ON THE EXISTENCE OR THE ABSENCE OF GLOBAL SOLUTIONS OF THE CAUCHY CHARACTERISTIC PROBLEM FOR SOME NONLINEAR HYPERBOLIC EQUATIONS

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For wave equations with power nonlinearity we investigate the problem of the existence or nonexistence of global solutions of the Cauchy characteristic problem in the light cone of the future.

1. Statement of the problem

Consider a nonlinear wave equation of the type

$$\Box u := \frac{\partial^2 u}{\partial t^2} - \Delta u = f(u) + F, \tag{1.1}$$

where f and F are the given real functions; note that f is a nonlinear and u is an unknown real function, $\Delta = \sum_{i=1}^{n} \partial^2/\partial x_i^2$.

For (1.1), we consider the Cauchy characteristic problem on finding in a truncated light cone of the future $D_T : |x| < t < T$, $x = (x_1,\dots,x_n)$, $n > 1$, $T = \text{const} > 0$, a solution $u(x,t)$ of that equation by the boundary condition

$$u|_{S_T} = g, \tag{1.2}$$

where g is the given real function on the characteristic conic surface $S_T : t = |x|$, $t \le T$. When considering the case $T = +\infty$ we assume that $D_\infty : t > |x|$ and $S_\infty = \partial D_\infty : t = |x|$.

Note that the questions on the existence or nonexistence of a global solution of the Cauchy problem for semilinear equations of type (1.1) with initial conditions $u|_{t=0} = u_0$, $\partial u/\partial t|_{t=0} = u_1$ have been considered in [1, 2, 6, 7, 8, 10, 13, 14, 15, 16, 17, 18, 22, 23, 26, 30, 31].

As for the characteristic problem in a linear case, that is, for problem (1.1)-(1.2) when the right-hand side of (1.1) does not involve the nonlinear summand $f(u)$, this problem is, as is known, formulated correctly, and the global solvability in the corresponding spaces of functions takes place [3, 4, 5, 11, 25].

Below we will distinguish the particular cases of the nonlinear function $f = f(u)$, when problem (1.1)-(1.2) is globally solvable in one case and unsolvable in the other one.

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2. Global solvability of the problem

Consider the case for $f(u) = -\lambda|u|^p u$, where $\lambda \neq 0$ and $p > 0$ are the given real numbers. In this case (1.1) takes the form

$$Lu := \frac{\partial^2 u}{\partial t^2} - \Delta u = -\lambda|u|^p u + F, \tag{2.1}$$

where for convenience we introduce the notation $L = \Box$. As is known, (2.1) appears in the relativistic quantum mechanics [13, 24, 28, 29].

For the sake of simplicity of our exposition we will assume that the boundary condition (1.2) is homogeneous, that is,

$$u|_{S_T} = 0. \tag{2.2}$$

Let $\overset{\circ}{W}{}_2^1(D_T,S_T) = \{u \in W_2^1(D_T) : u|_{S_T} = 0\}$, where $W_2^1(D_T)$ is the known Sobolev space.

Remark 2.1. The embedding operator $I : \overset{\circ}{W}{}_2^1(D_T,S_T) \to L_q(D_T)$ is the linear continuous compact operator for $1 < q < 2(n+1)/(n-1)$ when $n > 1$ [21, page 81]. At the same time, Nemytski's operator $K : L_q(D_T) \to L_2(D_T)$, acting by the formula $Ku = -\lambda|u|^p u$, is continuous and bounded if $q \geq 2(p+1)$ [19, page 349], [20, pages 66–67]. Thus if $p < 2/(n-1)$, that is, $2(p+1) < 2(n+1)/(n-1)$, then there exists the number q such that $1 < 2(p+1) \leq q < 2(n+1)/(n-1)$, and hence the operator

$$K_0 = KI : \overset{\circ}{W}{}_2^1(D_T,S_T) \longrightarrow L_q(D_T) \tag{2.3}$$

is continuous and compact and, more so, from $u \in \overset{\circ}{W}{}_2^1(D_T,S_T)$ follows $u \in L_{p+1}(D_T)$. As is mentioned above, here, and in the sequel it will be assumed that $p > 0$.

Definition 2.2. Let $F \in L_2(D_T)$ and $0 < p < 2/(n-1)$. The function $u \in \overset{\circ}{W}{}_2^1(D_T,S_T)$ is said to be a strong generalized solution of the nonlinear problem (2.1), (2.2) in the domain D_T if there exists a sequence of functions $u_m \in \overset{\circ}{C}{}^2(\overline{D}_T,S_T) = \{u \in \overset{\circ}{C}{}^2(\overline{D}_T) : u|_{S_T} = 0\}$ such that $u_m \to u$ in the space $\overset{\circ}{W}{}_2^1(D_T,S_T)$ and $[Lu_m + \lambda|u_m|^p u_m] \to F$ in the space $L_2(D_T)$. Moreover, the convergence of the sequence $\{\lambda|u_m|^p u_m\}$ to the function $\lambda|u|^p u$ in the space $L_2(D_T)$, as $u_m \to u$ in the space $\overset{\circ}{W}{}_2^1(D_T,S_T)$, follows from Remark 2.1, and since $|u|^{p+1} \in L_2(D_T)$, therefore on the strength of the boundedness of the domain D_T the function $u \in L_{p+1}(D_T)$.

Definition 2.3. Let $0 < p < 2/(n-1)$, $F \in L_{2,\mathrm{loc}}(D_\infty)$, and $F \in L_2(D_T)$ for any $T > 0$. It is said that problem (2.1), (2.2) is globally solvable if for any $T > 0$ this problem has a strong generalized solution in the domain D_T from the space $\overset{\circ}{W}{}_2^1(D_T,S_T)$.

Lemma 2.4. *Let $\lambda > 0$, $0 < p < 2/(n-1)$, and $F \in L_2(D_T)$. Then for any strong generalized solution $u \in \overset{\circ}{W}{}_2^1(D_T, S_T)$ of problem (2.1)-(2.2) in the domain D_T the estimate*

$$\|u\|_{\overset{\circ}{W}{}_2^1(D_T,S_T)} \le \sqrt{e}T\|F\|_{L_2(D_T)} \tag{2.4}$$

is valid.

Proof. Let $u \in \overset{\circ}{W}{}_2^1(D_T, S_T)$ be the strong generalized solution of problem (2.1)-(2.2). By Definition 2.2 and Remark 2.1 there exists a sequence of functions $u_m \in \overset{\circ}{C}{}^2(\overline{D}_T, S_T)$ such that

$$\begin{aligned} \lim_{m\to\infty} \|u_m - u\|_{\overset{\circ}{W}{}_2^1(D_T,S_T)} &= 0, \\ \lim_{m\to\infty} \|Lu_m + \lambda |u_m|^p u_m - F\|_{L_2(D_T)} &= 0. \end{aligned} \tag{2.5}$$

The function $u_m \in \overset{\circ}{C}{}^2(\overline{D}_T, S_T)$ can be considered as the solution of the following problem:

$$Lu_m + \lambda |u_m|^p u_m = F_m, \tag{2.6}$$

$$u_m|_{S_T} = 0. \tag{2.7}$$

Here

$$F_m = Lu_m + \lambda |u_m|^p u_m. \tag{2.8}$$

Multiplying both parts of (2.6) by $\partial u_m/\partial t$ and integrating with respect to the domain D_τ, $0 < \tau \le T$, we obtain

$$\begin{aligned} &\frac{1}{2}\int_{D_\tau} \frac{\partial}{\partial t}\left(\frac{\partial u_m}{\partial t}\right)^2 dx\,dt \\ &\quad - \int_{D_\tau} \Delta u_m \frac{\partial u_m}{\partial t} dx\,dt \\ &\quad + \frac{\lambda}{p+2}\int_{D_\tau} \frac{\partial}{\partial t} |u_m|^{p+2} dx\,dt \\ &= \int_{D_\tau} F_m \frac{\partial u_m}{\partial t} dx\,dt. \end{aligned} \tag{2.9}$$

Let $\Omega_\tau := D_T \cap \{t = \tau\}$ and denote by $\nu = (\nu_1, \dots, \nu_n, \nu_0)$ the unit vector of the outer normal to $S_T \setminus \{(0,\dots,0,0)\}$. Taking into account (2.7) and $\nu|_{\Omega_\tau} = (0,\dots,0,1)$, integration by parts results easily in

$$\begin{aligned}
&\int_{D_\tau}\frac{\partial}{\partial t}\left(\frac{\partial u_m}{\partial t}\right)^2 dx\,dt=\int_{\partial D_\tau}\left(\frac{\partial u_m}{\partial t}\right)^2\nu_0 ds=\int_{\Omega_\tau}\left(\frac{\partial u_m}{\partial t}\right)^2 dx+\int_{S_\tau}\left(\frac{\partial u_m}{\partial t}\right)^2\nu_0 ds,\\
&\int_{D_\tau}\frac{\partial}{\partial t}|u_m|^{p+2}dx\,dt=\int_{\partial D_\tau}|u_m|^{p+2}\nu_0 ds=\int_{\Omega_\tau}|u_m|^{p+2}dx,\\
&\int_{D_\tau}\frac{\partial^2 u_m}{\partial x_i^2}\frac{\partial u_m}{\partial t}dx\,dt=\int_{\partial D_\tau}\frac{\partial u_m}{\partial x_i}\frac{\partial u_m}{\partial t}\nu_i ds-\frac{1}{2}\int_{D_\tau}\frac{\partial}{\partial t}\left(\frac{\partial u_m}{\partial x_i}\right)^2 dx\,dt\\
&\qquad=\int_{\partial D_\tau}\frac{\partial u_m}{\partial x_i}\frac{\partial u_m}{\partial t}\nu_i ds-\frac{1}{2}\int_{\partial D_\tau}\left(\frac{\partial u_m}{\partial x_i}\right)^2\nu_0 ds\\
&\qquad=\int_{\partial D_\tau}\frac{\partial u_m}{\partial x_i}\frac{\partial u_m}{\partial t}\nu_i ds-\frac{1}{2}\int_{S_\tau}\left(\frac{\partial u_m}{\partial x_i}\right)^2\nu_0 ds-\frac{1}{2}\int_{\Omega_\tau}\left(\frac{\partial u_m}{\partial x_i}\right)^2 dx,
\end{aligned}\tag{2.10}$$

whence, by virtue of (2.9), it follows that

$$\begin{aligned}
\int_{D_\tau}F_m\frac{\partial u_m}{\partial t}dx\,dt&=\int_{S_\tau}\frac{1}{2\nu_0}\left[\sum_{i=1}^{n}\left(\frac{\partial u_m}{\partial x_i}\nu_0-\frac{\partial u_m}{\partial t}\nu_i\right)^2+\left(\frac{\partial u_m}{\partial t}\right)^2\left(\nu_0^2-\sum_{j=1}^{n}\nu_j^2\right)\right]ds\\
&\quad+\frac{1}{2}\int_{\Omega_\tau}\left[\left(\frac{\partial u_m}{\partial t}\right)^2+\sum_{i=1}^{n}\left(\frac{\partial u_m}{\partial x_i}\right)^2\right]dx+\frac{\lambda}{p+2}\int_{\Omega_\tau}|u_m|^{p+2}dx.
\end{aligned}\tag{2.11}$$

Since S_τ is the characteristic surface,

$$\left(\nu_0^2-\sum_{j=1}^{n}\nu_j^2\right)\Bigg|_{S_\tau}=0.\tag{2.12}$$

Taking into account that the operator $(\nu_0(\partial/\partial x_i)-\nu_i(\partial/\partial t))$, $i=1,2,\dots,n$, is the internal differential operator on S_τ, by means of (2.7) we have

$$\left(\frac{\partial u_m}{\partial x_i}\nu_0-\frac{\partial u_m}{\partial t}\nu_i\right)\Bigg|_{S_\tau}=0,\quad i=1,2,\dots,n.\tag{2.13}$$

By (2.12) and (2.13), from (2.11) we get

$$\int_{\Omega_\tau}\left[\left(\frac{\partial u_m}{\partial t}\right)^2+\sum_{i=1}^{n}\left(\frac{\partial u_m}{\partial x_i}\right)^2\right]dx+\frac{2\lambda}{p+2}\int_{\Omega_\tau}|u_m|^{p+2}dx=2\int_{D_\tau}F_m\frac{\partial u_m}{\partial t}dx\,dt.\tag{2.14}$$

In the notation $w(\delta)=\int_{\Omega_\delta}[(\partial u_m/\partial t)^2+\sum_{i=1}^{n}(\partial u_m/\partial x_i)^2]dx$, taking into account that $\lambda/(p+2)>0$ and also the inequality $2F_m(\partial u_m/\partial t)\le\varepsilon(\partial u_m/\partial t)^2+(1/\varepsilon)F_m^2$ which is valid for any $\varepsilon=\text{const}>0$, (2.14) yields

$$w(\delta)\le\varepsilon\int_0^\delta w(\sigma)d\sigma+\frac{1}{\varepsilon}\|F_m\|^2_{L_2(D_\delta)},\quad 0<\delta\le T.\tag{2.15}$$

From (2.15), if we take into account that the value $\|F_m\|^2_{L_2(D_\delta)}$ as the function of δ is nondecreasing, by Gronwall's lemma [12, page 13] we find that

$$w(\delta) \le \frac{1}{\varepsilon}\|F_m\|^2_{L_2(D_\delta)} \exp \delta\varepsilon. \tag{2.16}$$

Because $\inf_{\varepsilon>0}(\exp \delta\varepsilon/\varepsilon) = e\delta$, which is achieved for $\varepsilon = 1/\delta$, we obtain

$$w(\delta) \le e\delta\|F_m\|^2_{L_2(D_\delta)}, \quad 0 < \delta \le T. \tag{2.17}$$

From (2.17) in its turn it follows that

$$\begin{aligned} \|u_m\|^2_{\overset{\circ}{W}{}^1_2(D_T,S_T)} &= \int_{D_T}\left[\left(\frac{\partial u_m}{\partial t}\right)^2 + \sum_{i=1}^n \left(\frac{\partial u_m}{\partial x_i}\right)^2\right] dx\,dt \\ &= \int_0^T w(\delta)d\delta \le eT^2\|F_m\|^2_{L_2(D_T)} \end{aligned} \tag{2.18}$$

and hence

$$\|u_m\|_{\overset{\circ}{W}{}^1_2(D_T,S_T)} \le \sqrt{e}T\|F_m\|_{L_2(D_\tau)}. \tag{2.19}$$

Here we have used the fact that in the space $\overset{\circ}{W}{}^1_2(D_T,S_T)$ the norm

$$\|u\|_{W^1_2(D_T)} = \left\{\int_{D_T}\left[u^2 + \left(\frac{\partial u}{\partial t}\right)^2 + \sum_{i=1}^n \left(\frac{\partial u}{\partial x_i}\right)^2\right] dx\,dt\right\}^{1/2} \tag{2.20}$$

is equivalent to the norm

$$\|u\| = \left\{\int_{D_T}\left[\left(\frac{\partial u}{\partial t}\right)^2 + \sum_{i=1}^n \left(\frac{\partial u}{\partial x_i}\right)^2\right] dx\,dt\right\}^{1/2}, \tag{2.21}$$

since from the equalities $u|_{S_T} = 0$ and $u(x,t) = \int_{|x|}^t (\partial u(x,\tau)/\partial t)d\tau$, $(x,t) \in \overline{D}_T$, which are valid for any function $u \in \overset{\circ}{C}{}^2(\overline{D}_T, S_T)$, in a standard way we obtain the following inequality [21, page 63]:

$$\int_{D_T} u^2(x,t)dx\,dt \le T^2 \int_{D_T}\left(\frac{\partial u}{\partial t}\right)^2 dx\,dt. \tag{2.22}$$

By virtue of (2.5) and (2.8), passing to inequality (2.19) to the limit as $m \to \infty$, we obtain (2.4). Thus the lemma is proved. □

Remark 2.5. Before passing to the question on the solvability of the nonlinear problem (2.1), (2.2), we consider this question for a linear case in the form we need, when in (2.1) the parameter $\lambda = 0$, that is, for the problem

$$\begin{aligned} Lu(x,t) &= F(x,t), \quad (x,t) \in D_T, \\ u(x,t) &= 0, \quad (x,t) \in S_T. \end{aligned} \tag{2.23}$$

In this case for $F \in L_2(D_T)$, we analogously introduce the notion of a strong generalized solution u of problem (2.23) for which there exists the sequence of functions $u_m \in \overset{\circ}{C}{}^2(\overline{D}_T, S_T)$, such that $\lim_{m\to\infty} \|u_m - u\|_{\overset{\circ}{W}{}_2^1(D_T,S_T)} = 0$, $\lim_{m\to\infty} \|Lu_m - F\|_{L_2(D_T)} = 0$. It should be here noted that as we can see from the proof of Lemma 2.4, the a priori estimate (2.4) is likewise valid for the strong generalized solution of problem (2.23).

Since the space $C_0^\infty(D_T)$ of finite infinitely differentiable functions in D_T is dense in $L_2(D_T)$, for the given $F \in L_2(D_T)$ there exists the sequence of functions $F_m \in C_0^\infty(D_T)$ such that $\lim_{m\to\infty} \|F_m - F\|_{L_2(D_T)} = 0$. For the fixed m, if we continue the function F_m by zero outside the domain D_T and retain the same notation, we will find that $F_m \in C^\infty(R_+^{n+1})$ for which $\operatorname{supp} F_m \subset D_\infty$, where $R_+^{n+1} = R^{n+1} \cap \{t \geq 0\}$. Denote by u_m a solution of the Cauchy problem $Lu_m = F_m$, $u_m|_{t=0} = 0$, $\partial u_m/\partial t|_{t=0} = 0$, which, as is known, exists, is unique, and belongs to the space $C^\infty(R_+^{n+1})$ [9, page 192]. As far as $\operatorname{supp} F_m \subset D_\infty$, $u_m|_{t=0} = 0$, $\partial u_m/\partial t|_{t=0} = 0$, taking into account the geometry of the domain of dependence of a solution of the wave equation, we obtain $\operatorname{supp} F_m \subset D_\infty$ [9, page 191]. Retaining for the narrowing of the function u_m to the domain D_T the same notation, we can easily see that $u_m \in \overset{\circ}{C}{}^2(D_T, S_T)$, and by virtue of (2.4) we have

$$||u_m - u_k||_{\overset{\circ}{W}{}_2^1(D_T,S_T)} \leq \sqrt{e}T||F_m - F_k||_{L_2(D_T)}. \tag{2.24}$$

Since the sequence $\{F_m\}$ is fundamental in $L_2(D_T)$, the sequence $\{u_m\}$, owing to (2.24), is likewise fundamental in the complete space $\overset{\circ}{W}{}_2^1(D_T, S_T)$. Therefore there exists the function $u \in \overset{\circ}{W}{}_2^1(D_T, S_T)$ such that $\lim_{m\to\infty} \|u_m - u\|_{\overset{\circ}{W}{}_2^1(D_T,S_T)} = 0$, and since $Lu_m = F_m \to F$ in the space $L_2(D_T)$, this function will, by Remark 2.5, be the strong generalized solution of problem (2.23). The uniqueness of that solution from the space $\overset{\circ}{W}{}_2^1(D_T, S_T)$ follows from the a priori estimate (2.4). Consequently, for the solution u of problem (2.23) we can write $u = L^{-1}F$, where $L^{-1} : L_2(D_T) \to \overset{\circ}{W}{}_2^1(D_T, S_T)$ is the linear continuous operator whose norm, by virtue of (2.4), admits the estimate

$$||L^{-1}||_{L_2(D_T)\to \overset{\circ}{W}{}_2^1(D_T,S_T)} \leq \sqrt{e}T. \tag{2.25}$$

Remark 2.6. Taking into account (2.25) for $F \in L_2(D_T)$, $0 < p < 2/(n-1)$ and also Remark 2.1, it is not difficult to see that the function $u \in \overset{\circ}{W}{}_2^1(D_T, S_T)$ is the strong generalized solution of problem (2.1)-(2.2) if and only if u is the solution of the functional equation

$$u = L^{-1}(-\lambda|u|^p u + F) \tag{2.26}$$

in the space $\overset{\circ}{W}{}_2^1(D_T, S_T)$.

We rewrite (2.26) in the form

$$u = Au := L^{-1}(K_0 u + F), \tag{2.27}$$

where the operator $K_0 : \overset{\circ}{W}{}_2^1(D_T,S_T) \to L_2(D_T)$ from (2.3) is, by Remark 2.1, a continuous and compact one. Consequently, by virtue of (2.25) the operator $A : \overset{\circ}{W}{}_2^1(D_T,S_T) \to \overset{\circ}{W}{}_2^1(D_T,S_T)$ is likewise continuous and compact. At the same time, by Lemma 2.4, for any parameter $\tau \in [0,1]$ and any solution of the equation with the parameter $u = \tau Au$ the a priori estimate $\|u\|_{\overset{\circ}{W}{}_2^1(D_T,S_T)} \le c\|F\|_{L_2(D_T)}$ with the positive constant c, independent of u, τ, and F, is valid.

Therefore by Leray-Schauder theorem [32, page 375], (2.27), and hence problem (2.1)-(2.2), has at least one solution $u \in \overset{\circ}{W}{}_2^1(D_T,S_T)$.

Thus the following theorem is valid.

THEOREM 2.7. *Let $\lambda > 0$, $0 < p < 2/(n-1)$, $F \in L_{2,\mathrm{loc}}(D_\infty)$, and $F \in L_2(D_T)$ for any $T > 0$. Then problem (2.1)-(2.2) is globally solvable, that is, for any $T > 0$ this problem has the strong generalized solution $u \in \overset{\circ}{W}{}_2^1(D_T,S_T)$ in the domain D_T.*

3. Nonexistence of the global solvability

Below we will restrict ourselves to the case when in (2.1) the parameter $\lambda < 0$ and the space dimension $n = 2$.

Definition 3.1. Let $F \in C(\overline{D}_T)$. The function u is said to be a strong generalized continuous solution of problem (2.23) if $u \in \overset{\circ}{C}(\overline{D}_T,S_T) = \{u \in C(\overline{D}_T) : u|_{S_T} = 0\}$ and there exists a sequence of functions $u_m \in \overset{\circ}{C}{}^2(\overline{D}_T,S_T)$ such that $\lim_{m\to\infty} \|u_m - u\|_{C(\overline{D}_T)} = 0$ and $\lim_{m\to\infty} \|Lu_m - F\|_{C(\overline{D}_T)} = 0$.

We introduce into the consideration the domain $D_{x^0,t^0} = \{(x,t) \in R^3 : |x| < t < t^0 - |x - x^0|\}$ which for $(x^0,t^0) \in D_T$ is bounded below by a light cone of the future S_∞ with the vertex at the origin and above by the light cone of the past $S^-_{x^0,t^0} : t = t^0 - |x - x^0|$ with the vertex at the point (x^0,t^0).

LEMMA 3.2. *Let $n = 2$, $F \in \overset{\circ}{C}(\overline{D}_T,S_T)$. Then there exists the unique strong generalized continuous solution of problem (2.23) for which the integral representation*

$$u(x,t) = \frac{1}{2\pi}\int_{D_{x,t}} \frac{F(\xi,\tau)}{\sqrt{(t-\tau)^2 - |x-\xi|^2}}\,d\xi\,d\tau, \quad (x,t) \in D_T, \tag{3.1}$$

and the estimate

$$\|u\|_{C(\overline{D}_T)} \le c\|F\|_{C(\overline{D}_T)} \tag{3.2}$$

with the positive constant c, independent of F, are valid.

Proof. Without restriction of generality, we can assume that the function $F \in \overset{\circ}{C}(\overline{D}_T,S_T)$ is continuous in the domain $\overline{D}_\infty$ such that $F \in \overset{\circ}{C}(\overline{D}_\infty,S_\infty)$. Indeed, if $(x,t) \in \overline{D}_\infty \setminus \overline{D}_T$, then we can take $F(x,t) = F((T/t)x,T)$. Let $D_{T,\delta} : |x| + \delta < t < T$, where $0 < \delta = \text{const} < (1/2)T$. Obviously, $D_{T,\delta} \subset D_T$. Since $F \in C(\overline{D}_T)$ and $F|_{S_T} = 0$, for some strongly monotonically decreasing sequence of positive numbers $\{\delta_k\}$ there exists the sequence of functions $\{F_k\}$ such that

$$F_k \in C^\infty(\overline{D}_T), \quad \operatorname{supp} F_k \subset \overline{D}_{T,\delta_k}, \quad k = 1,2,\ldots, \\ \lim_{k\to\infty} ||F_k - F||_{C(\overline{D}_T)} = 0. \tag{3.3}$$

Indeed, let $\varphi_\delta \in C([0,+\infty))$ be the nondecreasing continuous function of one variable such that $\varphi_\delta(\tau) = 0$ for $0 \le \tau \le 2\delta$ and $\varphi_\delta(\tau) = 1$ for $t \ge 3\delta$. Let $\widetilde{F}_\delta(x,t) = \varphi_\delta(t - |x|)F(x,t)$, $(x,t) \in \overline{D}_T$. Since $F \in C(\overline{D}_T)$ and $F|_{S_T} = 0$, we can easily verify that

$$\widetilde{F}_\delta \in C(\overline{D}_T), \quad \operatorname{supp} \widetilde{F}_\delta \subset \overline{D}_{T,2\delta}, \quad \lim_{\delta\to\infty} ||\widetilde{F}_\delta - F||_{C(\overline{D}_T)} = 0. \tag{3.4}$$

Now we take advantage of the operation of averaging and let

$$G_\delta(x,t) = \varepsilon^{-n}\int_{R^3} \widetilde{F}_\delta(\xi,\tau)\rho\left(\frac{x-\xi}{\varepsilon},\frac{\tau}{\varepsilon}\right)d\xi\,d\tau, \quad \varepsilon = (\sqrt{2}-1)\delta, \tag{3.5}$$

where

$$\rho \in C_0^\infty(R^3), \quad \int_{R^3} \rho\,dx\,dt = 1, \quad \rho \ge 0, \\ \operatorname{supp} \rho = \{(x,t) \in R^3 : x^2 + t^2 \le 1\}. \tag{3.6}$$

From (3.4) and averaging properties [9, page 9] it follows that the sequence $F_k = G_{\delta_k}$, $k = 1,2,\ldots$, satisfies (3.3). Continuing the function F_k by zero to the strip $\Lambda_T : 0 < t < T$ and retaining the same notation, we have $F_k \in C^\infty(\overline{\Lambda}_T)$, where $\operatorname{supp} F_k \subset \overline{D}_{T,\delta_k} \subset \overline{D}_T$, $k = 1,2,\ldots$. Therefore, just in the same way as in proving Lemma 2.4, for the solution of the Cauchy problem $Lu_k = F_k$, $u_k|_{t=0} = 0$, $\partial u_k/\partial t|_{t=0} = 0$ in the strip Λ_T which exists, is unique, and belongs to the space $C^\infty(\overline{\Lambda}_T)$, we have $\operatorname{supp} u_k \subset D_T$ and, more so, $u_k \in \overset{\circ}{C}{}^2(\overline{D}_T,S_T)$, $k =,1,2\ldots$.

On the other hand, since $\operatorname{supp} F_k \subset \overline{D}_T$, $F_k \in C^\infty(\overline{\Lambda}_T)$ for the solution u_k of the Cauchy problem, by the Poisson formula the integral representation [33, page 227]

$$u_k(x,t) = \frac{1}{2\pi}\int_{D_{x,t}} \frac{F_k(\xi,\tau)}{\sqrt{(t-\tau)^2 - |x-\xi|^2}}d\xi\,d\tau, \quad (x,t) \in D_T, \tag{3.7}$$

is valid and the estimate [33, page 215]

$$||u_k||_{C(\overline{D}_T)} \le \frac{T^2}{2}||F_k||_{C(\overline{D}_T)} \tag{3.8}$$

holds.

By (3.4) and (3.8), the sequence $\{u_k\} \subset \overset{\circ}{C}{}^2(\overline{D}_T,S_T)$ is fundamental in the space $\overset{\circ}{C}(\overline{D}_T, S_T)$ and tends to some function u for which, by virtue of (3.7), the representation (3.1) is valid and the estimate (3.2) holds. Thus we have proved that problem (2.23) is solvable in the space $\overset{\circ}{C}(\overline{D}_T,S_T)$.

As for the uniqueness of the strong generalized continuous solution of problem (2.23), it follows from the following reasoning. Let $u \in \overset{\circ}{C}(\overline{D}_T, S_T)$ and $F = 0$ and there exists the sequence of functions $u_k \in \overset{\circ}{C}{}^2(\overline{D}_T, S_T)$ such that $\lim_{k\to\infty} \|u_k - u\|_{C(\overline{D}_T)} = 0$, $\lim_{k\to\infty} \|Lu_k\|_{C(\overline{D}_T)} = 0$. This implies that $\lim_{k\to\infty} \|u_k - u\|_{L_2(D_T)} = 0$ and $\lim_{k\to\infty} \|Lu_k\|_{L_2(D_T)} = 0$. Since the function $u_k \in \overset{\circ}{C}{}^2(\overline{D}_T, S_T)$ can be considered as the strong generalized solution of problem (2.23) for $F_k = Lu_k$ from the space $\overset{\circ}{W}{}_2^1(D_T, S_T)$, the estimate $\|u_k\|_{\overset{\circ}{W}{}_2^1(D_T,S_T)} \le \sqrt{e}T \|Lu_k\|_{L_2(D_T)}$ is valid according to Remark 2.5. Therefore $\lim_{k\to\infty} \|Lu_k\|_{L_2(D_T)} = 0$ implies that $\lim_{k\to\infty} \|u_k\|_{\overset{\circ}{W}{}_2^1(D_T,S_T)} = 0$, and hence $\lim_{k\to\infty} \|u_k\|_{L_2(D_T)} = 0$. Taking into account the fact that $\lim_{k\to\infty} \|u_k - u\|_{L_2(D_T)} = 0$, we obtain $u = 0$. Thus Lemma 3.2 is proved completely. □

Lemma 3.3. *Let $n = 2$, $\lambda < 0$, $F \in \overset{\circ}{C}(\overline{D}_T, S_T)$, and $F \ge 0$. Then if $u \in C^2(\overline{D}_T)$ is the classical solution of problem (2.1)-(2.2), then $u \ge 0$ in the domain D_T.*

Proof. If $u \in C^2(\overline{D}_T)$ is the classical solution of problem (2.1)-(2.2), then $u \in \overset{\circ}{C}{}^2(\overline{D}_T, S_T)$, and since $F \in \overset{\circ}{C}(\overline{D}_T, S_T)$, the right-hand side $G = -\lambda |u|^p u + F$ of (2.1) belongs to the space $\overset{\circ}{C}(\overline{D}_T, S_T)$. Considering the function $u \in \overset{\circ}{C}{}^2(\overline{D}_T, S_T)$ as the classical solution of problem (2.23) for $F = G$, that is,

$$Lu = G, \quad u|_{S_T} = 0, \tag{3.9}$$

it will, more so, be the strong generalized continuous solution of problem (3.9). Therefore, taking into account that $G \in \overset{\circ}{C}(\overline{D}_T, S_T)$, by Lemma 3.2, for the function u the integral representation

$$u(x,t) = -\frac{\lambda}{2\pi} \int_{D_{x,t}} \frac{|u|^p u}{\sqrt{(t-\tau)^2 - |x-\xi|^2}} d\xi\, d\tau + F_0(x,t) \tag{3.10}$$

holds. Here

$$F_0(x,t) = \frac{1}{2\pi} \int_{D_{x,t}} \frac{F(\xi,\tau)}{\sqrt{(t-\tau)^2 - |x-\xi|^2}} d\xi\, d\tau. \tag{3.11}$$

Consider now the integral equation

$$v(x,t) = \int_{D_{x,t}} \frac{g_0 v}{\sqrt{(t-\tau)^2 - |x-\xi|^2}} d\xi\, d\tau + F_0(x,t), \quad (x,t) \in \overline{D}_T, \tag{3.12}$$

with respect to an unknown function v, where $g_0 = -(\lambda/2\pi)|u|^p$.

Since $g_0, F_0 \in \overset{\circ}{C}(\overline{D}_T, S_T)$, and the operator in the right-hand side of (3.12) is an integral operator of Volterra type with a weak singularity, (3.12) is uniquely solvable in the space $C(\overline{D}_T)$. It should be noted that the solution v of (3.12) can be obtained by Picard's method of successive approximations:

$$\begin{gathered} v_0 = 0, \\ v_{k+1}(x,t) = \int_{D_{x,t}} \frac{g_0 v_k}{\sqrt{(t-\tau)^2 - |x-\xi|^2}} d\xi\, d\tau + F_0(x,t), \quad k = 1,2,\ldots. \end{gathered} \tag{3.13}$$

Indeed, let

$$\begin{gathered} \Omega_\tau = D_T \cap \{t = \tau\}, \qquad w_m|_{\overline{D}_T} = v_{m+1} - v_m \left(w_0|_{\overline{D}_T} = F_0\right), \\ w_m|_{\{0\le t\le T\}\setminus\overline{D}_T} = 0, \quad \lambda_m(t) = \max_{x\in\overline{\Omega}_t} \left| w_m(x,t) \right|, \quad m = 0,1,\ldots, \\ b = \int_{|\eta|<1} \frac{d\eta_1 d\eta_2}{\sqrt{1-|\eta|^2}} ||g_0||_{C(\overline{D}_T)} = 2\pi ||g_0||_{C(\overline{D}_T)}. \end{gathered} \tag{3.14}$$

Then, if

$$B_\beta \varphi(t) = b \int_0^t (t-\tau)^{\beta-1} \varphi(\tau) d\tau, \quad \beta > 0, \tag{3.15}$$

then taking into account the equality

$$B_\beta^m \varphi(t) = \frac{1}{\Gamma(m\beta)} \int_0^t \left(b\Gamma(\beta)\right)^m (t-\tau)^{m\beta-1} \varphi(\tau) d\tau \tag{3.16}$$

[12, page 206], by virtue of (3.13), we obtain

$$\begin{aligned} \left| w_m(x,t) \right| &= \left| \int_{D_{x,t}} \frac{g_0 w_{m-1}}{\sqrt{(t-\tau)^2 - |x-\xi|^2}} d\xi\, d\tau \right| \\ &\le \int_0^t d\tau \int_{|x-\xi|<t-\tau} \frac{|g_0|\, |w_{m-1}|}{\sqrt{(t-\tau)^2 - |x-\xi|^2}} d\xi\, d\tau \\ &\le ||g_0||_{C(\overline{D}_T)} \int_0^t d\tau \int_{|x-\xi|<t-\tau} \frac{\lambda_{m-1}(\tau)}{\sqrt{(t-\tau)^2 - |x-\xi|^2}} d\xi \\ &= ||g_0||_{C(\overline{D}_T)} \int_0^t (t-\tau)\lambda_{m-1}(\tau) d\tau \int_{|\eta|<1} \frac{d\eta_1 d\eta_2}{\sqrt{1-|\eta|^2}} \\ &= B_2 \lambda_{m-1}(t), \quad (x,t) \in D_T. \end{aligned} \tag{3.17}$$

It follows that

$$\begin{aligned} \lambda_m(t) &\le B_2\lambda_{m-1}(t) \le \cdots \le B_2^m \lambda_0(t) = \frac{1}{\Gamma(2m)} \int_0^t \left(b\Gamma(2)\right)^m (t-\tau)^{2m-1} \lambda_0(\tau) d\tau \\ &\le \frac{b^m}{\Gamma(2m)} \int_0^t (t-\tau)^{2m-1} ||w_0||_{C\left(\overline{D}_T\right)} d\tau = \frac{\left(bT^2\right)^m}{\Gamma(2m) 2m} ||F_0||_{C(\overline{D}_T)} = \frac{\left(bT^2\right)^m}{(2m)!} ||F_0||_{C(\overline{D}_T)} \end{aligned} \tag{3.18}$$

and hence

$$\|w_m\|_{C(\overline{D}_T)} = \|\lambda_m\|_{C([0,T])} \leq \frac{(bT^2)^m}{(2m)!}\|F_0\|_{C(\overline{D}_T)}. \tag{3.19}$$

Therefore the series $v = \lim_{m\to\infty} v_m = v_0 + \sum_{m=0}^{\infty} w_m$ converges in the class $C(\overline{D}_T)$ and its sum is the solution of (3.12). The uniqueness of the solution (3.12) in the space $C(\overline{D}_T)$ is proved analogously.

As far as $\lambda < 0$, we have $g_0 = -\lambda/2\pi \geq 0$, and by virtue of (3.11), the function $F_0 \geq 0$ because $F \geq 0$ by the condition. Therefore successive approximations v_k from (3.13) are nonnegative, and since $\lim_{k\to\infty}\|v_k - v\|_{C(\overline{D}_T)} = 0$, the solution $v \geq 0$ in the domain D_T, too. It now remains only to note that by virtue of (3.10), the function u is the solution of (3.12), and according to the unique solvability of that equation, $u = v \geq 0$ in the domain D_T. Thus the proof of Lemma 3.3 is complete. □

Remark 3.4. As it can be seen from the proof, Lemma 3.3 is likewise valid if instead of the condition $F \geq 0$ we will require the fulfillment of a more weak condition $F_0 \geq 0$, where the function F_0 is given by formula (3.11).

LEMMA 3.5. *Let $n = 2$, $F \in \overset{0}{C}(\overline{D}_T, S_T)$ and let $u \in C^2(\overline{D}_T)$ be the classical solution of problem (2.1)-(2.2). Then if for some point $(x^0,t^0) \in D_T$ the function $F|_{D_{x^0,t^0}} = 0$, then likewise $u|_{D_{x^0,t^0}} = 0$, where $D_{x^0,t^0} = \{(x,t) \in R^3 : |x| < t < t^0 - |x - x^0|\}$.*

Proof. Since $F|_{D_{x^0,t^0}} = 0$, by the representation (3.1) from Lemma 3.2, the solution u of problem (2.1)-(2.2) in the domain D_{x^0,t^0} satisfies the integral equation

$$u(x,t) = \frac{1}{2\pi}\int_{D_{x,t}} \frac{\widetilde{g}_0(\xi,\eta)u(\xi,\eta)}{\sqrt{(t-\tau)^2 - |x-\xi|^2}}\,d\xi\,d\tau, \quad (x,t) \in D_{x^0,t^0}, \tag{3.20}$$

where $\widetilde{g}_0 = -\lambda|u|^p$. Taking into account the fact that

$$\begin{aligned}\frac{1}{2\pi}\int_{D_{x,t}} \frac{\tau^m}{\sqrt{(t-\tau)^2 - |x-\xi|^2}}\,d\xi\,d\tau &\leq \frac{1}{2\pi}\int_0^t d\tau \int_{|x-\xi|<t-\tau} \frac{\tau^m}{\sqrt{(t-\tau)^2 - |x-\xi|^2}}\,d\xi \\ &= \frac{1}{2\pi}\int_0^t \tau^m(t-\tau)\,d\tau \int_{|\eta|<1} \frac{d\eta}{\sqrt{1-|\eta|^2}} \\ &= \frac{t^{m+2}}{(m+1)(m+2)}\end{aligned} \tag{3.21}$$

from (3.20) using the method of mathematical induction, we easily get

$$|u(x,t)| \leq MM_1^k \frac{t^{2k}}{(2k)!}, \quad (x,t) \in D_{x^0,t^0},\ k = 1,2,\ldots, \tag{3.22}$$

where $M = \max_{\overline{D}_T}|u(x,t)| = \|u\|_{C(\overline{D}_T)}$, $M_1 = \max_{\overline{D}_T}|\widetilde{g}_0(x,t)|$. Therefore, as $k \to +\infty$, we have $u|_{D_{x^0,t^0}} = 0$. Thus Lemma 3.5 is proved completely. □

Let c_R and $\varphi_R(x)$ be, respectively, the first eigenvalue and the eigenfunction of the Dirichlet problem in the circle $\Omega_R : x_1^2 + x_2^2 < R^2$. Consequently,

$$(\Delta\varphi_R + c_R\varphi_R)\,|_{\Omega_R} = 0, \quad \varphi_R\,|_{\partial\Omega_R} = 0. \tag{3.23}$$

As is known, $c_R > 0$, and if we change the sign and make the corresponding normalization, we will be able to get [27, page 25]

$$\varphi_R|_{\Omega_R} > 0, \qquad \int_{\Omega_R} \varphi_R dx = 1. \tag{3.24}$$

Theorem 3.6. *Let $n = 2$, $\lambda < 0$, $p > 0$, $F \in C(\overline{D}_\infty)$, $\operatorname{supp} F \cap S_\infty = \varnothing$, and $F \geq 0$. Then if the condition*

$$\overline{lim}_{T\to+\infty} T^{(p+2)/p} \int_0^T dt \int_{\Omega_1} F(2T\xi, t)\varphi_1(\xi)d\xi = +\infty \tag{3.25}$$

is fulfilled, then there exists the number $T_0 = T_0(F) > 0$ such that for $T \geq T_0$ problem (2.1)-(2.2) fails to have the classical solution $u \in C^2(\overline{D}_T)$ in the domain D_T.

Proof. Assume that problem (2.1)-(2.2) has the classical solution $u \in C^2(\overline{D}_T)$ in the domain D_T. Since $\operatorname{supp} F \cap S_\infty = \varnothing$, there exists the positive number $\delta < T/2$ such that $F|_{U_\delta(S_T)} = 0$, where $U_\delta(S_T) : |x| \leq t \leq |x| + \delta$, $t \leq T$. By Lemma 3.5, this implies that

$$u\,\big|_{U_\delta(S_T)} = 0. \tag{3.26}$$

Further, since by the condition $F \geq 0$, due to Lemma 3.3,

$$u|_{\overline{D}_T} \geq 0. \tag{3.27}$$

Therefore continuing the functions F and u by zero outside the domain D_T to the strip $\Lambda_T : 0 < t < T$ and retaining the same notation, we find that $u \in C^2(\overline{D}_T)$ is the classical solution of (2.1) in the strip Λ_T, which, by virtue of $\lambda < 0$ and (3.27), we can write in the form

$$u_{tt} - \Delta u = |\lambda| u^{p+1} + F(x,t), \quad (x,t) \in \Lambda_T. \tag{3.28}$$

Moreover, by (3.26),

$$\operatorname{supp} u \subset \overline{D}_{T,\delta}, \quad D_{T,\delta} = \{(x,t) \in R^3 : |x| + \delta < t < T\}. \tag{3.29}$$

Below, without restriction of generality we will assume that $\lambda = -1$, and hence $|\lambda| = 1$, since the case $\lambda < 0$, $\lambda \neq -1$ with regard to $p > 0$ is reduced to the case $\lambda = -1$ by introducing a new unknown function $v = |\lambda|^{1/p} u$. The function v will satisfy the equation

$$v_{tt} - \Delta v = v^{p+1} + |\lambda|^{1/p} F(x,t), \quad (x,t) \in \Lambda_T. \tag{3.30}$$

According to (3.30), below, instead of (2.1) we will consider the equation

$$u_{tt} - \Delta u = u^{p+1} + F(x,t), \quad (x,t) \in \Lambda_T. \tag{3.31}$$

We take $R \geq T$ and introduce into the consideration the functions

$$E(t) = \int_{\Omega_R} u(x,t)\varphi_R(x)dx,$$
$$f_R(t) = \int_{\Omega_R} F(x,t)\varphi_R(x)dx, \quad 0 \leq t \leq T. \tag{3.32}$$

It is clear that $E \in C^2([0,T])$, $f_R \in C([0,T])$, and, with regard to (3.27), the function $E \geq 0$. By (3.23), (3.29), and (3.32), the integration by parts results in

$$\int_{\Omega_R} \Delta u \varphi_R dx = \int_{\Omega_R} u \Delta \varphi_R dx = -c_R \int_{\Omega_R} u \varphi_R dx = -c_R E. \tag{3.33}$$

By (3.24), (3.27), and $p > 0$, and using Jensen's inequality [27, page 26], we obtain

$$\int_{\Omega_R} u^{p+1} \varphi_R dx \geq \left(\int_{\Omega_R} u \varphi_R dx \right)^{p+1} = E^{p+1}. \tag{3.34}$$

From (3.29), (3.31), (3.32), (3.33), and (3.34) it follows that

$$E'' + c_R E \geq E^{p+1} + f_R, \qquad 0 \leq t \leq T,$$
$$E(0) = 0, \qquad E'(0) = 0. \tag{3.35}$$

To investigate problem (3.35), we will use the method of test functions [26, pages 10–12]. To this end, we take T_1, $0 < T_1 < T$ and consider the nonnegative test function $\psi \in C^2([0,T])$ such that

$$0 \leq \psi \leq 1, \quad \psi(t) = 1, \quad 0 \leq t \leq T_1, \qquad \psi^{(k)}(T) = 0, \quad k = 0,1,2. \tag{3.36}$$

From (3.35) and (3.36) it easily follows that

$$\int_0^T E^{p+1}(t)\psi(t)dt \leq \int_0^T E(t)[\psi''(t) + c_R\psi(t)]dt - \int_0^T f_R(t)\psi(t)dt. \tag{3.37}$$

If, in Young's inequality with parameter $\varepsilon > 0$

$$ab \leq \frac{\varepsilon}{\alpha}a^\alpha + \frac{1}{\alpha' \varepsilon^{\alpha'-1}} b^{\alpha'}, \quad a,b \geq 0, \ \alpha' = \frac{\alpha}{\alpha - 1}, \tag{3.38}$$

we put $\alpha = p+1$, $\alpha' = (p+1)/p$, $a = E\psi^{1/(p+1)}$, $b = |\psi'' + c_R\psi|/\psi^{1/(p+1)}$ and take into account that $\alpha'/\alpha = 1/(\alpha - 1) = \alpha' - 1$, then we will get

$$E|\psi'' + c_R\psi| = E\psi^{1/\alpha}\frac{|\psi'' + c_R\psi|}{\psi^{1/\alpha}} \leq \frac{\varepsilon}{\alpha}E^\alpha\psi + \frac{1}{\alpha'\varepsilon^{\alpha'-1}}\frac{|\psi'' + c_R\psi|^{\alpha'}}{\psi^{\alpha'-1}}. \tag{3.39}$$

By (3.39), from (3.37) we have

$$\left(1 - \frac{\varepsilon}{\alpha}\right)\int_0^T E^\alpha \psi \, dt \leq \frac{1}{\alpha'\varepsilon^{\alpha'-1}}\int_0^T \frac{|\psi'' + c_R\psi|^{\alpha'}}{\psi^{\alpha'-1}}dt - \int_0^T f_R(t)\psi(t)dt. \tag{3.40}$$

Taking into account that $\min_{0<\varepsilon<\alpha}[((\alpha-1)/(\alpha-\varepsilon))(1/\varepsilon^{\alpha'-1})]=1$ which can be achieved for $\varepsilon=1$, from (3.40) by means of (3.36), we find that

$$\int_0^{T_1} E^{\alpha}dt \le \int_0^T \frac{\left|\psi''+c_R\psi\right|^{\alpha'}}{\psi^{\alpha'-1}}dt-\alpha'\int_0^T f_R(t)\psi(t)dt. \tag{3.41}$$

Now in the capacity of the test function ψ we take the function of the type

$$\psi(t)=\psi_0(\tau),\quad \tau=\frac{t}{T_1},\ 0\le\tau\le\tau_1=\frac{T}{T_1}, \tag{3.42}$$

where

$$\begin{gathered}\psi_0\in C^2([0,\tau_1]),\quad 0\le\psi_0\le 1,\\ \psi_0(\tau)=1,\quad 0\le\tau\le 1,\qquad \psi_0^{(k)}(\tau_1)=0,\quad k=0,1,2.\end{gathered} \tag{3.43}$$

It is not difficult to see that

$$c_R=\frac{c_1}{R^2}\le\frac{c_1}{T^2}\le\frac{c_1}{T_1^2},\qquad \varphi_R(x)=\frac{1}{R^2}\varphi_1\left(\frac{x}{R}\right). \tag{3.44}$$

By virtue of (3.42), (3.43), and (3.44), taking into account that $\psi''(t)=0$ for $0\le t\le T_1$ and $f_R\ge 0$, since $F\ge 0$, as well as the well-known inequality $|a+b|^{\alpha'}\le 2^{\alpha'-1}(|a|^{\alpha'}+|b|^{\alpha'})$, from (3.41) we obtain

$$\begin{aligned}\int_0^{T_1}E^{\alpha}dt&\le\int_0^{T_1}\frac{c_R^{\alpha'}\psi^{\alpha'}}{\psi^{\alpha'-1}}dt+\int_{T_1}^{T}\frac{\left|\psi''+c_R\psi\right|^{\alpha'}}{\psi^{\alpha'-1}}dt-\alpha'\int_0^T f_R(t)\psi(t)dt\\ &\le c_R^{\alpha'}\int_0^{T_1}\psi\,dt+T_1\int_1^{\tau_1}\frac{\left|(1/T_1^2)\psi_0''(\tau)+c_R\psi_0(\tau)\right|^{\alpha'}}{(\psi_0(\tau))^{\alpha'-1}}d\tau-\alpha'\int_0^{T_1}f_R(t)dt\\ &\le c_R^{\alpha'}T_1+\frac{2^{\alpha'-1}}{T_1^{2\alpha'-1}}\int_1^{\tau_1}\frac{\left|\psi_0''(\tau)\right|^{\alpha'}}{(\psi_0(\tau))^{\alpha'-1}}d\tau+T_1 2^{\alpha'-1}c_R^{\alpha'}\int_1^{\tau_1}\psi_0(\tau)d\tau-\alpha'\int_0^{T_1}f_R(t)dt\\ &\le\frac{c_1^{\alpha'}}{T_1^{2\alpha'-1}}+\frac{2^{\alpha'-1}}{T_1^{2\alpha'-1}}\int_1^{\tau_1}\frac{\left|\psi_0''(\tau)\right|^{\alpha'}}{(\psi_0(\tau))^{\alpha'-1}}d\tau+\frac{2^{\alpha'-1}c_1^{\alpha'}}{T_1^{2\alpha'-1}}(\tau_1-1)-\alpha'\int_0^{T_1}f_R(t)dt.\end{aligned} \tag{3.45}$$

Now we put $R=T$, $\tau_1=2$, that is, $T_1=(1/2)T$. Then inequality (3.45) takes the form

$$\begin{aligned}\int_0^{(1/2)T}E^{\alpha}dt\le\left(\frac{1}{2}T\right)^{1-2\alpha'}&\Bigg[c_1^{\alpha'}(1+2^{\alpha'-1})+2^{\alpha'-1}\int_1^2\frac{\left|\psi_0''(\tau)\right|^{\alpha'}}{(\psi_0(\tau))^{\alpha'-1}}d\tau\\ &\quad-\alpha'\left(\frac{1}{2}T\right)^{2\alpha'-1}\int_0^{(1/2)T}f_T(t)dt\Bigg],\quad 2\alpha'-1=\frac{p+2}{p}.\end{aligned} \tag{3.46}$$

As is known, the function ψ_0 with the properties (3.43) for which the integral

$$\varkappa(\psi_0)=\int_1^2\frac{\left|\psi_0''(\tau)\right|^{\alpha'}}{(\psi_0(\tau))^{\alpha'-1}}d\tau<+\infty \tag{3.47}$$

is finite does exist [26, page 11].

With regard to (3.32) and (3.44), we have

$$\begin{aligned}\beta(T) &= \int_0^{(1/2)T} f_T(t)dt = \int_0^{(1/2)T} dt \int_{\Omega_T} F(x,t)\varphi_T(x)dx \\ &= \int_0^{(1/2)T} dt \int_{\Omega_T} F(x,t)\frac{1}{T^2}\varphi_1\left(\frac{x}{T}\right)dx \\ &= \int_0^{(1/2)T} dt \int_{\Omega_1} F(T\xi,t)\varphi_1(\xi)d\xi.\end{aligned} \tag{3.48}$$

If condition (3.25) is fulfilled, then by virtue of (3.46), (3.47), and (3.48) there exists the number $T = T_0 > 0$ for which the right-hand side of inequality (3.46) is negative, but this is impossible because the left-hand side of inequality (3.46) is nonnegative. Thus for $T = T_0$, and hence for $T \geq T_0$, problem (2.1)-(2.2) fails to have the classical solution $u \in C^2(\overline{D}_T)$ in the domain D_T. Thus Theorem 3.6 is proved completely. □

Corollary 3.7. *Let $n = 2$, $\lambda < 0$, $F \in C(\overline{D}_\infty)$, $\operatorname{supp} F \cap S_\infty = \varnothing$, $F \not\equiv 0$, and $F \geq 0$. If $0 < p < 2$, then there exists the number $T_0 = T_0(F) > 0$ such that for $T \geq T_0$ problem (2.1)-(2.2) fails to have the classical solution $u \in C^2(\overline{D}_T)$ in the domain D_T.*

Indeed, since $F \not\equiv 0$ and $F \geq 0$, there exists the point $P_0(x^0, t^0) \in D_\infty$ such that $F(x^0, t^0) > 0$. Without restriction of generality, we can assume that the point P_0 lies on the axis t, that is, $x^0 = 0$, since, otherwise, this can be achieved by the Lorentz transformation for which (2.1) is invariant and which leaves the characteristic cone $S_\infty : t = |x|$ unchanged [5, page 744]. Since $F(0,t^0) > 0$ and $F \in C(\overline{D}_\infty)$, there exist the numbers $t^0 > \delta$, $\varepsilon_0 > 0$, and $\sigma > 0$ such that $F(x,t) \geq \sigma$ for $|x| < \varepsilon_0$, $|t - t^0| < \varepsilon_0$. Take $T > 2(t^0 + \varepsilon_0)$. Then for $|x| < \varepsilon_0$ it is evident that $|x/T| < 1/2$, and if we introduce the notation $m_0 = \inf_{|\eta|<1/2} \varphi_1(\eta)$, then if $\varphi_1(x) > 0$ in the unit circle $\Omega_1 : |x| < 1$, we find that $m_0 > 0$. Hence by virtue of (3.48), we have

$$\begin{aligned}\beta(T) &= \frac{1}{T^2}\int_0^{(1/2)T} dt \int_{\Omega_T} F(x,t)\varphi_1\left(\frac{x}{T}\right)dx \\ &\geq \frac{1}{T^2}\int_{t^0-\varepsilon}^{t^0+\varepsilon} dt \int_{|x|<\varepsilon_0} F(x,t)\varphi_1\left(\frac{x}{T}\right)dx \\ &\geq \frac{1}{T^2}\int_{t^0-\varepsilon}^{t^0+\varepsilon} dt \int_{|x|<\varepsilon_0} \sigma m_0 dx = \frac{2\pi\varepsilon_0^3\sigma m_0}{T^2}\end{aligned} \tag{3.49}$$

and, consequently,

$$T^{(p+2)/p}\int_0^T dt \int_{\Omega_1} F(2T\xi,t)\varphi_1(\xi)d\xi = T^{(p+2)/p}\beta(2T) \geq \frac{1}{2}\pi\varepsilon_0^3\sigma m_0 T^{(2-p)/p}. \tag{3.50}$$

From the last inequality for $0 < p < 2$ we immediately obtain (3.25) and, according to Theorem 3.6, problem (2.1)-(2.2) fails to have the classical solution $u \in C^2(\overline{D}_T)$ in the domain D_T for $T \geq T_0$.

Corollary 3.8. *Let $n = 2$, $\lambda < 0$, $F \in C(\overline{D}_\infty)$, $\operatorname{supp} F \cap S_\infty = \varnothing$, and $F \geq 0$. Suppose next that $F(x,t) \geq \gamma(t) \geq 0$ for $|x| < \varepsilon(t) < t$, $t > \delta$, and $\sup_{t>\delta}(\varepsilon(t)/t) = \varepsilon_0 < 1$, where $\gamma(t)$ and $\varepsilon(t)$ are the given continuous functions with $\gamma(t) \geq 0$ and $\varepsilon(t) > 0$. If the condition*

$$\overline{\lim}_{T\to+\infty} T^{(2-p)/p} \int_\delta^T \varepsilon^2(t)\gamma(t)dt = +\infty \tag{3.51}$$

is fulfilled, then there exists the number $T_0 = T_0(F) > 0$ such that for $T \geq T_0$ problem (2.1)-(2.2) fails to have the classical solution $u \in C^2(\overline{D}_T)$ in the domain D_T.

Indeed, for $|x| < \varepsilon(t)$, $t \leq (1/2)T$, we have $|x/T| < \varepsilon(t)/T = (\varepsilon(t)/t)(t/T) \leq (1/2)\varepsilon_0$. Since $\int_{|\eta|<(1/2)\varepsilon_0} \varphi_1(\eta) = m_0 > 0$, by virtue of (3.48) we have

$$\begin{aligned}
\beta(T) &= \frac{1}{T^2}\int_0^{(1/2)T} dt \int_{\Omega_T} F(x,t)\varphi_1\left(\frac{x}{T}\right)dx \\
&\geq \frac{1}{T^2}\int_\delta^{(1/2)T} dt \int_{|x|<\varepsilon(t)} \gamma(t)\varphi_1\left(\frac{x}{T}\right)dx \\
&\geq \frac{m_0}{T^2}\int_\delta^{(1/2)T} dt \int_{|x|<\varepsilon(t)} \gamma(t)dx = \frac{\pi m_0}{T^2}\int_\delta^{(1/2)T} \varepsilon^2(t)\gamma(t)dt.
\end{aligned} \tag{3.52}$$

Therefore with regard to (3.48),

$$T^{(p+2)/p}\int_0^T dt \int_{\Omega_1} F(2T\xi,t)\varphi_1(\xi)d\xi = T^{(p+2)/p}\beta(2T) \geq \frac{\pi m_0}{4} T^{(2-p)/p}\int_\delta^T \varepsilon^2(t)\gamma(t)dt, \tag{3.53}$$

whence by (3.51) we obtain (3.25) and hence the validity of Corollary 3.8.

Remark 3.9. Inequality (3.46) allows us to estimate the time interval after which the solution fails. Indeed, let

$$\begin{aligned}
\chi(T) &= \sup_{0<t<T} \alpha'\left(\frac{1}{2}t\right)^{2\alpha'-1}\int_0^{(1/2)t} f_t(\tau)d\tau, \\
\chi_0 &= c_1^{\alpha'}\left(1 + 2^{\alpha'-1}\right) + 2^{\alpha'-1}\varkappa(\psi_0),
\end{aligned} \tag{3.54}$$

where $\alpha' = (p+1)/p$, and the finite positive number $\varkappa(\psi_0)$ is given by (3.47). Since $F \in C(\overline{D}_\infty)$, the function $\chi(T)$ in the interval $0 < T < +\infty$ is continuous and nondecreasing, while by virtue of (3.25) and (3.48) we have $\lim_{T\to+\infty}\chi(T) = +\infty$. Hence since $\lim_{T\to 0}\chi(T) = 0$, the equation $\chi(T) = \chi_0$ is solvable. Denote by $T = T_1$ the root of the above-mentioned equation for which $\chi(T) > \chi(T_1)$ for $T_1 < T < T_1 + \varepsilon$, where ε is a sufficiently small positive number. Now it is clear that problem (2.1)-(2.2) has no classical solution in the domain D_T for $T > T_1$, since in this case the right-hand side of inequality (3.46) is negative.

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