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Research Article

Signorini Cylindrical Waves and Shannon Wavelets

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Hyperelastic materials based on Signorini's strain energy density are studied by using Shannon wavelets. Cylindrical waves propagating in a nonlinear elastic material from the circular cylindrical cavity along the radius are analyzed in the following by focusing both on the main nonlinear effects and on the method of solution for the corresponding nonlinear differential equation. Cylindrical waves' solution of the resulting equations can be easily represented in terms of this family of wavelets. It will be shown that Hankel functions can be linked with Shannon wavelets, so that wavelets can have some physical meaning being a good approximation of cylindrical waves. The nonlinearity is introduced by Signorini elastic energy density and corresponds to the quadratic nonlinearity relative to displacements. The configuration state of elastic medium is defined through cylindrical coordinates but the deformation is considered as functionally depending only on the radial coordinate. The physical and geometrical nonlinearities arising from the wave propagation are discussed from the point of view of wavelet analysis.

1. Introduction

In this paper, cylindrical waves arising from the nonlinear equation of hyperelastic Signorini materials [1–6] are studied. In particular, it will be shown that cylindrical waves can be easily given in terms of Shannon wavelets.

Hyperelastic materials based on Signorini's strain energy density [7, 8] were recently investigated [1–6, 9, 10], because of the simple form of the Signorini potential, which has the main advantage to be dependent only on three constants, including the two classical Lamé constants (λ, μ). Hyperelastic materials and composites are interesting for the many recent advances both in theoretical approaches and in practical discoveries of new composites, having extreme behaviors under deformation [9].

However, Signorini hyperelastic materials, as a drawback, lead to some nonlinear equations, to be studied in cylindrical coordinates [11–13]. The starting point, for searching the solution of these equations, is the Weber equation, which is classically solved by the special functions of Bessel type. Thus the main advantage of three parameters' potential is counterbalanced by the Bessel function approximation. It has been recently shown [14] that Bessel functions locally coincide with Shannon wavelets, thus enabling us to represent cylindrical waves by the multiscale approach [9, 15, 16] of Shannon wavelets [14, 17–20]. In this way, Shannon wavelets might have some physical meaning through the cylindrical waves propagation.

In recent years wavelets have been successfully applied to the wavelet representation of integrodifferential operators [16–24], thus giving rise to the so-called wavelet solutions of PDE (see, e.g., [16, 21, 22]) and integral equations (see, e.g., [20, 23, 24]).

In fact, wavelets enjoy many interesting features such as the localization, the multiscale representation, and the fast decay to zero (either in space or in frequency domain), which are a useful tool in many different applications, (see, e.g., [17–20] and references therein).

Usually wavelets have been used only as any other kind of orthogonal functions, with some additional features but seldom they have shown to have also some physical meanings [15, 25].

We will see that Shannon wavelets can approximate very well the Bessel functions, thus being the most suitable tool for investigating cylindrical waves. Shannon wavelets are analytically defined functions, infinitely differentiable, and sharply bounded in the frequency domain. Their derivatives can be defined to any order by a simple analytical function [17–20], thus enabling us to approximate a function and its derivatives and easily performing the projection of differential operators.

This paper is organized as follows. Section 2 deals with some preliminary remarks on the elastic materials in generalized coordinates. In Section 3, Signorini density energy is defined and the basic equations in cylindrical coordinates for wave propagation in materials are given. The main properties of Shannon wavelets, reconstruction of a function, and connection coefficients are shortly described in Section 4. In Section 5 the similarities and distinctions between Bessel functions and Shannon wavelets are given. Section 6 deals with some remarks on perturbation method. In the same section the Shannon wavelet solution of the nonlinear wave propagation is given and the corresponding nonlinear effects are commented.

2. Preliminary Remarks

Let $(\theta^1, \theta^2, \theta^3)$, be the (Lagrangian) cylindrical coordinate system $\theta^1 = r,\ \theta^2 = \vartheta,\ \theta^3 = z$, and $(ds)^2 = g_{ik} d\theta^i d\theta^k = (dr)^2 + r^2 (d\vartheta)^2 + (dz)^2$ with

$$\left\| g_{ik} \right\| = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{Vmatrix}, \qquad \left\| g^{ik} \right\| = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & \dfrac{1}{r^2} & 0 \\ 0 & 0 & 1 \end{Vmatrix}, \tag{2.1}$$

being the corresponding vector length and metric.

The Cauchy-Green strain tensor is defined as

$$\varepsilon_{ik} = \frac{1}{2}\left(\nabla_i u_k + \nabla_k u_i + \nabla_i u_j \nabla_k u^j \right), \tag{2.2}$$

with $\vec{u} = \{u_i\}$ being the displacement vector (in each point of the continuum).

The covariant derivatives of a vector $\{v_i\}$ are

$$\nabla_i v^k = \frac{\partial v^k}{\partial \theta^i} + v^j \Gamma^k_{ji}, \qquad \nabla_i v_j = \frac{\partial v_j}{\partial \theta^i} - v_k \Gamma^k_{ji} \tag{2.3}$$

and can be easily computed by means of the Christoffel's symbols

$$\Gamma^m_{ki} = \frac{1}{2} g^{mn} \left(\frac{\partial g_{kn}}{\partial \theta^i} + \frac{\partial g_{in}}{\partial \theta^k} - \frac{\partial g_{ki}}{\partial \theta^n} \right) \tag{2.4}$$

and the metric values (2.1). Thanks to (2.1) the only nonvanishing components of these symbols are

$$\Gamma^1_{22} = -r, \qquad \Gamma^2_{12} = \Gamma^2_{21} = \left(\frac{1}{r} \right). \tag{2.5}$$

Concerning the deformation, it can be classified according to the nonvanishing components of the displacement vector. We have cylindrical waves [4, 9, 10, 26–29] when

$$\vec{u}\left(\theta^1, \theta^2, \theta^3\right) = \vec{u}(r, \vartheta, z) = \{u_1 = u_r(r),\ u_2 = r \cdot u_\vartheta = u_3 = u_z = 0\}. \tag{2.6}$$

When the components of the Cauchy-Green tensor are known, we can easily evaluate the three invariants:

$$\begin{aligned}
I_1(\varepsilon_{ik}) &= \varepsilon_{ik} g^{ik} = \varepsilon_{11} \cdot 1 + \varepsilon_{22} \cdot \frac{1}{r^2} + \varepsilon_{33} \cdot 1, \\
I_2(\varepsilon_{ik}) &= \varepsilon_{im} \varepsilon_{nk} g^{ik} g^{nm} \\
&= (\varepsilon_{11} \cdot 1)^2 + \left(\varepsilon_{22} \cdot \frac{1}{r^2} \right)^2 + (\varepsilon_{33} \cdot 1)^2 + \left(\varepsilon_{12} \cdot \frac{1}{r} \right)^2 + \left(\varepsilon_{23} \cdot \frac{1}{r} \right)^2 + (\varepsilon_{13} \cdot 1)^2, \\
I_3(\varepsilon_{ik}) &= \varepsilon_{pm} \varepsilon_{in} \varepsilon_{kq} g^{im} g^{pq} g^{kn} \\
&= (\varepsilon_{11})^3 + \left(\varepsilon_{22} \frac{1}{r^2} \right)^3 + (\varepsilon_{33})^3 + (\varepsilon_{13} \cdot 1)\left(\varepsilon_{13}\varepsilon_{11} + \varepsilon_{23}\varepsilon_{12} \frac{1}{r^2} + \varepsilon_{13}\varepsilon_{33} \right) \\
&\quad + \left(\varepsilon_{12} \cdot \frac{1}{r^2} \right)\left(\varepsilon_{12}\varepsilon_{11} + \varepsilon_{12}\varepsilon_{22} \frac{1}{r^2} + \varepsilon_{13}\varepsilon_{23} \right) + \left(\varepsilon_{23} \cdot \frac{1}{r^2} \right)\left(\varepsilon_{12}\varepsilon_{13} + \varepsilon_{23}\varepsilon_{22} \frac{1}{r^2} + \varepsilon_{23}\varepsilon_{33} \right),
\end{aligned} \tag{2.7}$$

which, in dealing with hyperelastic materials, enable us to compute the potential.

In the case of cylindrical waves (2.6), by taking into account (2.2), the only nonzero components of the strain tensor are

$$\varepsilon_{11} = \varepsilon_{rr} = u_{r,r} + \frac{1}{2}(u_{r,r})^2, \qquad \varepsilon_{22} = r^2 \varepsilon_{\vartheta\vartheta} = r u_r + \frac{1}{2}(u_r)^2, \tag{2.8}$$

$$
\begin{aligned}
I_1(\varepsilon_{ik}) &= \varepsilon_{ik} g^{ik} = \varepsilon_{11} + \varepsilon_{22} \\
&= u_{r,r} + \frac{1}{2}(u_{r,r})^2 + r u_r + \frac{1}{2}(u_r)^2, \\
I_2(\varepsilon_{ik}) &= \varepsilon_{im}\varepsilon_{nk} g^{ik} g^{nm} = \varepsilon_{11}^2 + \frac{1}{r^4}\varepsilon_{22}^2 \\
&= (u_{r,r})^2 + (u_{r,r})^3 + \frac{1}{r^2}(u_r)^2 + \frac{1}{r^3}(u_r)^3 + \frac{1}{4}(u_{r,r})^4 + \frac{1}{4r^4}(u_r)^4, \\
I_3(\varepsilon_{ik}) &= \varepsilon_{pm}\varepsilon_{in}\varepsilon_{kq} g^{im} g^{pq} g^{kn} = \varepsilon_{11}^3 + \frac{1}{r^6}\varepsilon_{22}^3 = (u_{r,r})^3 + \frac{1}{r^3}(u_r)^3 \\
&\quad + \frac{3}{2}\left[(u_{r,r})^4 + \frac{1}{r^4}(u_r)^4\right] + \frac{3}{4}\left[(u_{r,r})^5 + \frac{1}{r^5}(u_r)^5\right] + \frac{1}{8}\left[(u_{r,r})^6 + \frac{1}{r^6}(u_r)^6\right],
\end{aligned}
\tag{2.9}
$$

so that, by neglecting displacements of order higher than three, we have

$$
\begin{aligned}
I_1(\varepsilon_{ik}) &= \varepsilon_{ik} g^{ik} = \varepsilon_{11} + \varepsilon_{22} \\
&= u_{r,r} + \frac{1}{2}(u_{r,r})^2 + r u_r + \frac{1}{2}(u_r)^2, \\
I_2(\varepsilon_{ik}) &= \varepsilon_{im}\varepsilon_{nk} g^{ik} g^{nm} = \varepsilon_{11}^2 + \frac{1}{r^4}\varepsilon_{22}^2 \cong (u_{r,r})^2 + (u_{r,r})^3 + \frac{1}{r^2}(u_r)^2 + \frac{1}{r^3}(u_r)^3, \\
I_3(\varepsilon_{ik}) &= \varepsilon_{pm}\varepsilon_{in}\varepsilon_{kq} g^{im} g^{pq} g^{kn} = \varepsilon_{11}^3 + \frac{1}{r^6}\varepsilon_{22}^3 \cong (u_{r,r})^3 + \frac{1}{r^3}(u_r)^3.
\end{aligned}
\tag{2.10}
$$

Signorini potential, which belongs to the polynomial hyperelastic model (also called generalized Rivlin model) [26–29], is defined as [1–9, 14]

$$
W = \left(\frac{1}{\sqrt{I_{A3}}}\right)\left[c I_{A2} + \left(\frac{1}{2}\right)\left(\lambda + \mu - \left(\frac{c}{2}\right)\right)(I_{A1})^2 + \left(\lambda + \left(\frac{c}{2}\right)\right)(1 - I_{A1})\right] - \left(\mu + \left(\frac{c}{2}\right)\right)
\tag{2.11}
$$

with

$$
\begin{aligned}
I_{A1} &= \frac{I_1 + 2(I_1)^2 - 2I_2 + 2(I_1)^3 - 6I_1I_2 + 4I_3}{1 + 2I_1 + 2(I_1)^2 - 2I_2 + (4/3)(I_1)^3 - 4I_1I_2 + (8/3)I_3}, \\
I_{A2} &= \frac{1}{2}(I_1)^2 - \frac{(1/2)(I_1)^2 - (1/2)I_2 + (I_1)^3 - 3I_1I_2 + 2I_3}{1 + 2I_1 + 2(I_1)^2 - 2I_2 + (4/3)(I_1)^3 - 4I_1I_2 + (8/3)I_3}, \\
I_{A3} &= \frac{2}{3}I_1I_2 - \frac{1}{4\sqrt{3}}(I_1)^3 \frac{(I_1)^3 - I_1I_2 + 2I_3}{1 + 2I_1 + 2(I_1)^2 - 2I_2 + (4/3)(I_1)^3 - 4I_1I_2 + (8/3)I_3}.
\end{aligned}
\tag{2.12}
$$

Therefore from the previous equation, by taking into account (2.10), we have the approximation

$$
\begin{aligned}
I_{A1} &\cong (\varepsilon_{11}+\varepsilon_{22}) + 2(\varepsilon_{11}+\varepsilon_{22})^2 - 2\left(\varepsilon_{11}^2 + \frac{1}{r^4}\varepsilon_{22}^2\right) + 2(\varepsilon_{11}+\varepsilon_{22})^3 \\
&\quad - 6(\varepsilon_{11}+\varepsilon_{22})\left(\varepsilon_{11}^2 + \frac{1}{r^4}\varepsilon_{22}^2\right) + 4\left(\varepsilon_{11}^3 + \frac{1}{r^6}\varepsilon_{22}^3\right) \\
&= (\varepsilon_{11}+\varepsilon_{22}) + 2\varepsilon_{11}\varepsilon_{22}, \\
I_{A2} &\cong -\frac{1}{2}\left(\varepsilon_{11}^2 + \varepsilon_{22}^2\right) + (\varepsilon_{11}+\varepsilon_{22})^3 - 3(\varepsilon_{11}+\varepsilon_{22})\left(\varepsilon_{11}^2 + \frac{1}{r^4}\varepsilon_{22}^2\right) + 2\left(\varepsilon_{11}^3 + \frac{1}{r^6}\varepsilon_{22}^3\right) \\
&= -\frac{1}{2}\left(\varepsilon_{11}^2 + \frac{1}{r^4}\varepsilon_{22}^2\right), \\
I_{A3} &\cong \frac{2}{3}(\varepsilon_{11}+\varepsilon_{22})\left(\varepsilon_{11}^2 + \frac{1}{r^4}\varepsilon_{22}^2\right),
\end{aligned}
\tag{2.13}
$$

that is

$$
\begin{aligned}
I_{A1} &\cong (\varepsilon_{11}+\varepsilon_{22}) + 2\varepsilon_{11}\varepsilon_{22}, \\
I_{A2} &\cong -\frac{1}{2}\left(\varepsilon_{11}^2 + \frac{1}{r^4}\varepsilon_{22}^2\right), \\
I_{A3} &\cong \frac{2}{3}(\varepsilon_{11}+\varepsilon_{22})\left(\varepsilon_{11}^2 + \frac{1}{r^4}\varepsilon_{22}^2\right),
\end{aligned}
\tag{2.14}
$$

and, according to (2.8),

$$
\begin{aligned}
I_{A1} &\cong \left(u_{r,r} + \frac{1}{2}(u_{r,r})^2 + ru_r + \frac{1}{2}(u_r)^2\right) + 2\left[u_{r,r} + \frac{1}{2}(u_{r,r})^2\right]\left[ru_r + \frac{1}{2}(u_r)^2\right], \\
I_{A2} &\cong -\frac{1}{2}\left(\left[u_{r,r} + \frac{1}{2}(u_{r,r})^2\right]^2 + \frac{1}{r^4}\left[ru_r + \frac{1}{2}(u_r)^2\right]^2\right), \\
I_{A3} &\cong \frac{2}{3}\left(u_{r,r} + \frac{1}{2}(u_{r,r})^2 + ru_r + \frac{1}{2}(u_r)^2\right)\left(\left[u_{r,r} + \frac{1}{2}(u_{r,r})^2\right]^2 + \frac{1}{r^4}\left[ru_r + \frac{1}{2}(u_r)^2\right]^2\right).
\end{aligned}
\tag{2.15}
$$

3. Cylindrical Waves Equation

The basic equations of motion are [1–5, 9, 10, 14, 26–29]

$$
\nabla_i T^{ik} - \rho \nabla_i \varepsilon^{ik} = \frac{\partial^2 u^k}{\partial t^2}, \tag{3.1}
$$

where T^{ik} is the Piola-Kirchoff stress tensor. For hyperelastic materials it is $T^{ik} = (\partial W/\partial \varepsilon_{ik})$ where W is given by (2.11), for Signorini's materials.

Taking into account that

$$\frac{\partial W}{\partial \varepsilon_{ik}} = \sum_{h=1}^{3} \frac{\partial W}{\partial I_{Ah}} \frac{\partial I_{Ah}}{\partial \varepsilon_{ik}}, \tag{3.2}$$

and, according to (2.11), it is

$$\begin{gathered}
\frac{\partial W}{\partial I_{A1}} = \left(\frac{1}{\sqrt{I_{A3}}}\right)\left[\left(\lambda + \mu - \left(\frac{c}{2}\right)\right)I_{A1} - \left(\lambda + \left(\frac{c}{2}\right)\right)\right], \\
\frac{\partial W}{\partial I_{A2}} = \frac{c}{2\sqrt{I_{A3}}}, \\
\frac{\partial W}{\partial I_{A3}} = \frac{1}{2\sqrt{I_{A3}}}\left[cI_{A2} + \left(\frac{1}{2}\right)\left(\lambda + \mu - \left(\frac{c}{2}\right)\right)(I_{A1})^2 + \left(\lambda + \left(\frac{c}{2}\right)\right)(1 - I_{A1})\right],
\end{gathered} \tag{3.3}$$

and, by (2.14), the only unvanishing derivatives are

$$\begin{gathered}
\frac{\partial I_{A1}}{\partial \varepsilon_{11}} = 1 + 2\varepsilon_{22}, \qquad \frac{\partial I_{A1}}{\partial \varepsilon_{22}} = 1 + 2\varepsilon_{11}, \\
\frac{\partial I_{A2}}{\partial \varepsilon_{11}} = -\varepsilon_{11}, \qquad \frac{\partial I_{A2}}{\partial \varepsilon_{22}} = -\frac{1}{r^4}\varepsilon_{22}, \\
\frac{\partial I_{A3}}{\partial \varepsilon_{11}} = 2\varepsilon_{11} + \frac{2}{3r^4}\varepsilon_{22}^2 + \frac{4}{3}\varepsilon_{11}\varepsilon_{22}, \qquad \frac{\partial I_{A3}}{\partial \varepsilon_{22}} = \frac{2}{3}\varepsilon_{11}^2 + \frac{2}{r^4}\varepsilon_{22}^2 + \frac{4}{3}\varepsilon_{11}\varepsilon_{22}.
\end{gathered} \tag{3.4}$$

The Piola-Kirchoff tensor for the Signorini model (see also [3–5, 9]) is

$$T^{ik} = \left[\lambda I_{A1} + cI_{A2} + \frac{1}{2}\left(\lambda + \mu - \frac{c}{2}\right)(I_{A1})^2\right]g^{ik} + 2\left[\mu - \left(\lambda + \mu + \frac{c}{2}\right)I_{A1}\right]\varepsilon^{ik} + 2c\left(\varepsilon^{ij}\varepsilon_j^k\right). \tag{3.5}$$

In the strain components, we will neglect those terms with order higher than 3, so that the only unvanishing components of T^{ik} are

$$\begin{aligned}
T^{11} &= \left[\lambda I_{A1} + cI_{A2} + \frac{1}{2}\left(\lambda + \mu - \frac{c}{2}\right)(I_{A1})^2\right] + 2\left[\mu - \left(\lambda + \mu + \frac{c}{2}\right)I_{A1}\right]\varepsilon_{11} + 2c(\varepsilon_{11})^2, \\
T^{22} &= \frac{1}{r^2}\left[\lambda I_{A1} + cI_{A2} + \frac{1}{2}\left(\lambda + \mu - \frac{c}{2}\right)(I_{A1})^2\right] + 2\left[\mu - \left(\lambda + \mu + \frac{c}{2}\right)I_{A1}\right]\frac{\varepsilon_{22}}{r^2} + 2c\left(\frac{\varepsilon_{22}}{r^2}\right)^2, \\
T^{33} &= \left[\lambda I_{A1} + cI_{A2} + \frac{1}{2}\left(\lambda + \mu - \frac{c}{2}\right)(I_{A1})^2\right].
\end{aligned} \tag{3.6}$$

By using (2.8),(2.10), and (3.6) we finally get the Kirchoff tensor in terms of displacements:

$$\begin{aligned}
T^{11} &= T^{rr} \\
&= (\lambda + 2\mu) u_{r,r} + \lambda \frac{u_r}{r} + \frac{1}{4}(-10\lambda - 4\mu + 5c)(u_{r,r})^2 + \frac{1}{2}(2\lambda - 2\mu - 5c)\frac{1}{r}\, u_r u_{r,r} \\
&\quad + \frac{1}{4}(6\lambda + 2\mu + c)\frac{1}{r^2}(u_r)^2 + \frac{1}{2}(6\lambda + 13c)(u_{r,r})^3 + \frac{1}{4}(70\lambda - 18\mu + c)\frac{1}{r}\, u_r u_{r,r} \\
&\quad + \frac{1}{4}(-42\lambda + 10\mu + 15c)\frac{1}{r^2}(u_r)^2 + \frac{1}{2}(4\lambda - 2\mu + 3c)\frac{1}{r^3}(u_r)^3, \\
r^2 T^{22} &= T_{\vartheta\vartheta} \\
&= (\lambda + 2\mu)\frac{u_r}{r} + \lambda u_{r,r} + \frac{1}{4}(-2\lambda + 2\mu + c)(u_{r,r})^2 + \frac{1}{2}(2\lambda - 2\mu - 5c)\frac{1}{r}\, u_r u_{r,r} \\
&\quad + \frac{1}{4}(-2\lambda - 4\mu + 5c)\frac{1}{r^2}(u_r)^2 + (2\lambda - \mu + 4c)\ (u_{r,r})^3 + \frac{1}{4}(-42\lambda + 10\mu + 15c)\frac{1}{r} u_r u_{r,r} \\
&\quad + \frac{1}{4}(70\lambda - 18\mu + c)\frac{1}{r^2}(u_r)^2 + (-\lambda + 4c)\frac{1}{r^3}(u_r)^3.
\end{aligned} \tag{3.7}$$

From (3.1) the only nontrivial equation is the first one:

$$\begin{aligned}
(\lambda + 2\mu)\left(u_{r,rr} + \frac{u_{r,r}}{r} + u_r - \frac{u_r}{r^2}\right) - \rho \ddot{u}_r &= S_1 u_{r,rr} u_{r,r} + S_2 \frac{1}{r} u_{r,rr} u_r + S_3 \frac{1}{r}(u_{r,r})^2 + S_4 \frac{1}{r^2} u_{r,r} u_r \\
&\quad + S_5 \frac{1}{r^3}(u_r)^2 + S_6 u_{r,rr}(u_{r,r})^2 + S_7 \frac{1}{r^3} u_{r,rr}(u_r)^2 \\
&\quad + S_8 \frac{1}{r} u_{r,rr} u_{r,r} u_r + S_9 \frac{1}{r}(u_{r,r})^3 + S_{10} \frac{1}{r^4}(u_r)^3 \\
&\quad + S_{11} \frac{1}{r^2}(u_{r,r})^2 u_r + S_{12} \frac{1}{r^3} u_{r,r}(u_r)^2,
\end{aligned} \tag{3.8}$$

where the coefficients $S_1, S_2, \ldots, S_{12}$ depend on Signorini parameters $\lambda, \mu,$ and c:

$$\begin{gathered}
S_1 = \frac{1}{2}(-6\lambda + 4\mu + 5c), \qquad S_2 = \frac{1}{2}(4\lambda - 2\mu - 5c), \qquad S_3 = \frac{1}{2}(2\lambda - \mu - 3c), \\
S_4 = \frac{1}{2}(2\mu - 5c), \qquad S_5 = \frac{1}{2}(5\mu - 3c), \qquad S_6 = \frac{1}{4}(9\lambda - 12\mu + 93c), \\
S_7 = \frac{1}{2}(24\lambda - 4\mu - 7c), \qquad S_8 = 36\lambda - 10\mu - 2c, \qquad S_9 = \frac{1}{2}(32\lambda - 13\mu - 2c), \\
S_{10} = -\frac{1}{4}(10\lambda + c), \qquad S_{11} = \frac{1}{4}(-74\lambda + 26\mu + 33c), \qquad S_{12} = \frac{1}{4}(22\lambda - 18\mu + 7c).
\end{gathered} \tag{3.9}$$

In the following, we will search solutions in the following form:

$$u_r = e^{i\omega t} u(r), \tag{3.10}$$

where time-harmonic waves $e^{i\omega t}$ are separated by the longitudinal waves $u(r)$, so that

$$\ddot{u}_r = -\omega^2 u_r, \qquad \omega = \sqrt{\frac{1-(\lambda+2\mu)}{\rho}}, \tag{3.11}$$

and $u(r)$ is the solution of the following equation:

$$\begin{aligned}\left(u_{,rr} + \frac{u_{,r}}{r} + u - \frac{u}{r^2}\right) &= a_1 u_{,rr}u_{,r} + a_2\frac{1}{r}u_{,rr}u + a_3\frac{1}{r}(u_{,r})^2 + a_4\frac{1}{r^2}u_{,r}u + a_5\frac{1}{r^3}(u)^2 \\ &+ a_6 u_{,rr}(u_{,r})^2 + a_7\frac{1}{r^3}u_{,rr}(u)^2 + a_8\frac{1}{r}u_{,rr}u_{,r}u \\ &+ a_9\frac{1}{r}(u_{,r})^3 + a_{10}\frac{1}{r^4}(u)^3 + a_{11}\frac{1}{r^2}(u_{,r})^2 u + a_{12}\frac{1}{r^3}u_{,r}(u)^2,\end{aligned} \tag{3.12}$$

with $a_i = S_i/(\lambda+2\mu),\ i = 1,\ldots,12$.

Equation (3.12) gives the more general model of cylindrical wave propagation for Signorini hyperelastic materials. At the r.h.s. there appear nonlinear terms up to the third order in $u, u_{,r},$ and $u_{,rr}$ while the coefficients depend on both inverse r up to the 4th power and the physical parameters $\lambda, \mu,$ and c. In the following we will search the Shannon wavelet solution of (3.12), by neglecting $O(r^{-1})$ terms in the r.h.s., by showing that Shannon wavelets are linked with Bessel functions.

3.1. Linear Equation

If we neglect the nonlinear terms of the right-hand side, from (3.12) we simply get the linear equation:

$$\left(u_{,rr} + \frac{u_{,r}}{r} + u - \frac{u}{r^2}\right) = 0, \tag{3.12$'$}$$

which is the (homogeneous) Weber equation [30, 31], classically solved by Bessel functions.

In fact, Bessel function $J_n(x)$ of order n is defined as the solution of the Weber equation:

$$x^2 y'' + xy' + \left(x^2 - n^2\right)y = 0, \quad n \in \mathbb{C}. \tag{3.13}$$

In particular, when $n = 1$, the more general solution of

$$x^2 y'' + xy' + \left(x^2 - 1\right)y = 0 \tag{3.14}$$

is

$$y(x) = c_1 J_1(x) + c_2 J_2(x). \tag{3.15}$$

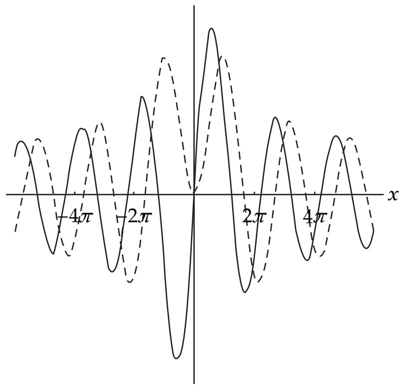

Figure 1: Bessel Functions $J_1(x)$ (bold) and $J_2(x)$ (dashed).

The Taylor series for Bessel function is

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(n+k+1)} \left(\frac{1}{2}x\right)^{2k+n}, \quad x \in (-\varepsilon, \varepsilon) \tag{3.16}$$

with $\Gamma(n)$ being gamma function.

So, for integer values of n, being $\Gamma(n+1) = n!$, there result

$$\begin{aligned} J_1(x) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+1)!} \left(\frac{1}{2}x\right)^{2k+1} \\ &= \left(\frac{1}{2}x\right) - \frac{1}{2!}\left(\frac{1}{2}x\right)^3 + \frac{1}{2!3!}\left(\frac{1}{2}x\right)^5 - \frac{1}{3!4!}\left(\frac{1}{2}x\right)^7 \cdots, \\ J_2(x) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+2)!} \left(\frac{1}{2}x\right)^{2k+2} \\ &= \frac{1}{2!}\left(\frac{1}{2}x\right)^2 - \frac{1}{3!}\left(\frac{1}{2}x\right)^4 + \frac{1}{2!4!}\left(\frac{1}{2}x\right)^6 + \cdots. \end{aligned} \tag{3.17}$$

It can be easily seen that $J_{2n}(x)$ ($n \in \mathbb{N}$) are even functions and $J_{2n+1}(x)$ ($n \in \mathbb{N}$) are odd functions, while both are localized functions with some decay to zero (Figure 1).

A good approximation of $J_1(x)$ in the interval $[-\pi/2, \pi/2]$ can be already obtained by the third-order polynomial, while with the 7th power polynomial we can have a good approximation in $[-\pi, \pi]$:

$$J_1(x) \cong \left(\frac{1}{2}x\right) - \frac{1}{2!}\left(\frac{1}{2}x\right)^3 + \frac{1}{2!3!}\left(\frac{1}{2}x\right)^5 - \frac{1}{3!4!}\left(\frac{1}{2}x\right)^7. \tag{3.16'}$$

Analogously a good approximation of $J_2(x)$ is obtained in the interval $[-\pi/2, \pi/2]$ by a second order polynomial, whereas with the 6th power polynomial we have a good approximation in $[-\pi, \pi]$

$$J_2(x) \cong \frac{1}{2!}\left(\frac{1}{2}x\right)^2 - \frac{1}{3!}\left(\frac{1}{2}x\right)^4 + \frac{1}{2!4!}\left(\frac{1}{2}x\right)^6. \tag{3.16$''$}$$

3.2. *Second-Order Equation*

Equation (3.12) gives rise to many interesting nonlinear equations for cylindrical waves. In fact, up to the second-order nonlinearities, it becomes

$$\left(u_{,rr} + \frac{u_{,r}}{r} + u - \frac{u}{r^2}\right) = u_{,r}\left(a_1 u_{,rr} + a_3 \frac{u_{,r}}{r} - a_4 \frac{u}{r^2}\right) + \frac{1}{r} u\left(a_2 u_{,rr} - a_5 \frac{u}{r^2}\right). \tag{3.18}$$

So, by keeping only the first term of the right-hand side, which is equivalent to neglect terms $O(1/r)$, we have

$$\left(u_{,rr} + \frac{u_{,r}}{r} + u - \frac{u}{r^2}\right) = a_1 u_{,r} u_{,rr}. \tag{3.12$''$}$$

3.3. *Third-Order Equation*

Up to the third-order nonlinearities, and neglecting all terms $O(1/r)$, (3.12) gives

$$\left(u_{,rr} + \frac{u_{,r}}{r} + u - \frac{u}{r^2}\right) = u_{,rr}\left[a_1 u_{,r} + a_6 (u_{,r})^2\right]. \tag{3.12$'''$}$$

We will give the solutions of (3.12′), (3.12″), and (3.12‴) by using Shannon wavelets. In order to do so, we need first to show that, in a sufficient large neighborhood of zero, Shannon wavelets are equivalent to the Bessel function. We can also see that at the same approximation the Taylor polynomial for Shannon wavelets is one order lower than the Taylor polynomial for the corresponding Bessel function, so that Shannon wavelets are more efficient from computational point of view.

4. Shannon Wavelet

In this section Shannon wavelets and their differential properties are shortly summarized (for further readings and explicit computations see, e.g., [14, 17–20] and references therein).

Shannon scaling function $\varphi(x)$ and wavelet function $\psi(x)$ are localized functions with some decay to zero (like Bessel functions), defined as

$$\begin{aligned}\varphi(x) &= \operatorname{sinc} x = \frac{\sin \pi x}{\pi x} = \frac{e^{\pi i x} - e^{-\pi i x}}{2\pi i x},\\ \psi(x) &= \frac{\sin \pi(x-1/2) - \sin 2\pi(x-1/2)}{\pi(x-1/2)}\\ &= \frac{e^{-2i\pi x}\left(-i + e^{i\pi x} + e^{3\, i\pi x} + i\ e^{4\, i\pi x}\right)}{(\pi - 2\pi x)}.\end{aligned} \tag{4.1}$$

The corresponding families of translated and dilated instances wavelet [17–20], on which is based the multiscale analysis, are

$$\begin{aligned}\varphi_k^n(x) &= 2^{n/2}\varphi(2^n x - k) = 2^{n/2}\frac{\sin \pi(2^n x - k)}{\pi(2^n x - k)}\\ &= 2^{n/2}\frac{e^{\pi i(2^n x - k)} - e^{-\pi i(2^n x - k)}}{2\pi i(2^n x - k)},\\ \psi_k^n(x) &= 2^{n/2}\ \frac{\sin \pi(2^n x - k - 1/2) - \sin 2\pi(2^n x - k - 1/2)}{\pi(2^n x - k - 1/2)},\\ &= \frac{2^{n/2}}{2\pi(2^n x - k + 1/2)}\sum_{s=1}^{2} i^{1+s} e^{s\pi i(2^n x - k)} - i^{1-s} e^{-s\pi i(2^n x - k)},\end{aligned} \tag{4.2}$$

with $\varphi_k^0(x) = \varphi_k(x)$ and $\psi_k^0(x) = \psi_k(x)$. In the following, we will denote

$$\varphi_0^0(x) = \varphi(x), \qquad \psi_0^0(x) = \psi(x). \tag{4.3}$$

Both families of Shannon scaling and wavelet are $L_2(\mathbb{R})$-functions, with a slow decay to zero, so that

$$\lim_{x \to \pm\infty} \varphi_k^n(x) = 0, \qquad \lim_{x \to \pm\infty} \psi_k^n(x) = 0. \tag{4.4}$$

For each $f(x) \in L_2(\mathbb{R})$ and $g(x) \in L_2(\mathbb{R})$, the inner product is defined as

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(x)\overline{g(x)}dx, \tag{4.5}$$

where the bar stands for the complex conjugate.

With respect to this inner product, Shannon wavelets are orthogonal functions so that [18–20]

$$\begin{aligned}\left\langle \psi_k^n(x), \psi_h^m(x) \right\rangle &= \delta^{nm}\delta_{hk},\\ \left\langle \varphi_k^0(x), \varphi_h^0(x) \right\rangle &= \delta_{kh},\\ \left\langle \varphi_k^0(x), \psi_h^m(x) \right\rangle &= 0, \quad m \geq 0,\end{aligned} \tag{4.6}$$

with δ^{nm} and δ_{hk} being the Kroenecker symbols.

Let $f(x) \in L_2(\mathbb{R})$ be a function such that the integrals

$$\begin{aligned} \alpha_k &= \left\langle f(x), \varphi_k^0(x) \right\rangle = \int_{-\infty}^{\infty} f(x)\overline{\varphi_k^0(x)}dx, \\ \beta_k^n &= \left\langle f(x), \psi_k^n(x) \right\rangle = \int_{-\infty}^{\infty} f(x)\overline{\psi_k^n(x)}dx \end{aligned} \tag{4.7}$$

exist and have finite values; it can be shown that the series

$$f(x) = \sum_{h=-\infty}^{\infty} \alpha_h \varphi_h^0(x) + \sum_{n=0}^{\infty} \sum_{k=-\infty}^{\infty} \beta_k^n \psi_k^n(x) \tag{4.8}$$

on the right side converges to $f(x)$. For a fixed upper bound we simply have the approximation (for the error estimate see [20])

$$f(x) \cong \sum_{h=-K}^{K} \alpha_h \varphi_h^0(x) + \sum_{n=0}^{N} \sum_{k=-S}^{S} \beta_k^n \psi_k^n(x). \tag{4.9}$$

4.1. Differentiable Properties of Shannon Wavelets

The derivatives of the Shannon wavelets are [19, 20]

$$\begin{aligned} \frac{d^\ell}{d\,x^\ell}\varphi_h^0(x) &= \sum_{k=-\infty}^{\infty} \lambda_{hk}^{(\ell)}\ \varphi_k^0(x), \\ \frac{d^\ell}{dx^\ell}\psi_h^m(x) &= \sum_{n=0}^{\infty} \sum_{k=-\infty}^{\infty} \gamma^{(\ell)mn}_{\ \ hk}\psi_k^n(x)\ , \end{aligned} \tag{4.10}$$

with

$$\lambda_{kh}^{(\ell)} \stackrel{\text{def}}{=} \left\langle \frac{d^\ell}{dx^\ell}\varphi_k^0(x), \varphi_h^0(x) \right\rangle, \qquad \gamma^{(\ell)mn}_{\ \ hk} \stackrel{\text{def}}{=} \left\langle \frac{d^\ell}{dx^\ell}\psi_k^n(x), \psi_h^m(x) \right\rangle \tag{4.11}$$

being the connection coefficients [18–20].

It has been shown [19, 20] that

$$\lambda_{kh}^{(\ell)} = \begin{cases} (-1)^{k-h}\dfrac{i^\ell}{2\pi}\displaystyle\sum_{s=1}^{\ell}\frac{\ell!\pi^s}{s![i(k-h)]^{\ell-s+1}}\left[(-1)^s - 1\right], & k \neq h, \\ \dfrac{i^\ell \pi^{\ell+1}}{2\pi(\ell+1)}\left[1 + (-1)^\ell\right], & k = h, \end{cases} \tag{4.12}$$

when $\ell \geq 1$, and $\lambda_{kh}^{(0)} = \delta_{kh}$, and

$$\gamma^{(\ell)mn}_{\ \ hk} = \mu(h-k)\delta^{nm}\left\{\sum_{s=1}^{\ell+1}(-1)^{[1+\mu(h-k)](2\ell-s+1)/2}\frac{\ell! i^{\ell-s}\ \pi^{\ell-s}}{(\ell-s+1)!|h-k|^{s}}(-1)^{-s-2(h+k)}2^{n\ell-s-1}\right.$$
$$\left.\times\left\{2^{\ell+1}\left[(-1)^{4h+s}+(-1)^{4k+\ell}\right]-2^{s}\left[(-1)^{3k+h+\ell}+(-1)^{3h+k+s}\right]\right\}\right\},\quad k\neq h,$$
$$\gamma^{(\ell)mn}_{\ \ hk} = \delta^{nm}\left[i^{\ell}\frac{\pi^{\ell}2^{n\ell-1}}{\ell+1}\left(2^{\ell+1}-1\right)\left(1+(-1)^{\ell}\right)\right],\quad k=h, \tag{4.13}$$

with

$$\mu(m) = \operatorname{sign}(m) = \begin{cases} 1, & m>0, \\ -1, & m<0 \\ 0, & m=0. \end{cases}, \tag{4.14}$$

According to (4.10) the Taylor series of the scaling and Shannon wavelet, nearby the origin, are

$$\varphi_h^0(x) = \sum_{\ell=0}^{\infty}\frac{1}{\ell!}\lambda_{h0}^{(\ell)}x^{\ell},\quad |x|<\varepsilon,$$
$$\psi_h^m(x) = \sum_{\ell=0}^{\infty}\frac{1}{\ell!}\gamma^{(\ell)mn}_{\ \ h0}2^{m/2}\left(x-2^{-m-1}\right)^{\ell},\quad \left|x-2^{-m-1}\right|<\varepsilon. \tag{4.15}$$

5. Similarities between Bessel Functions and Shannon Wavelets

Since Bessel functions are $L_2(\mathbb{R})$, they can be easily represented in terms of Shannon wavelets as follows:

$$J_1(x) \cong -\frac{1}{2}\psi\left(\frac{x}{3\sqrt{2}}+\frac{1}{5}\right)-0.08,\quad x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right),$$
$$J_2(x) \cong -\frac{1}{2}\varphi\left(\frac{x}{2\sqrt{2}}\right)+\frac{1}{2},\quad x\in(-\pi,\pi). \tag{5.1}$$

In particular, around $x = 0$ they nearly coincide with the Shannon scaling and wavelet, so that the even $J_{2n}(x)$, $(n \in \mathbb{N})$ can be well approximated by the scaling Shannon functions (Figure 2), while the odd Bessel functions $J_{2n+1}(x)$ and $(n \in \mathbb{N})$ can be approximated by the Shannon wavelets (Figure 3).

Although this approximation for both is restricted to an interval, we can assume that in the interval $|\varepsilon| \leq \pi/2$, where the perturbation method is applied, Bessel functions substantially coincide with the Shannon wavelet families; in other words, Shannon scaling functions and Shannon wavelets are solution of the Weber equation in the interval $|\varepsilon| \leq \pi/2$.

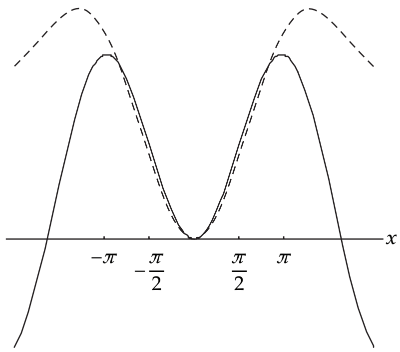

Figure 2: Bessel function $J_2(x)$ and (dashed) the Shannon scaling function $-(1/2)\varphi(x/2\sqrt{2}) + 1/2$.

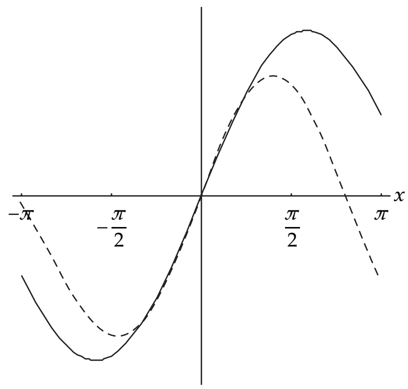

Figure 3: Bessel function $J_1(x)$ and (dashed) the Shannon wavelet function $-(1/2)\psi(x/3\sqrt{2} + 1/5) - 0.08$.

According to (4.13) and (4.15) the Taylor expansion (in $x = 0$) for the scaling wavelet is

$$\varphi(x) = \sum_{k=0}^{\infty} (-1)^k \frac{\pi^{2k} x^{2k}}{(2k+1)!}, \tag{5.2}$$

so that at the sixth order

$$\varphi(x) = 1 - \frac{\pi^2 x^2}{3!} + \frac{\pi^4 x^4}{5!} - \frac{\pi^6 x^6}{7!} \cdots . \tag{5.3}$$

Analogously, for the Shannon wavelet $\psi(x)$, in $x = 0$, it is up to the sixth order;

$$\psi(x) = -1 + \frac{1}{2!}\frac{7}{3}\pi^2\left(x - \frac{1}{2}\right)^2 - \frac{1}{4!}\frac{31}{5}\pi^4\left(x - \frac{1}{2}\right)^4 + \frac{1}{6!}\frac{127}{7}\pi^6\left(x - \frac{1}{2}\right)^6. \tag{5.4}$$

By comparing the Taylor expansion for Bessel functions, as given by (3.16′), and (3.16″) and the Taylor expansion of Shannon wavelets (5.3) and (5.4), we can see that a good

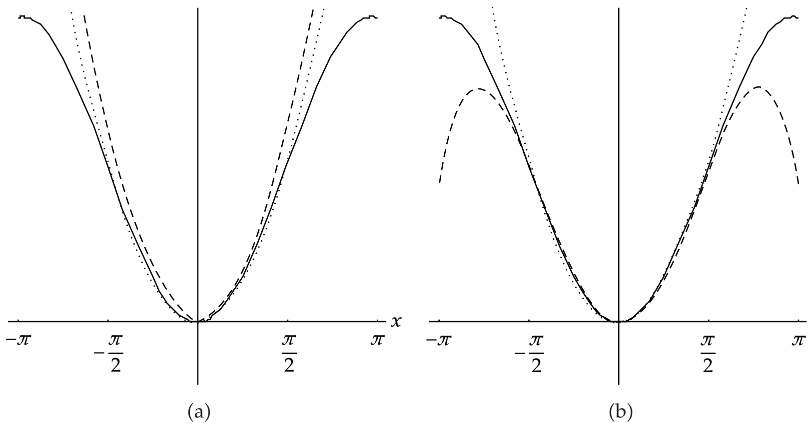

Figure 4: Approximation of $J_2(x)$ with a second-order polynomial from (5.3) (dotted) and, dashed, the Taylor polynomial (3.16″) at the second order (a) and fouth order (b).

approximation of the Bessel can be obtained by a lower-order polynomial approximation of the Shannon wavelet (Figures 4 and 5).

Taking into account (4.15) and (5.2) it can be easily shown that, for $x \in (-\pi, \pi)$, the error of the approximation in (5.1) tends to zero for $k \to \infty$. For instance, it is

$$\left| J_2(x) + \frac{1}{2}\varphi\left(\frac{x}{2\sqrt{2}}\right) - \frac{1}{2} \right| \le \left| J_2(x) + \frac{1}{2}\varphi\left(\frac{x}{2\sqrt{2}}\right) \right|,$$
$$\left| J_2(x) + \frac{1}{2}\varphi\left(\frac{x}{2\sqrt{2}}\right) \right| \overset{(4.15),(5.2)}{=} \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{2^{2k+1}} \left(\frac{x^2}{2k!(k+2)!} + \frac{\pi^{2k}}{2^k(2k+1)!} \right), \tag{5.5}$$

so that for $|x| \le \pi$ it is

$$\left| J_2(x) + \frac{1}{2}\varphi\left(\frac{x}{2\sqrt{2}}\right) \right| \le \sum_{k=0}^{\infty} (-1)^k \frac{\pi^{2k}}{2^{2k+1}} \left(\frac{\pi^2}{2k!(k+2)!} + \frac{\pi^{2k}}{2^k(2k+1)!} \right). \tag{5.6}$$

The series at the r.h.s is an alternating series which converges to zero, since, according to Leibniz rule, it is

$$\lim_{k\to\infty} \frac{\pi^{2k}}{2^{2k+1}} \left(\frac{\pi^2}{2k!(k+2)!} + \frac{\pi^{2k}}{2^k(2k+1)!} \right) = 0. \tag{5.7}$$

Analogously, we can show the same result for the wavelet approximation $(4.15)_1$ of the Bessel function $J_1(x)$.

By using the approximation (5.1) we can assume as solution of the Weber equation (3.12′) the Shannon wavelet

$$-\frac{1}{2}\varphi\left(\frac{x}{3\sqrt{2}} + \frac{1}{5}\right) - 0.08, \quad x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right). \tag{5.8}$$

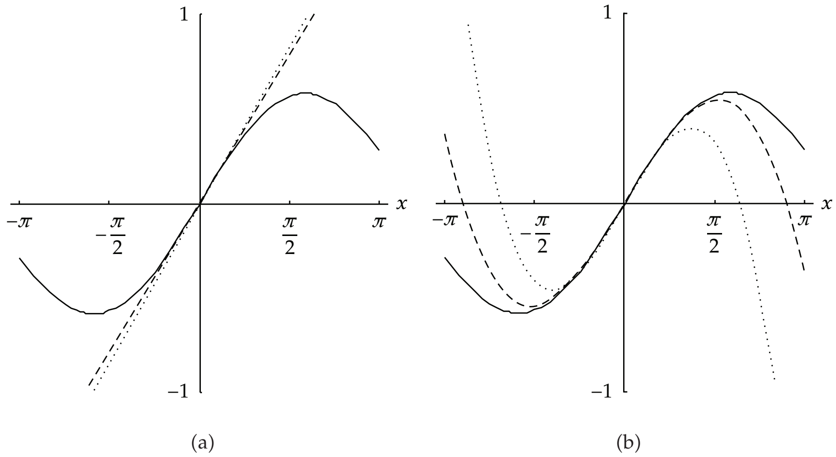

Figure 5: Approximation of $J_1(x)$ with a first-order (a) and third-order (b) polynomial from (5.4) (dotted) and, dashed, the Taylor polynomial (3.16′).

The derivatives of this function, according to (4.10), are

$$\frac{d^{\ell}}{dx^{\ell}}\left[-\frac{1}{2}\varphi\left(\frac{x}{3\sqrt{2}}+\frac{1}{5}\right)-0.08\right]=-\frac{1}{2\left(3\sqrt{2}\right)^{\ell}}\sum_{k=-\infty}^{\infty}\gamma^{(\ell)00}{}_{0k}\varphi_k^0\left(\frac{x}{3\sqrt{2}}+\frac{1}{5}\right), \tag{5.9}$$

and up to the second order,

$$\begin{aligned}\frac{d}{dx}\left[-\frac{1}{2}\varphi\left(\frac{x}{3\sqrt{2}}+\frac{1}{5}\right)-0.08\right]&\cong-\frac{1}{2\left(3\sqrt{2}\right)}\left[\frac{1}{4}\varphi_1^0\left(\frac{x}{3\sqrt{2}}+\frac{1}{5}\right)\right],\\ \frac{d^2}{dx^2}\left[-\frac{1}{2}\varphi\left(\frac{x}{3\sqrt{2}}+\frac{1}{5}\right)-0.08\right]&\cong-\frac{1}{2\left(3\sqrt{2}\right)^2}\left[-\frac{7}{3}\varphi_0^0\left(\frac{x}{3\sqrt{2}}+\frac{1}{5}\right)+\frac{1}{8}\varphi_1^0\left(\frac{x}{3\sqrt{2}}+\frac{1}{5}\right)\right],\end{aligned} \tag{5.10}$$

where, the explicit values of the connection coefficients are (4.13):

$$\begin{gathered}\gamma^{(1)00}{}_{00}=0,\qquad \gamma^{(1)00}{}_{01}=\frac{1}{4},\qquad \gamma^{(1)00}{}_{02}=\frac{1}{8},\ldots,\\ \gamma^{(2)00}{}_{00}=-\frac{7}{3},\qquad \gamma^{(2)00}{}_{01}=\frac{1}{8},\qquad \gamma^{(2)00}{}_{02}=\frac{1}{32},\ldots.\end{gathered} \tag{4.13′}$$

The derivatives (5.10) have two components along two orthogonal functions, so that the projection with respect to $\psi(x/3\sqrt{2}+1/5)$ gives

$$\frac{d}{dx}\left[-\frac{1}{2}\psi\left(\frac{x}{3\sqrt{2}}+\frac{1}{5}\right)-0.08\right]\cong 0,$$
$$\frac{d^{2}}{dx^{2}}\left[-\frac{1}{2}\psi\left(\frac{x}{3\sqrt{2}}+\frac{1}{5}\right)-0.08\right]\cong \frac{7}{\left(18\sqrt{2}\right)}\psi\left(\frac{x}{3\sqrt{2}}+\frac{1}{5}\right), \tag{5.11}$$

while, with respect to $\psi_1(x/3\sqrt{2}+1/5)$, we get

$$\frac{d}{dx}\left[-\frac{1}{2}\psi\left(\frac{x}{3\sqrt{2}}+\frac{1}{5}\right)-0.08\right]\cong -\frac{1}{24\sqrt{2}}\left[\psi_1^0\left(\frac{x}{3\sqrt{2}}+\frac{1}{5}\right)\right],$$
$$\frac{d^2}{dx^2}\left[-\frac{1}{2}\psi\left(\frac{x}{3\sqrt{2}}+\frac{1}{5}\right)-0.08\right]\cong -\frac{1}{288}\left[\psi_1^0\left(\frac{x}{3\sqrt{2}}+\frac{1}{5}\right)\right]. \tag{5.12}$$

It can be easily shown by a direct computation that it is also

$$\frac{d}{dx}\left[\psi\left(\frac{x}{3\sqrt{2}}+\frac{1}{5}\right)-0.08\right]=\frac{1}{12\sqrt{2}}\left[\psi_1^0\left(\frac{x}{3\sqrt{2}}+\frac{1}{5}\right)\right],$$
$$\frac{d}{dx}\psi_1^0\left(\frac{x}{3\sqrt{2}}+\frac{1}{5}\right)=\frac{1}{6\sqrt{2}}\psi_1^0\left(\frac{x}{3\sqrt{2}}+\frac{1}{5}\right),$$
$$\frac{d^2}{dx^2}\psi_1^0\left(\frac{x}{3\sqrt{2}}+\frac{1}{5}\right)=\frac{1}{72}\psi_1^0\left(\frac{x}{3\sqrt{2}}+\frac{1}{5}\right). \tag{5.10'}$$

6. Perturbation Method

In order to compute the cylindrical waves solution of the nonlinear equations (3.12″) and (3.12‴) we will consider the perturbation method [9]. This method is based on the assumption that the solution of the nonlinear problem

$$Lu(x)=Nu(x), \tag{6.1}$$

with L and N being the linear and nonlinear parts of the differential operator, can be expressed as a converging series, which depends on a small parameter $0\le\varepsilon\le 1$:

$$u(x,\varepsilon)=\sum_{n=0}^{\infty}\varepsilon^n u^{(n)}(x) \tag{6.2}$$

such that $u^{(0)}(x)$ is the solution of the linear problem:

$$Lu^{(0)}(x)=0. \tag{6.3}$$

The other terms of the series are computed by solving the recursive set of (of linear) equations:

$$Lu^{(n+1)}(x) = Nu^{(n)}(x), \quad n \geq 0. \tag{6.4}$$

6.1. Second-Order Nonlinearity

Let us search the solution of the second-order nonlinear equation (3.12′) (where for convenience $r \to x$):

$$\left(u_{,xx} + \frac{u_{,x}}{x} + u - \frac{u}{x^2}\right) = a_1 u_{,xx} u_{,x} \tag{6.5}$$

by assuming that

$$u(x) = u^{(0)}(x) + \varepsilon u^{(1)}(x), \tag{6.6}$$

where $u^{(0)}(x)$ is the solution of the linear equation:

$$\left(u_{,xx} + \frac{u_{,x}}{x} + u - \frac{u}{x^2}\right) = 0. \tag{6.7}$$

When $u^{(0)}(x)$ is known, $u^{(1)}(x)$ is computed as the solution of

$$\left(u_{,xx} + \frac{u_{,x}}{x} + u - \frac{u}{x^2}\right) = a_1 u^{(0)}_{,xx} u^{(0)}_{,x}. \tag{6.8}$$

Moreover as initial condition is taken, $u(x,0) = u^{(0)}(x)$ and the perturbation is on time so that the small parameter can be identified with time $\varepsilon \to t$ and the solution of (3.12″) can be written as

$$u(x,t) = e^{-i\omega t}\left[u^{(0)}(x) + t u^{(1)}(x)\right]. \tag{6.9}$$

The general solution of (6.8) implies some cumbersome hypergeometric series and Laguerre polynomials (see, e.g., [14]); however, it should be noticed that since the r.h.s. of (6.8) is obtained from (3.12), by neglecting all terms $O(1/r)$ we can approximate also the l.h.s with the same hypotheses so that $u^{(1)}(x)$ can be searched as solution of

$$u_{,xx} + u = a_1 u^{(0)}_{,xx} u^{(0)}_{,x}. \tag{6.10}$$

The solution of (6.7) is (5.8), so that by inserting this wavelet function in the right-hand side of (6.8) and taking into account (5.9) and (5.10′), the function $u^{(1)}(x)$ will be obtained by solving

$$\begin{aligned} u_{,xx} + u &= \frac{a_1}{4\left(3\sqrt{2}\right)^3} \sum_{k=-\infty}^{\infty} \gamma^{(2)00}{}_{0k}\varphi_k^0\left(\frac{x}{3\sqrt{2}} + \frac{1}{5}\right) \sum_{k=-\infty}^{\infty} \gamma^{(1)00}{}_{0k}\varphi_k^0\left(\frac{x}{3\sqrt{2}} + \frac{1}{5}\right) \\ &= \frac{a_1}{4\left(3\sqrt{2}\right)^3}\left[\gamma^{(2)00}{}_{00}\varphi_0^0\left(\frac{x}{3\sqrt{2}} + \frac{1}{5}\right) + \gamma^{(2)00}{}_{01}\varphi_1^0\left(\frac{x}{3\sqrt{2}} + \frac{1}{5}\right) + \cdots\right] \\ &\quad \times \left[\gamma^{(1)00}{}_{00}\varphi_0^0\left(\frac{x}{3\sqrt{2}} + \frac{1}{5}\right) + \gamma^{(1)00}{}_{01}\varphi_1^0\left(\frac{x}{3\sqrt{2}} + \frac{1}{5}\right) + \cdots\right]. \end{aligned} \tag{6.11}$$

By using the values of the connection coefficients (4.13′), for $-1 \leq k \leq 1$ and the orthogonality property of wavelets, we have

$$u_{,xx} + u = \frac{a_1}{4\left(3\sqrt{2}\right)^3}\frac{1}{32}\left[\varphi_1^0\left(\frac{x}{3\sqrt{2}} + \frac{1}{5}\right)\right]^2. \tag{6.12}$$

The solution $u^{(1)}(x)$ of (6.12) is searched in the form

$$u^{(1)}(x) = f(x)\left[\varphi_1^0\left(\frac{x}{3\sqrt{2}} + \frac{1}{5}\right)\right]^2. \tag{6.13}$$

By deriving and taking into account (5.10′),

$$\begin{aligned} u_{,x}^{(1)}(x) &= \left(f_{,x} + \frac{1}{3\sqrt{2}}f\right)\left[\varphi_1^0\left(\frac{x}{3\sqrt{2}} + \frac{1}{5}\right)\right]^2, \\ u_{,xx}^{(1)}(x) &= \left[f_{,xx} + \frac{2}{3\sqrt{2}}f_{,x} + \frac{1}{18}f\right]\left[\varphi_1^0\left(\frac{x}{3\sqrt{2}} + \frac{1}{5}\right)\right]^2. \end{aligned} \tag{6.14}$$

Equation (6.12) becomes

$$\left[f_{,xx} + \frac{2}{3\sqrt{2}}f_{,x} + \frac{19}{18}f\right]\left[\varphi_1^0\left(\frac{x}{3\sqrt{2}} + \frac{1}{5}\right)\right]^2 = \frac{a_1}{4\left(3\sqrt{2}\right)^3}\frac{1}{32}\left[\varphi_1^0\left(\frac{x}{3\sqrt{2}} + \frac{1}{5}\right)\right]^2, \tag{6.15}$$

that is,

$$f_{,xx} + \frac{2}{3\sqrt{2}}f_{,x} + \frac{19}{18}f = \frac{a_1}{128\left(3\sqrt{2}\right)^3}. \tag{6.16}$$

The solution is

$$f(x) = \frac{a_1}{7296\sqrt{2}} + e^{-x/(3\sqrt{2})}(c_1 \cos x + c_2 \sin x) \tag{6.17}$$

so that

$$u^{(1)}(x) = \left[\frac{a_1}{7296\sqrt{2}} + e^{-x/3\sqrt{2}}(c_1 \cos x + c_2 \sin x)\right]\left[\varphi_1^0\left(\frac{x}{3\sqrt{2}} + \frac{1}{5}\right)\right]^2. \tag{6.18}$$

If we assume that at the initial time $t = 0$, the nonlinear effect is neglectable, in a such a way that $u^{(1)}(0) = 0$ so that

$$0 = \left[\frac{a_1}{7296\sqrt{2}} + c_1\right]\left[\varphi_1^0\left(\frac{1}{5}\right)\right]^2, \tag{6.19}$$

which simplifies the previous form of $u^{(1)}(x)$ into

$$u^{(1)}(x) = \frac{a_1}{7296\sqrt{2}}\left(1 - e^{-x/(3\sqrt{2})} cos\, x\right)\left[\varphi_1^0\left(\frac{x}{3\sqrt{2}} + \frac{1}{5}\right)\right]^2. \tag{6.20}$$

There follows that the general solution of (6.5) is

$$u(x) = -\frac{1}{2}\psi\left(\frac{x}{3\sqrt{2}} + \frac{1}{5}\right) - 0.08 + \frac{a_1}{7296\sqrt{2}}\left(1 - e^{-x/(3\sqrt{2})} \cos x\right)\left[\varphi_1^0\left(\frac{x}{3\sqrt{2}} + \frac{1}{5}\right)\right]^2, \tag{6.21}$$

and the explicit solution of (3.12″) becomes (see Figure 6)

$$u(x,t) = \left\{-\frac{1}{2}\psi\left(\frac{x}{3\sqrt{2}} + \frac{1}{5}\right) - 0.08 + t\frac{a_1}{7296\sqrt{2}}\left(1 - e^{-x/(3\sqrt{2})} \cos x\right)\left[\varphi_1^0\left(\frac{x}{3\sqrt{2}} + \frac{1}{5}\right)\right]^2\right\}e^{-i\omega t}. \tag{6.22}$$

As expected the evolution of the initial profile (Figure 6) shows the main nonlinear effect of large (increasing) amplitude. The initial profile is deformed by showing the increasing amplitude.

6.2. Third-Order Nonlinearity

Let us search the solution of the third-order nonlinear equation (3.12″) (where $r \to x$):

$$\left(u_{,xx} + \frac{u_{,x}}{x} + u - \frac{u}{x^2}\right) = a_6 u_{,xx}\ (u_{,x})^2. \tag{6.23}$$

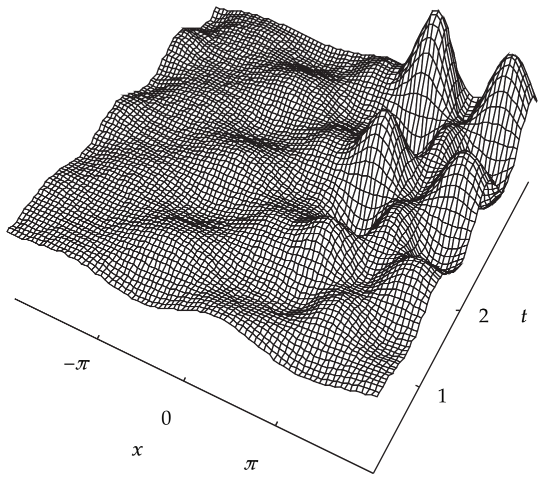

Figure 6: Wave solution with $a_1 = 10^4$ and $\omega = 5$.

The solution of (6.12) can be written as

$$u(x,t) = e^{-i\omega t}\left[u^{(0)}(x) + tu^{(1)}(x)\right], \tag{6.24}$$

where $u^{(0)}(x)$ is given by (5.8). Inserting this wavelet function in the right-hand side of (6.23), with the same approximation as in the previous case, and taking into account (5.9) and (5.10′), the function $u^{(1)}(x)$ will be obtained by solving

$$\begin{aligned} u_{,xx} + u &= -\frac{a_6}{2592}\sum_{k=-\infty}^{\infty} \gamma^{(2)00}{}_{0k}\varphi_k^0\left(\frac{x}{3\sqrt{2}} + \frac{1}{5}\right)\left[\sum_{k=-\infty}^{\infty} \gamma^{(1)00}{}_{0k}\varphi_k^0\left(\frac{x}{3\sqrt{2}} + \frac{1}{5}\right)\right]^2 \\ &= -\frac{a_6}{2592}\left[\gamma^{(2)00}{}_{00}\varphi_0^0\left(\frac{x}{3\sqrt{2}} + \frac{1}{5}\right) + \gamma^{(2)00}{}_{01}\varphi_1^0\left(\frac{x}{3\sqrt{2}} + \frac{1}{5}\right) + \cdots\right] \\ &\quad\times \left[\gamma^{(1)00}{}_{00}\varphi_0^0\left(\frac{x}{3\sqrt{2}} + \frac{1}{5}\right) + \gamma^{(1)00}{}_{01}\varphi_1^0\left(\frac{x}{3\sqrt{2}} + \frac{1}{5}\right) + \cdots\right], \end{aligned} \tag{6.25}$$

that is,

$$u_{,xx} + u = -\frac{a_6}{2592}\gamma^{(2)00}{}_{01}\left[\gamma^{(1)00}{}_{01}\right]^2\left[\varphi_1^0\left(\frac{x}{3\sqrt{2}} + \frac{1}{5}\right)\right]^3. \tag{6.26}$$

By taking into account the values of the connection coefficients (4.13′), we have

$$u_{,xx} + u = -\frac{a_6}{331776}\left[\varphi_1^0\left(\frac{x}{3\sqrt{2}} + \frac{1}{5}\right)\right]^3. \tag{6.27}$$

The solution $u^{(1)}(x)$ of (6.23) is searched in the form

$$u^{(1)}(x) = f(x)\left[\varphi_1^0\left(\frac{x}{3\sqrt{2}} + \frac{1}{5}\right)\right]^3. \tag{6.28}$$

By deriving and taking into account (5.10′),

$$\begin{gathered} u_{,x}^{(1)}(x) = \left(f_{,x} + \frac{1}{2\sqrt{2}}f\right)\left[\varphi_1^0\left(\frac{x}{3\sqrt{2}} + \frac{1}{5}\right)\right]^3, \\ u_{,xx}^{(1)}(x) = \left[f_{,xx} + \frac{3}{2\sqrt{2}}f_{,x} + \frac{1}{8}f\right]\left[\varphi_1^0\left(\frac{x}{3\sqrt{2}} + \frac{1}{5}\right)\right]^3. \end{gathered} \tag{6.29}$$

Equation (6.27) becomes

$$\left[f_{,xx} + \frac{3}{2\sqrt{2}}f_{,x} + \frac{9}{8}f\right]\left[\varphi_1^0\left(\frac{x}{3\sqrt{2}} + \frac{1}{5}\right)\right]^3 = -\frac{a_6}{331776}\left[\varphi_1^0\left(\frac{x}{3\sqrt{2}} + \frac{1}{5}\right)\right]^3. \tag{6.30}$$

So, by assuming the same hypotheses of the previous quadratic case, and with the same computations, we have

$$u^{(1)}(x) = -\frac{a_6}{373248}\left(1 - e^{-3x/(4\sqrt{2})} cos\frac{3}{4}\sqrt{\frac{3}{2}}x\right)\left[\varphi_1^0\left(\frac{x}{3\sqrt{2}} + \frac{1}{5}\right)\right]^3. \tag{6.31}$$

The general solution of (6.23) is

$$u(x) = -\frac{1}{2}\psi\left(\frac{x}{3\sqrt{2}} + \frac{1}{5}\right) - 0.08 - \frac{a_6}{373248}\left(1 - e^{-3x/(4\sqrt{2})}\cos\frac{3}{4}\sqrt{\frac{3}{2}}x\right)\left[\varphi_1^0\left(\frac{x}{3\sqrt{2}} + \frac{1}{5}\right)\right]^3, \tag{6.32}$$

and the explicit solution of (3.12‴) becomes (Figure 7)

$$\begin{aligned} u(x,t)\Bigg\{ & -\frac{1}{2}\psi\left(\frac{x}{3\sqrt{2}} + \frac{1}{5}\right) - 0.08 - \frac{a_6}{373248}t \\ & \times \left(1 - e^{-3x/(4\sqrt{2})}\cos\frac{3}{4}\sqrt{\frac{3}{2}}x\right)\left[\varphi_1^0\left(\frac{x}{3\sqrt{2}} + \frac{1}{5}\right)\right]^3\Bigg\} e^{-i\omega t}. \end{aligned} \tag{6.33}$$

As in the previous case we can observe the rapid growing of the amplitude, together with a splitting of the peak.

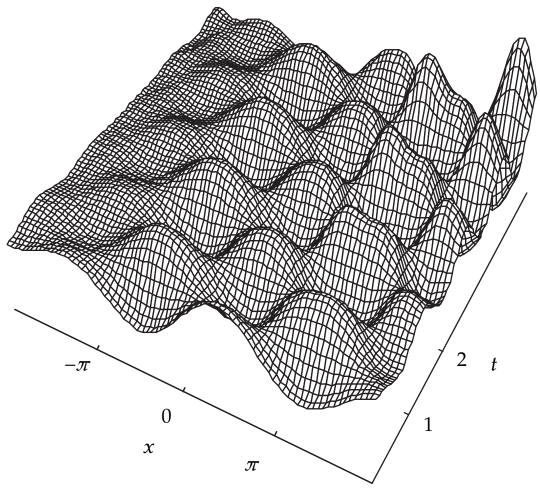

Figure 7: Wave solution with $a_6 = 10^5$ and $\omega = 5$.

7. Conclusions

It has been shown that cylindrical waves in a quadratic nonlinear Signorini structural model can be easily investigated by using Shannon wavelets. The initial profile, (solution of the linear equation) can be represented by Shannon wavelets and the evolution in time is described by the deforming wavelet profile thus giving a physical meaning to these kinds of wavelets. Shannon wavelets are equivalent to the Bessel function, at least in a quite sufficiently large neighborhood of 0. We have also noticed that, at the same approximation of the cylindrical wave, the Taylor polynomial for Shannon wavelets is one order lower than the Taylor polynomial for the corresponding Bessel function, so that Shannon wavelets are more efficient from computational point of view. It should be also noticed that Shannon wavelets are only the real part of the Newland harmonic wavelets [9, 17], so that also the Hankel functions, which are obtained by complex combination of Bessel functions, might have the same good approximation by harmonic wavelets.

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