}$$

Then, for solution (44), we have

$$\begin{aligned}&u_\delta-\lambda_2(u_+,\rho_+)\\ &\quad=\frac{[\rho u]-\sqrt{[\rho u]^2-[\rho][\rho u^2+u]}}{[\rho]}-\lambda_2(u_+,\rho_+)\\ &\quad=\frac{[\rho](\lambda_2(u_+,\rho_+))^2-2[\rho u]\lambda_2(u_+,\rho_+)+[\rho u^2+u]}{(-[\rho]\lambda_2(u_+,\rho_+)+[\rho u])+\sqrt{[\rho u]^2-[\rho][\rho u^2+u]}}\\ &\quad\geq 0,\\ &u_\delta-\lambda_1(u_-,\rho_-)\\ &\quad=\frac{[\rho u]-\sqrt{[\rho u]^2-[\rho][\rho u^2+u]}}{[\rho]}-\lambda_1(u_-,\rho_-)\\ &\quad=\frac{[\rho](\lambda_1(u_-,\rho_-))^2-2[\rho u]\lambda_1(u_-,\rho_-)+[\rho u^2+u]}{(-[\rho]\lambda_1(u_-,\rho_-)+[\rho u])+\sqrt{[\rho u]^2-[\rho][\rho u^2+u]}}\\ &\quad\leq 0,\end{aligned} \tag{47}$$

which imply that the entropy condition $(33')$ is valid. One can easily observe especially that, in the two inequalities above, both of the signs "=" are valid if and only if $\lambda_1(u_-,\rho_-)=\lambda_2(u_+,\rho_+)$. In this case, we have $u_\delta=\lambda_1(u_-,\rho_-)=\lambda_2(u_+,\rho_+)$.

For solution (45), when $\rho_- < \rho_+$,

$$\begin{aligned} u_\delta - \lambda_2(u_+, \rho_+) &= \frac{[\rho u] + \sqrt{[\rho u]^2 - [\rho][\rho u^2 + u]}}{[\rho]} - \lambda_2(u_+, \rho_+) \\ &= \frac{-[\rho]\lambda_2(u_+, \rho_+) + [\rho u] + \sqrt{[\rho u]^2 - [\rho][\rho u^2 + u]}}{[\rho]} \\ &< 0, \end{aligned} \tag{48}$$

and when $\rho_- > \rho_+$,

$$\begin{aligned} u_\delta - \lambda_1(u_-, \rho_-) &= \frac{[\rho u] + \sqrt{[\rho u]^2 - [\rho][\rho u^2 + u]}}{[\rho]} - \lambda_1(u_-, \rho_-) \\ &= \frac{-[\rho]\lambda_1(u_-, \rho_-) + [\rho u] + \sqrt{[\rho u]^2 - [\rho][\rho u^2 + u]}}{[\rho]} \\ &> 0. \end{aligned} \tag{49}$$

These show that the solution (45) does not satisfy the entropy condition $(33')$.

Thus we have proved the following result.

Theorem 4. *Let $(u_+, \rho_+) \in V\ (u_-, \rho_-)$. Then Riemann problem (8) for (7) admits one and only one entropy solution in the sense of measures of the form*

$$(u, \rho)(x, t) = \begin{cases} (u_-, \rho_-), & x < u_\delta t, \\ (u_\delta, w(t)\,\delta(x - u_\delta t)), & x = u_\delta t, \\ (u_+, \rho_+), & x > u_\delta t, \end{cases} \tag{50}$$

where u_δ and $w(t)$ are shown in (42) for $\rho_- = \rho_+$ or (44) for $\rho_- \neq \rho_+$.

At last, associating with the results in Section 2, we can conclude the following.

Theorem 5. *For Riemann problem (7)-(8), there exists a unique entropy solution, which contains two different (or just one) contact discontinuities when $u_- < u_+ + (1/\rho_+)$ and a delta shock wave when $u_- \geq u_+ + (1/\rho_+)$.*

4. Numerical Simulations for Riemann Solutions

In this section, by employing the Nessyahu-Tadmor scheme [25] with 300×300 cells and $CFL = 0.475$, we simulate the obtained Riemann solutions in two cases $u_- < u_+ + (1/\rho_+)$ and $u_- \geq u_+ + (1/\rho_+)$. We take $(u_-, \rho_-) = (2, 1)$.

For the case $u_- < u_+ + (1/\rho_+)$, we take $(u_+, \rho_+) = (4, 2)$, and from (24), we can calculate $(u^*, \rho^*) = (2, 0.4)$. For the case $u_- \geq u_+ + (1/\rho_+)$, we take $(u_+, \rho_+) = (-1, 3,)$. The numerical results are presented in Figures 4 and 5, respectively.

It can be clearly observed that the contact discontinuities develop in Figure 4 while a delta shock wave is developed in Figure 5. All the numerical results are in complete agreement with the theoretical analysis.

5. Two kinds of Interactions of Elementary Waves

In this section, we investigate two kinds of interesting interactions of elementary waves by considering the initial-value problem (9) for system (7). The first case is that the interaction of two contact discontinuities results in a delta shock wave. The second case is that a delta shock wave vanishes after its interaction with a contact discontinuity.

Case (i). Assume that the initial data in (9) satisfy

$$u_- + \frac{1}{\rho_-} = u_m + \frac{1}{\rho_m}, \quad u_m = u_+. \tag{51}$$

At this moment, a contact discontinuity $\vec{J}$ emits from $(0, 0)$ with a speed $u_m + (1/\rho_m)$ and a contact discontinuity $\overleftarrow{J}$ emits from $(a, 0)$ with a speed u_m, respectively. From $u_m + (1/\rho_m) > u_m$, it is known that $\vec{J}$ will overtake $\overleftarrow{J}$ at a point (x_0, t_0), which can be calculated to be $(x_0, t_0) = (a(\rho_m u_m + 1), a\rho_m)$. At the moment $t = t_0$, a new Riemann problem is formed with (u_-, ρ_-) and (u_+, ρ_+) on both sides. We consider the following situation for ρ:

$$\frac{1}{\rho_m} - \frac{1}{\rho_-} \geq \frac{1}{\rho_+}, \tag{52}$$

which means that from (51)

$$u_- \geq u_+ + \frac{1}{\rho_+}. \tag{53}$$

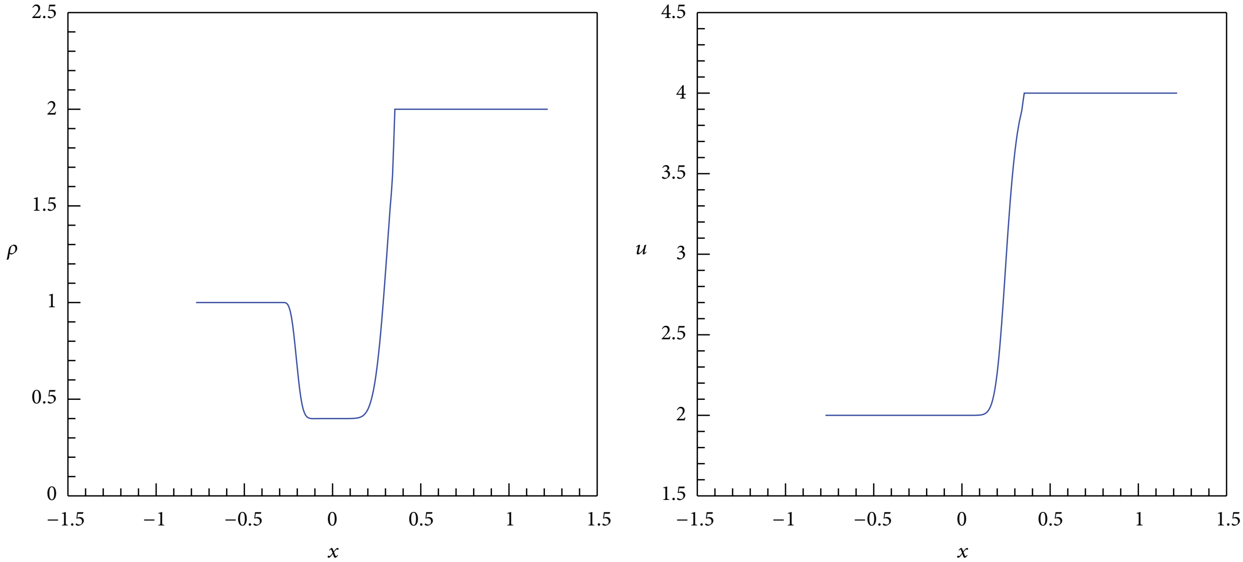
Then from Theorem 5, it can be concluded that a delta shock wave δ emits from the moment $t = t_0$, whose speed, location, and weight can be obtained by solving (30) under (33) with initial data

$$t = t_0 : \ x(t_0) = x_0, \qquad w(t_0) = 0. \tag{54}$$

See Figure 6(a), in which ① $= (u_i, \rho_i)\ (i = -, m, +)$.

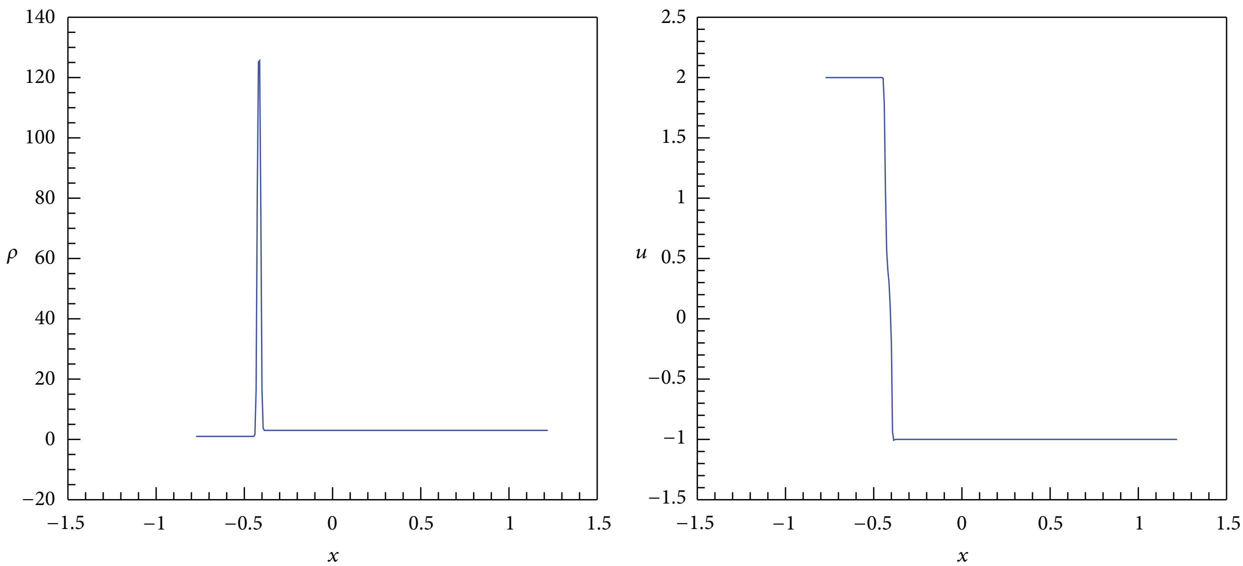
Case (ii). Suppose the initial data (9) in satisfy

$$u_- \geq u_m + \frac{1}{\rho_m}, \quad u_m = u_+. \tag{55}$$



(a) (b)

FIGURE 4: Numerical results for the case $u_- < u_+ + (1/\rho_+)$ at $t = 0.2$.



(a) (b)

FIGURE 5: Numerical results for the case $u_- \geq u_+ + (1/\rho_+)$ at $t = 0.5$.

At this time, a delta shock wave δ emits from $(0,0)$ with the speed σ satisfying $u_- \geq \sigma \geq u_m + (1/\rho_m)$ and a contact discontinuity $\overleftarrow{J}$ emits from $(a,0)$ with the speed u_m. Since $\sigma \geq u_m + (1/\rho_m) > u_m$, δ will interact with $\overleftarrow{J}$ at a point $(x_0, t_0) = (a\sigma/(\sigma - u_+), a/(\sigma - u_+))$. As before, at the moment $t = t_0$, a new Riemann problem is formed with (u_-, ρ_-) and (u_+, ρ_+) on both sides. Let us suppose that the initial data furthermore satisfy

$$u_- < u_+ + \frac{1}{\rho_+}, \tag{56}$$

then from Theorem 5, we can conclude that two contact discontinuities emit form $t = t_0$ as follows:

$$\overleftarrow{J} : \frac{dx}{dt} = u_-, \qquad \vec{J} : \frac{dx}{dt} = u_+ + \frac{1}{\rho_+}, \tag{57}$$

and the intermediate states (u^*, ρ^*) connecting two contact discontinuities satisfy

$$u^* = u_-, \qquad u^* + \frac{1}{\rho^*} = u_+ + \frac{1}{\rho_+}. \tag{58}$$

See Figure 6(b), in which $\circledast = (u^*, \rho^*)$.

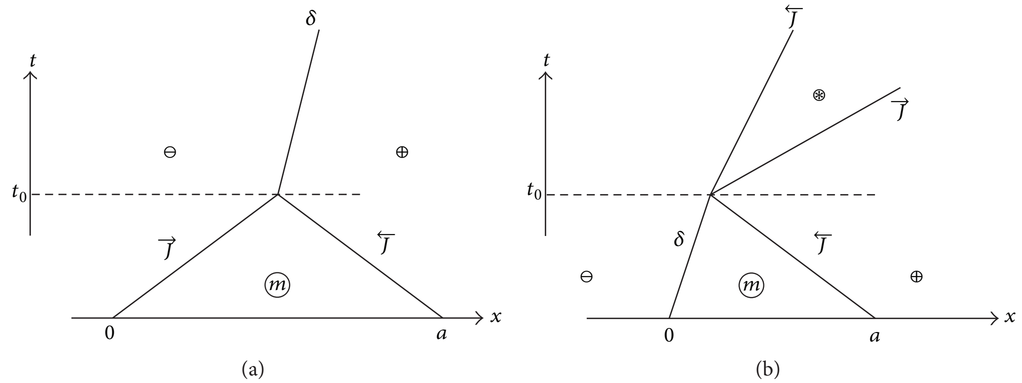


FIGURE 6: Two kinds of interactions of elementary waves.

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