ON A BOUNDARY VALUE PROBLEM FOR NONLINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

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We consider the problem $u'(t) = H(u)(t) + Q(u)(t)$, $u(a) = h(u)$, where $H, Q : C([a,b];R) \to L([a,b];R)$ are, in general, nonlinear continuous operators, $H \in \mathcal{K}_{ab}^{\alpha\beta}(g_0, g_1, p_0, p_1)$, and $h : C([a,b];R) \to R$ is a continuous functional. Efficient conditions sufficient for the solvability and unique solvability of the problem considered are established.

1. Notation

The following notation is used throughout the paper:

N is the set of all natural numbers.

R is the set of all real numbers, $R_+ = [0,+\infty[$, $[x]_+ = (1/2)(|x| + x)$, $[x]_- = (1/2)(|x| - x)$.

$C([a,b];R)$ is the Banach space of continuous functions $u : [a,b] \to R$ with the norm $\|u\|_C = \max\{|u(t)| : t \in [a,b]\}$.

$\widetilde{C}([a,b];R)$ is the set of absolutely continuous functions $u : [a,b] \to R$.

$L([a,b];R)$ is the Banach space of Lebesgue integrable functions $p : [a,b] \to R$ with the norm $\|p\|_L = \int_a^b |p(s)|ds$.

$L([a,b];R_+) = \{p \in L([a,b];R) : p(t) \geq 0 \text{ for } t \in [a,b]\}$.

$\mathcal{M}_{ab}$ is the set of measurable functions $\tau : [a,b] \to [a,b]$.

$\mathcal{K}_{ab}$ is the set of continuous operators $F : C([a,b];R) \to L([a,b];R)$ satisfying the Carathèodory condition, that is, for each $r > 0$ there exists $q_r \in L([a,b];R_+)$ such that

$$\big|F(v)(t)\big| \leq q_r(t) \quad \text{for } t \in [a,b],\ v \in C\big([a,b];R\big),\ \|v\|_C \leq r. \tag{1.1}$$

$K([a,b] \times A;B)$, where $A \subseteq R^2$, $B \subseteq R$, is the set of functions $f : [a,b] \times A \to B$ satisfying the Carathèodory conditions, that is, $f(\cdot,x) : [a,b] \to B$ is a measurable function for all $x \in A$, $f(t,\cdot) : A \to B$ is a continuous function for almost all $t \in [a,b]$, and for each $r > 0$ there exists $q_r \in L([a,b];R_+)$ such that

$$\big|f(t,x)\big| \leq q_r(t) \quad \text{for } t \in [a,b],\ x \in A,\ \|x\| \leq r. \tag{1.2}$$

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2. Statement of the problem

We consider the equation

$$u'(t) = H(u)(t) + Q(u)(t), \tag{2.1}$$

where $H \in \mathcal{H}_{ab}^{\alpha\beta}(g_0, g_1, p_0, p_1)$ (see Definition 2.1) and $Q \in \mathcal{K}_{ab}$. By a solution of (2.1) we understand a function $u \in \widetilde{C}([a,b];R)$ satisfying the equality (2.1) almost everywhere in $[a,b]$.

Definition 2.1. We will say that an operator H belongs to the set $\mathcal{H}_{ab}^{\alpha\beta}(g_0, g_1, p_0, p_1)$, where $g_0, g_1, p_0, p_1 \in L([a,b];R_+)$ and $\alpha, \beta \in [0,1[$, if $H \in \mathcal{K}_{ab}$ is such that, on the set $C([a,b];R)$, the inequalities

$$-mg_0(t) - \mu(m,\alpha)M^{1-\alpha}g_1(t) \le H(v)(t) \le Mp_0(t) + \mu(M,\beta)m^{1-\beta}p_1(t) \quad \text{for } t \in [a,b] \tag{2.2}$$

are fulfilled, where

$$M = \max\{[v(t)]_+ : t \in [a,b]\}, \qquad m = \max\{[v(t)]_- : t \in [a,b]\}, \tag{2.3}$$

and the function $\mu : R_+ \times [0,1[\to R_+$ is defined by

$$\mu(x,y) = \begin{cases} 1 & \text{if } x = 0, \quad y = 0, \\ x^y & \text{otherwise.} \end{cases} \tag{2.4}$$

The class $\mathcal{H}_{ab}^{\alpha\beta}(g_0, g_1, p_0, p_1)$ contains all the positively homogeneous operators H and, in particular, those defined by the formula

$$\begin{aligned} H(u)(t) = {} & p_0(t)[u(\tau_1(t))]_+ + p_1(t)[u(\tau_2(t))]_+^{\beta}[u(\tau_3(t))]_-^{1-\beta} \\ & - g_0(t)[u(\nu_1(t))]_- + g_1(t)[u(\nu_2(t))]_-^{\alpha}[u(\nu_3(t))]_+^{1-\alpha}, \end{aligned} \tag{2.5}$$

where $\tau_i, \nu_i \in \mathcal{M}_{ab}$ $(i = 1,2,3)$, $\alpha \ne 0$, $\beta \ne 0$.

The class of equations (2.1) contains various equations with "maxima" studied, for example, in [3, 4, 33, 35, 36, 38, 41]. For example, the equations

$$u'(t) = p(t)\max\{u(s) : \tau_1(t) \le s \le \tau_2(t)\} + q_0(t), \tag{2.6}$$

$$\begin{aligned} u'(t) = {} & p(t)\max\{u(s) : \tau_1(t) \le s \le \tau_2(t)\} \\ & + f\big(t, u(t), \max\{u(s) : \nu_1(t) \le s \le \nu_2(t)\}\big), \end{aligned} \tag{2.7}$$

where $p, q_0 \in L([a,b];R)$, $\tau_i, \nu_i \in \mathcal{M}_{ab}$ $(i = 1,2)$, $\tau_1(t) \le \tau_2(t)$, $\nu_1(t) \le \nu_2(t)$ for $t \in [a,b]$, and $f \in K([a,b] \times R^2; R)$, can be rewritten in form (2.1) with $H \in \mathcal{H}_{ab}^{00}([p]_+, [p]_-, [p]_+, [p]_-)$.

Another type of (2.1) is an equation where H is a linear operator. In that case the results presented coincide with those obtained in [5, 6]. Other conditions guaranteeing the solvability of (2.1), (2.8) with a linear operator H can be found, for example, in [10, 11, 13, 15]. Conditions for the solvability and unique solvability of other types of boundary value problems for (2.1) with a linear operator H are established, for example, in [8, 12, 14, 16, 17, 18, 19, 20, 21, 22, 23, 26, 27, 34, 39].

We will study the problem on the existence and uniqueness of a solution of (2.1) satisfying the condition

$$u(a) = h(u), \tag{2.8}$$

where $h : C([a,b];R) \to R$ is a continuous operator such that for each $r > 0$ there exists $M_r \in R_+$ such that

$$|h(v)| \le M_r \quad \text{for } v \in C([a,b];R),\ \|v\|_C \le r. \tag{2.9}$$

There are many interesting results concerning the solvability of general boundary value problems for functional differential equations (see, e.g., [1, 2, 7, 9, 24, 25, 28, 29, 30, 31, 32, 37, 40, 42] and the references therein). In spite of this, the general theory of boundary value problems for functional differential equations is not still complete. Here, we try to fill this gap in a certain way. More precisely, in Section 3, we establish unimprovable efficient conditions sufficient for the solvability and unique solvability of problem (2.1), (2.8). In Section 4, some auxiliary propositions are proved. Sections 5 and 6 are devoted to the proof of the main results and the examples demonstrating their optimality, respectively.

3. Main results

Throughout the paper, $q \in K([a,b] \times R_+; R_+)$ is a function nondecreasing in the second argument and such that

$$\lim_{x \to +\infty} \frac{1}{x} \int_a^b q(s,x)\,ds = 0. \tag{3.1}$$

Theorem 3.1. *Let there exist $c \in R_+$ such that, on the set $C([a,b];R)$, the inequality*

$$h(v)\operatorname{sgn} v(a) \le c \tag{3.2}$$

is fulfilled and, on the set $\{v \in C([a,b];R) : |v(a)| \le c\}$, the inequality

$$|Q(v)(t)| \le q(t, \|v\|_C) \quad \text{for } t \in [a,b] \tag{3.3}$$

is satisfied. If, moreover,

$$\int_a^b g_0(s)\,ds < 1, \qquad \int_a^b p_0(s)\,ds < 1, \tag{3.4}$$

and, for $t \in [a,b]$, the inequalities

$$\left(\frac{\int_a^t g_1(s)ds}{1-\int_a^t g_0(s)ds}\right)^{(1-\beta)/(1-\alpha)} \int_t^b p_1(s)ds - \left(\frac{\int_a^t g_1(s)ds}{1-\int_a^t g_0(s)ds}\right)^{1/(1-\alpha)} < 1 - \int_t^b p_0(s)ds, \quad (3.5)$$

$$\left(\frac{\int_a^t p_1(s)ds}{1-\int_a^t p_0(s)ds}\right)^{(1-\alpha)/(1-\beta)} \int_t^b g_1(s)ds - \left(\frac{\int_a^t p_1(s)ds}{1-\int_a^t p_0(s)ds}\right)^{1/(1-\beta)} < 1 - \int_t^b g_0(s)ds \quad (3.6)$$

hold, then problem (2.1), (2.8) has at least one solution.

Theorem 3.1 is unimprovable in the sense that neither of the strict inequalities in (3.4)–(3.6) can be replaced by the nonstrict one (see Remark 6.1).

Theorem 3.2. *Let there exist $c \in R_+$ such that, on the set $C([a,b];R)$, inequality (3.2) is fulfilled and, on the set $\{v \in C([a,b];R) : |v(a)| \le c\}$, the inequality*

$$Q(v)(t)\operatorname{sgn} v(t) \le q(t, \|v\|_C) \quad \text{for } t \in [a,b] \quad (3.7)$$

is satisfied. If, moreover, (3.4) holds and

$$\left(\frac{\int_a^t g_1(s)ds}{1-\int_a^t g_0(s)ds}\right)^{(1-\beta)/(1-\alpha)} \int_t^b p_1(s)ds < 1 - \int_t^b p_0(s)ds \quad \text{for } t \in [a,b], \quad (3.8)$$

$$\left(\frac{\int_a^t p_1(s)ds}{1-\int_a^t p_0(s)ds}\right)^{(1-\alpha)/(1-\beta)} \int_t^b g_1(s)ds < 1 - \int_t^b g_0(s)ds \quad \text{for } t \in [a,b], \quad (3.9)$$

then the problem (2.1), (2.8) has at least one solution.

Theorem 3.2 is unimprovable in the sense that neither of the strict inequalities in (3.4), (3.8), and (3.9) can be replaced by the nonstrict one (see Remark 6.4).

Theorem 3.3. *Assume that the operators H_z, $z \in \{v \in C([a,b];R) : v(a) = h(v)\}$, defined by the formula*

$$H_z(v)(t) \overset{\text{def}}{=} H(v+z)(t) - H(z)(t) \quad \text{for } t \in [a,b] \quad (3.10)$$

belong to the set $\mathcal{H}_{ab}^{\alpha\beta}(g_0, g_1, p_0, p_1)$. Let, moreover, for all v and w from the set $C([a,b];R)$, the inequality

$$[h(v) - h(w)]\operatorname{sgn}(v(a) - w(a)) \le 0 \quad (3.11)$$

hold, and let

$$Q(v) \equiv q^* \quad \text{for } v \in C([a,b];R), \quad |v(a)| \le |h(0)|, \quad (3.12)$$

where $q^* \in L([a,b];R)$. If, moreover, condition (3.4) holds and for $t \in [a,b]$ the inequalities (3.5) and (3.6) are fulfilled, then the problem (2.1), (2.8) has a unique solution.

Theorem 3.4. *Let the operators* H_z, $z \in \{v \in C([a,b];R) : v(a) = h(v)\}$, *defined by (3.10) belong to the set* $\mathcal{H}_{ab}^{\alpha\beta}(g_0, g_1, p_0, p_1)$. *Assume also that, on the set* $C([a,b];R)$ *the inequality (3.11) is fulfilled, and, on the set* $\{v \in C([a,b];R) : |v(a)| \le |h(0)|\}$, *the inequality*

$$[Q(v)(t) - Q(w)(t)] \operatorname{sgn}(v(t) - w(t)) \le 0 \quad \text{for } t \in [a,b] \tag{3.13}$$

holds. If, moreover, inequalities (3.4), (3.8), and (3.9) are fulfilled, then problem (2.1), (2.8) has a unique solution.

Remark 3.5. The inclusions $H_z \in \mathcal{H}_{ab}^{\alpha\beta}(g_0, g_1, p_0, p_1)$, where H_z are defined by (3.10), are fulfilled, for example, if H is a strongly bounded linear operator. In this case, the optimality of obtained results was proved in [21] (see Remark 4.2 on page 97 and Remark 12.2 on page 243 therein). More precisely, Theorems 3.3 and 3.4 are unimprovable in the sense that neither of the strict inequalities (3.4)–(3.6), (3.8), and (3.9) can be replaced by the nonstrict one.

The following corollary gives conditions sufficient for the solvability of problem (2.7), (2.8).

Corollary 3.6. *Let there exist* $c \in R_+$ *such that on the set* $C([a,b];R)$ *the inequality (3.2) is fulfilled and*

$$|f(t,x,y)| \le q(t,|x|) \quad \text{for } t \in [a,b],\ x,y \in R. \tag{3.14}$$

If, moreover,

$$\int_a^b [p(s)]_+ ds < 1, \tag{3.15}$$

$$\int_a^b [p(s)]_- ds < 1 + 2\sqrt{1 - \int_a^b [p(s)]_+ ds}, \tag{3.16}$$

then problem (2.7), (2.8) has at least one solution.

Corollary 3.7. *Let inequality (3.11) be fulfilled on the set* $C([a,b];R)$. *If, moreover, (3.15) and (3.16) hold, then problem (2.6), (2.8) has a unique solution.*

Remark 3.8. Corollaries 3.6 and 3.7 are unimprovable in the sense that neither of the strict inequalities (3.15) and (3.16) can be replaced by the nonstrict one. Indeed, if $\tau_1 \equiv \tau_2$ and $\nu_1 \equiv \nu_2$, then (2.6) and (2.7) are differential equations with deviating arguments. In that case, the optimality of obtained results was established in [21] (see Remark 4.2 on page 97 and Proposition 10.1 on page 190 therein).

Corollary 3.9. *Let there exist* $c \in R_+$ *such that on the set* $C([a,b];R)$ *the inequality (3.2) is fulfilled and*

$$f(t,x,y) \operatorname{sgn} x \le q(t,|x|) \quad \text{for } t \in [a,b],\ x,y \in R. \tag{3.17}$$

If, moreover, (3.15) and

$$\int_a^b [p(s)]_- ds < 2\sqrt{1 - \int_a^b [p(s)]_+ ds} \tag{3.18}$$

hold, then the problem (2.7), (2.8) has at least one solution.

The following corollary gives conditions sufficient for the unique solvability of problem (2.7), (2.8).

Corollary 3.10. *Let inequality (3.11) be fulfilled on the set $C([a,b];R)$ and, in addition,*

$$[f(t,x_1,y_1) - f(t,x_2,y_2)]\operatorname{sgn}(x_1 - x_2) \leq 0 \quad \text{for } t \in [a,b],\ x_1,x_2,y_1,y_2 \in R. \tag{3.19}$$

If, moreover, (3.15) and (3.18) hold, then the problem (2.7), (2.8) has a unique solution.

Remark 3.11. Corollaries 3.9 and 3.10 are unimprovable in the sense that neither of the strict inequalities (3.15) and (3.18) can be replaced by the nonstrict one. Indeed, when $\tau_1 \equiv \tau_2$ and $\nu_1 \equiv \nu_2$, (2.7) is a differential equation with deviating arguments and, in this case, the optimality of obtained results is proved in [21] (see Remark 12.2 on page 243 therein).

4. Auxiliary propositions

First we formulate a result from [25] in a suitable for us form.

Lemma 4.1. *Let there exist a number $\rho > 0$ such that, for every $\delta \in]0,1[$, an arbitrary function $u \in \widetilde{C}([a,b];R)$ satisfying*

$$u'(t) = \delta[H(u)(t) + Q(u)(t)] \quad \text{for } t \in [a,b],\ u(a) = \delta h(u), \tag{4.1}$$

admits the estimate

$$\|u\|_C \leq \rho. \tag{4.2}$$

Then problem (2.1), (2.8) has at least one solution.

Definition 4.2. We will say that an operator $H \in \mathcal{K}_{ab}$ belongs to the set $\mathcal{U}$, if there exists a number $r > 0$ such that for any $q^* \in L([a,b];R_+)$, $c \in R_+$, and $\delta \in]0,1]$, every function $u \in \widetilde{C}([a,b];R)$ satisfying the inequalities $|u(a)| \leq c$ and

$$|u'(t) - \delta H(u)(t)| \leq q^*(t) \quad \text{for } t \in [a,b] \tag{4.3}$$

admits the estimate

$$\|u\|_C \leq r\left(c + \|q^*\|_L\right). \tag{4.4}$$

Lemma 4.3. *Let there exist $c \in R_+$ such that inequalities (3.2) and (3.3) are fulfilled on the sets $C([a,b];R)$ and $\{v \in C([a,b];R) : |v(a)| \leq c\}$, respectively. If, moreover, $H \in \mathcal{U}$, then problem (2.1), (2.8) has at least one solution.*

Proof. Let r be the number appearing in Definition 4.2. According to (3.1), there exists $\rho > 2rc$ such that

$$\frac{1}{x}\int_a^b q(s,x)ds < \frac{1}{2r} \quad \text{for } x > \rho. \tag{4.5}$$

Now assume that a function $u \in \widetilde{C}([a,b];R)$ satisfies (4.1) for some $\delta \in]0,1[$. Then, according to (3.2), u satisfies the inequality $|u(a)| \le c$. By (3.3) we obtain that the inequality (4.3) is fulfilled with $q^*(t) = q(t,\|u\|_C)$ for $t \in [a,b]$. Hence, by the condition $H \in \mathcal{U}$ and the definition of the number ρ, we get estimate (4.2).

Since ρ depends neither on u nor on δ, it follows from Lemma 4.1 that problem (2.1), (2.8) has at least one solution. □

Let $h_i \in L([a,b];R_+)$ $(i = 1,2,3,4)$, $\overline{\alpha},\overline{\beta} \in [0,1[$. For an arbitrary fixed $t \in [a,b]$, consider the systems of inequalities

$$\begin{aligned} m &\le m\int_a^t h_1(s)ds + \mu(m,\overline{\alpha})M^{1-\overline{\alpha}}\int_a^t h_2(s)ds, \\ m + M &\le M\int_t^b h_3(s)ds + \mu(M,\overline{\beta})m^{1-\overline{\beta}}\int_t^b h_4(s)ds \end{aligned} \tag{4.6$_t$}$$

and

$$\begin{aligned} M &\le M\int_a^t h_3(s)ds + \mu(M,\overline{\beta})m^{1-\overline{\beta}}\int_a^t h_4(s)ds, \\ M + m &\le m\int_t^b h_1(s)ds + \mu(m,\overline{\alpha})M^{1-\overline{\alpha}}\int_t^b h_2(s)ds, \end{aligned} \tag{4.7$_t$}$$

where $\mu : R_+ \times [0,1[\to R_+$ is defined by (2.4). By a solution of system $((4.6)_t)_t$ (resp., $((4.7)_t)_t$), we understand a pair $(M,m) \in R_+ \times R_+$ satisfying $((4.6)_t)_t$ (resp., $((4.7)_t)_t$).

Definition 4.4. Let $h_i \in L([a,b];R_+)$ $(i = 1,2,3,4)$ and $\overline{\alpha},\overline{\beta} \in [0,1[$. We will say that a 4-tuple (h_1,h_2,h_3,h_4) belongs to the set $\mathcal{A}_{ab}(\overline{\alpha},\overline{\beta})$, if for every $t \in [a,b]$ the systems $((4.6)_t)_t$ and $((4.7)_t)_t$ have only the trivial solution.

Lemma 4.5. *Let $H \in \mathcal{H}_{ab}^{\alpha\beta}(g_0,g_1,p_0,p_1)$. If*

$$(g_0,g_1,p_0,p_1) \in \mathcal{A}_{ab}(\alpha,\beta), \tag{4.8}$$

then $H \in \mathcal{U}$.

Proof. Assume on the contrary that, for every $n \in N$, there exist $q_n^* \in L([a,b];R_+)$, $c_n \in R_+$, $\delta_n \in]0,1]$, and $u_n \in \widetilde{C}([a,b];R)$ such that

$$|u_n(a)| \le c_n, \tag{4.9}$$

$$|u_n'(t) - \delta_n H(u_n)(t)| \le q_n^*(t) \quad \text{for } t \in [a,b], \tag{4.10}$$

$$||u_n||_C > n(c_n + ||q_n^*||_L). \tag{4.11}$$

Put

$$v_n(t) = \frac{u_n(t)}{||u_n||_C} \quad \text{for } t \in [a,b],\ n \in N. \tag{4.12}$$

Obviously,

$$||v_n||_C = 1 \quad \text{for } n \in N, \tag{4.13}$$

$$v_n'(t) = \frac{\delta_n}{||u_n||_C} H(u_n)(t) + q_n(t) \quad \text{for } t \in [a,b],\ n \in N, \tag{4.14}$$

where

$$q_n(t) \stackrel{\text{def}}{=} v_n'(t) - \frac{\delta_n}{||u_n||_C} H(u_n)(t) \quad \text{for } t \in [a,b],\ n \in N. \tag{4.15}$$

By virtue of (4.9) and (4.12), we get

$$|v_n(a)| \le \frac{c_n}{||u_n||_C} \quad \text{for } n \in N. \tag{4.16}$$

Note also that, in view of (4.10), (4.12), and (4.15), we have

$$||q_n||_L \le \frac{||q_n^*||_L}{||u_n||_C} \quad \text{for } n \in N. \tag{4.17}$$

Furthermore, for $n \in N$, put

$$M_n = \max\{[v_n(t)]_+ : t \in [a,b]\}, \qquad m_n = \max\{[v_n(t)]_- : t \in [a,b]\}. \tag{4.18}$$

Evidently, $M_n \ge 0$, $m_n \ge 0$ for $n \in N$, and on account of (4.13), we have

$$M_n + m_n \ge 1 \quad \text{for } n \in N. \tag{4.19}$$

According to (4.18) and (4.19), for every $n \in N$, the points s_n, $t_n \in [a,b]$ can be chosen in the following way:

(i) if $M_n = 0$, then let $t_n = a$ and let $s_n \in [a,b]$ be such that

$$v_n(s_n) = -m_n, \tag{4.20}$$

(ii) if $m_n = 0$, then let $s_n = a$ and let $t_n \in [a,b]$ be such that

$$v_n(t_n) = M_n, \tag{4.21}$$

(iii) if $M_n > 0$ and $m_n > 0$, then let s_n, $t_n \in [a,b]$ be such that (4.20) and (4.21) are satisfied.

By virtue of (4.13) and (4.18) we have that the sequences $\{M_n\}_{n=1}^{+\infty}$ and $\{m_n\}_{n=1}^{+\infty}$ are bounded. Obviously, also the sequences $\{s_n\}_{n=1}^{+\infty}$ and $\{t_n\}_{n=1}^{+\infty}$ are bounded, and, moreover, for every $n \in N$ we have either

$$a \le s_n \le t_n \le b, \tag{4.22}$$

or

$$a \le t_n \le s_n \le b. \tag{4.23}$$

Therefore, without loss of generality we can assume that there exist M_0, $m_0 \in R_+$ and s_0, $t_0 \in [a,b]$ such that

$$\lim_{n\to+\infty} M_n = M_0, \qquad \lim_{n\to+\infty} m_n = m_0, \qquad \lim_{n\to+\infty} s_n = s_0, \qquad \lim_{n\to+\infty} t_n = t_0, \tag{4.24}$$

$$a \le s_n \le t_n \le b \quad \text{for } n \in N, \tag{4.25}$$

or, instead of (4.25),

$$a \le t_n \le s_n \le b \quad \text{for } n \in N. \tag{4.26}$$

Furthermore, on account of (4.19), we have

$$M_0 + m_0 \ge 1. \tag{4.27}$$

Let (4.25) be fulfilled. Then the integration of (4.14) from a to s_n and from s_n to t_n, respectively, for every $n \in N$ yields

$$\begin{aligned} v_n(s_n) &= v_n(a) + \frac{\delta_n}{\|u_n\|_C}\int_a^{s_n} H(u_n)(\xi)d\xi + \int_a^{s_n} q_n(\xi)d\xi, \\ v_n(t_n) - v_n(s_n) &= \frac{\delta_n}{\|u_n\|_C}\int_{s_n}^{t_n} H(u_n)(\xi)d\xi + \int_{s_n}^{t_n} q_n(\xi)d\xi. \end{aligned} \tag{4.28}$$

From (4.28), in view of (4.11), (4.12), (4.16)–(4.18), the assumptions $\delta_n \in]0,1]$ and $H \in \mathcal{H}_{ab}^{\alpha\beta}(g_0,g_1,p_0,p_1)$, and the choice of points s_n and t_n, for every $n \in N$ we get

$$\begin{aligned} m_n &\le m_n\int_a^{s_n} g_0(\xi)d\xi + \mu(m_n,\alpha)M_n^{1-\alpha}\int_a^{s_n} g_1(\xi)d\xi + \frac{1}{n}, \\ M_n + m_n &\le M_n\int_{s_n}^{b} p_0(\xi)d\xi + \mu(M_n,\beta)m_n^{1-\beta}\int_{s_n}^{b} p_1(\xi)d\xi + \frac{1}{n}. \end{aligned} \tag{4.29}$$

Therefore, according to (2.4) and (4.24), from (4.29) as $n \to +\infty$ we obtain

$$\begin{aligned} m_0 &\le m_0\int_a^{s_0} g_0(\xi)d\xi + \mu(m_0,\alpha)M_0^{1-\alpha}\int_a^{s_0} g_1(\xi)d\xi, \\ M_0 + m_0 &\le M_0\int_{s_0}^{b} p_0(\xi)d\xi + \mu(M_0,\beta)m_0^{1-\beta}\int_{s_0}^{b} p_1(\xi)d\xi. \end{aligned} \tag{4.30}$$

Consequently, the pair (M_0,m_0) is a solution of system $((4.6)_t)_{s_0}$ with $h_1 \equiv g_0$, $h_2 \equiv g_1$, $h_3 \equiv p_0$, $h_4 \equiv p_1$, $\overline{\alpha} = \alpha$, and $\overline{\beta} = \beta$. However, inequality (4.27) contradicts inclusion (4.8).

If (4.26) is fulfilled, then it can be shown analogously that (M_0,m_0) is a solution of inequalities $((4.7)_t)_{t_0}$ with $h_1 \equiv g_0$, $h_2 \equiv g_1$, $h_3 \equiv p_0$, $h_4 \equiv p_1$, $\overline{\alpha} = \alpha$, and $\overline{\beta} = \beta$. Also in this case the inequality (4.27) contradicts (4.8). □

Lemma 4.6. *Let condition (3.4) be satisfied and let for $t \in [a,b]$ the inequalities (3.5) and (3.6) be fulfilled. Then the inclusion (4.8) holds.*

Proof. Assume on the contrary that there exists $t_0 \in [a,b]$ such that either the system $((4.6)_t)_{t_0}$ or the system $((4.7)_t)_{t_0}$ with $h_1 \equiv g_0$, $h_2 \equiv g_1$, $h_3 \equiv p_0$, $h_4 \equiv p_1$, $\overline{\alpha} = \alpha$, and $\overline{\beta} = \beta$ has a nontrivial solution.

First suppose that (M_0, m_0) is a nontrivial solution of $((4.6)_t)_{t_0}$. Put

$$
\begin{aligned}
G_0(t) &= \int_a^t g_0(s)ds, \quad G_1(t) = \int_a^t g_1(s)ds \quad \text{for } t \in [a,b], \\
P_0(t) &= \int_t^b p_0(s)ds, \quad P_1(t) = \int_t^b p_1(s)ds \quad \text{for } t \in [a,b].
\end{aligned}
\tag{4.31}
$$

Then, according to the assumptions, (M_0, m_0) satisfies

$$m_0 \le m_0 G_0(t_0) + \mu(m_0,\alpha) M_0^{1-\alpha} G_1(t_0), \tag{4.32}$$

$$m_0 + M_0 \le M_0 P_0(t_0) + \mu(M_0,\beta) m_0^{1-\beta} P_1(t_0). \tag{4.33}$$

If $M_0 = 0$, then $m_0 > 0$, and from (4.32), in view of (3.4) and (4.31), we get a contradiction $m_0 < m_0$. If $m_0 = 0$, then $M_0 > 0$, and from (4.33), in view of (3.4) and (4.31), we get a contradiction $M_0 < M_0$. Therefore assume that

$$M_0 > 0, \qquad m_0 > 0. \tag{4.34}$$

In this case, according to (2.4), we have

$$\mu(m_0,\alpha) = m_0^{\alpha}, \qquad \mu(M_0,\beta) = M_0^{\beta}. \tag{4.35}$$

Then from (4.32) and (4.33), in view of (3.4), (4.31), (4.34), and (4.35), we obtain

$$0 < \frac{m_0}{M_0} \le \left(\frac{G_1(t_0)}{1 - G_0(t_0)} \right)^{1/(1-\alpha)}, \tag{4.36}$$

$$0 < 1 - P_0(t_0) \le \left(\frac{m_0}{M_0} \right)^{1-\beta} P_1(t_0) - \frac{m_0}{M_0}. \tag{4.37}$$

If $\beta = 0$, then multiplying (4.36) by (4.37) we get

$$1 - P_0(t_0) \le \left(\frac{G_1(t_0)}{1 - G_0(t_0)} \right)^{1/(1-\alpha)} \left(P_1(t_0) - 1\right), \tag{4.38}$$

which, in view of (4.31), contradicts (3.5) with $t = t_0$.

Suppose that $\beta \neq 0$. Since the function $x \mapsto x^{1-\beta}A - x$, defined on $[0,+\infty[$, $A \in R_+$, achieves the maximal value at the point $x = (1-\beta)^{1/\beta}A^{1/\beta}$, from (4.37) we obtain

$$1 - P_0(t_0) \le (1-\beta)^{(1-\beta)/\beta} \left(P_1(t_0)\right)^{(1-\beta)/\beta} P_1(t_0) - (1-\beta)^{1/\beta} \left(P_1(t_0)\right)^{1/\beta}. \tag{4.39}$$

The last inequality results in

$$\left(\frac{1-\beta}{\beta}\right)^{\beta}(1-P_0(t_0))^{\beta} \leq (1-\beta)P_1(t_0). \tag{4.40}$$

On the other hand, according to (3.4) and (4.31), the inequalities (4.36) and (4.37) imply $0 < G_1(t_0) \leq G_1(b)$, $0 < P_1(t_0) \leq P_1(a)$, and

$$\frac{G_1(a)}{1-G_0(a)} = 0, \qquad \frac{G_1(b)}{1-G_0(b)} > 0, \qquad (1-\beta)P_1(a) > 0, \qquad (1-\beta)P_1(b) = 0. \tag{4.41}$$

Therefore, since the functions G_0, G_1, and P_1 are continuous, there exists $t_1 \in]a,b[$ such that

$$\left(\frac{G_1(t_1)}{1-G_0(t_1)}\right)^{1/(1-\alpha)} = ((1-\beta)P_1(t_1))^{1/\beta}. \tag{4.42}$$

Using the last equality in (3.5) for $t = t_1$, on account of (4.31), it yields

$$((1-\beta)P_1(t_1))^{(1-\beta)/\beta}P_1(t_1) - ((1-\beta)P_1(t_1))^{1/\beta} < 1 - P_0(t_1), \tag{4.43}$$

whence we get

$$(1-\beta)P_1(t_1) < \left(\frac{1-\beta}{\beta}\right)^{\beta}(1-P_0(t_1))^{\beta}. \tag{4.44}$$

Since the functions P_0 and P_1 are nonincreasing in $[a,b]$, the last inequality implies

$$(1-\beta)P_1(t) < \left(\frac{1-\beta}{\beta}\right)^{\beta}(1-P_0(t))^{\beta} \quad \text{for } t \in [t_1,b]. \tag{4.45}$$

According to (4.40) and (4.45) we have

$$t_0 < t_1. \tag{4.46}$$

Furthermore, since the functions G_0 and G_1 are nondecreasing in $[a,b]$ and the function P_1 is nonincreasing in $[a,b]$, the equality (4.42), on account of (4.46), results in

$$\left(\frac{G_1(t_0)}{1-G_0(t_0)}\right)^{1/(1-\alpha)} \leq ((1-\beta)P_1(t_0))^{1/\beta}. \tag{4.47}$$

However, since the function $x \mapsto x^{1-\beta}A - x$, defined on $[0,+\infty[$ with $A > 0$, is nondecreasing in $[0,((1-\beta)A)^{1/\beta}]$, from (4.36) and (4.37), by virtue of (4.47), we obtain

$$1 - P_0(t_0) \leq \left(\frac{G_1(t_0)}{1-G_0(t_0)}\right)^{(1-\beta)/(1-\alpha)} P_1(t_0) - \left(\frac{G_1(t_0)}{1-G_0(t_0)}\right)^{1/(1-\alpha)}, \tag{4.48}$$

which, on account of (4.31), contradicts (3.5) with $t = t_0$.

In analogous way it can be shown that assuming (M_0, m_0) to be a nontrivial solution of $((4.7)_t)_{t_0}$ we obtain a contradiction to (3.6) with $t = t_0$. □

Definition 4.7. We will say that an operator $H \in \mathcal{K}_{ab}$ belongs to the set $\mathcal{V}$, if there exists a number $r > 0$ such that for any $q^* \in L([a,b];R_+)$, $c \in R_+$, and $\delta \in]0,1]$, every function $u \in \widetilde{C}([a,b];R)$ satisfying the inequalities $|u(a)| \le c$ and

$$[u'(t) - \delta H(u)(t)]\operatorname{sgn} u(t) \le q^*(t) \quad \text{for } t \in [a,b] \tag{4.49}$$

admits the estimate (4.4).

Lemma 4.8. *Let there exist $c \in R_+$ such that on the set $C([a,b];R)$ the inequality (3.2) is satisfied and on the set $\{v \in C([a,b];R) : |v(a)| \le c\}$ the inequality (3.7) is fulfilled. If, moreover, $H \in \mathcal{V}$, then the problem (2.1), (2.8) has at least one solution.*

Proof. Let r be the number appearing in Definition 4.7. According to (3.1), there exists $\rho > 2rc$ such that

$$\frac{1}{x}\int_a^b q(s,x)\,ds < \frac{1}{2r} \quad \text{for } x > \rho. \tag{4.50}$$

Now assume that a function $u \in \widetilde{C}([a,b];R)$ satisfies (4.1) for some $\delta \in]0,1[$. Then, according to (3.2), u satisfies the inequality $|u(a)| \le c$. By (3.7) we obtain that the inequality (4.49) is fulfilled with $q^*(t) = q(t, \|u\|_C)$ for $t \in [a,b]$. Hence, by the condition $H \in \mathcal{V}$ and the definition of the number ρ we get the estimate (4.2).

Since ρ depends neither on u nor on δ, it follows from Lemma 4.1 that the problem (2.1), (2.8) has at least one solution. □

Let $h_i \in L([a,b];R_+)$ $(i = 1,2,3,4)$, $\overline{\alpha}, \overline{\beta} \in [0,1[$. For arbitrarily fixed $t \in [a,b]$ consider the systems of inequalities

$$\begin{aligned} m &\le m\int_a^t h_1(s)\,ds + \mu(m,\overline{\alpha})M^{1-\overline{\alpha}}\int_a^t h_2(s)\,ds, \\ M &\le M\int_t^b h_3(s)\,ds + \mu(M,\overline{\beta})m^{1-\overline{\beta}}\int_t^b h_4(s)\,ds \end{aligned} \tag{4.51$_t$}$$

and

$$\begin{aligned} M &\le M\int_a^t h_3(s)\,ds + \mu(M,\overline{\beta})m^{1-\overline{\beta}}\int_a^t h_4(s)\,ds, \\ m &\le m\int_t^b h_1(s)\,ds + \mu(m,\overline{\alpha})M^{1-\overline{\alpha}}\int_t^b h_2(s)\,ds, \end{aligned} \tag{4.52$_t$}$$

where $\mu : R_+ \times [0,1[\to R_+$ is defined by (2.4).

By a solution of the system $((4.51)_t)_t$, respectively, $((4.52)_t)_t$, we will understand a pair $(M,m) \in R_+ \times R_+$ satisfying $((4.51)_t)_t$, respectively, $((4.52)_t)_t$.

Definition 4.9. Let $h_i \in L([a,b];R_+)$ $(i = 1,2,3,4)$ and $\overline{\alpha},\overline{\beta} \in [0,1[$. We will say that a 4-tuple (h_1,h_2,h_3,h_4) belongs to the set $\mathcal{B}_{ab}(\overline{\alpha},\overline{\beta})$, if for every $t \in [a,b]$ the systems $((4.51)_t)_t$ and $((4.52)_t)_t$ have only the trivial solution.

Lemma 4.10. *Let $H \in \mathcal{H}_{ab}^{\alpha\beta}(g_0,g_1,p_0,p_1)$. If*

$$(g_0,g_1,p_0,p_1) \in \mathcal{B}_{ab}(\alpha,\beta), \tag{4.53}$$

then $H \in \mathcal{V}$.

Proof. Assume on the contrary that for every $n \in N$ there exist $q_n^* \in L([a,b];R_+)$, $c_n \in R_+$, $\delta_n \in]0,1]$, and $u_n \in \widetilde{C}([a,b];R)$ such that the inequalities (4.9),

$$[u_n'(t) - \delta_n H(u_n)(t)]\operatorname{sgn} u_n(t) \le q_n^*(t) \quad \text{for } t \in [a,b] \tag{4.54}$$

and (4.11) are fulfilled. Define the functions v_n by (4.12). Obviously, the equalities (4.13) and (4.14) are satisfied, where q_n are defined by (4.15). By virtue of (4.9) and (4.12) we have the inequality (4.16). Furthermore, on account of (4.12), (4.15), and (4.54), we have

$$q_n(t)\operatorname{sgn} v_n(t) \le \frac{q_n^*(t)}{\|u_n\|_C} \quad \text{for } t \in [a,b],\ n \in N. \tag{4.55}$$

For $n \in N$ define numbers M_n and m_n by (4.18). Evidently, $M_n \ge 0$, $m_n \ge 0$ for $n \in N$, and on account of (4.13), the inequality (4.19) holds.

According to (4.18) and (4.19), for every $n \in N$ the points σ_n, s_n, ξ_n, $t_n \in [a,b]$ can be chosen in the following way:

(i) if $M_n = 0$, then let $\xi_n = a$, $t_n = a$, $s_n \in [a,b]$ be such that (4.20) is fulfilled, and let

$$\sigma_n = \begin{cases} a & \text{if } s_n = a \\ \inf\{t \in [a,s_n[\, : v_n(s) < 0 \text{ for } s \in]t,s_n]\} & \text{if } s_n \ne a, \end{cases} \tag{4.56}$$

(ii) if $m_n = 0$, then let $\sigma_n = a$, $s_n = a$, $t_n \in [a,b]$ be such that (4.21) is fulfilled, and let

$$\xi_n = \begin{cases} a & \text{if } t_n = a \\ \inf\{t \in [a,t_n[\, : v_n(s) > 0 \text{ for } s \in]t,t_n]\} & \text{if } t_n \ne a, \end{cases} \tag{4.57}$$

(iii) if $M_n > 0$ and $m_n > 0$, then let s_n, $t_n \in [a,b]$ be such that (4.20) and (4.21) are fulfilled, and let σ_n and ξ_n be defined by (4.56) and (4.57), respectively.

Note that for every $n \in N$ the following holds:

$$\begin{aligned} &\text{if } \sigma_n \ne s_n, \quad \text{then } v_n(s) < 0 \quad \text{for } s \in]\sigma_n,s_n], \\ &\text{if } \xi_n \ne t_n, \quad \text{then } v_n(s) > 0 \quad \text{for } s \in]\xi_n,t_n]. \end{aligned} \tag{4.58}$$

Furthermore, with respect to (4.16), we get

$$|v_n(\sigma_n)| \le \frac{c_n}{\|u_n\|_C}, \qquad |v_n(\xi_n)| \le \frac{c_n}{\|u_n\|_C} \quad \text{for } n \in N. \tag{4.59}$$

By virtue of (4.13) and (4.18) we have that the sequences $\{M_n\}_{n=1}^{+\infty}$ and $\{m_n\}_{n=1}^{+\infty}$ are bounded. Obviously, also the sequences $\{s_n\}_{n=1}^{+\infty}$ and $\{t_n\}_{n=1}^{+\infty}$ are bounded, and, moreover, for every $n \in N$ we have either

$$a \le \sigma_n \le s_n \le \xi_n \le t_n \le b, \tag{4.60}$$

or

$$a \le \xi_n \le t_n \le \sigma_n \le s_n \le b. \tag{4.61}$$

Therefore, without loss of generality we can assume that there exist M_0, $m_0 \in R_+$ and s_0, $t_0 \in [a,b]$ such that (4.24) is fulfilled, and either

$$a \le \sigma_n \le s_n \le \xi_n \le t_n \le b \quad \text{for } n \in N, \tag{4.62}$$

or

$$a \le \xi_n \le t_n \le \sigma_n \le s_n \le b \quad \text{for } n \in N. \tag{4.63}$$

Furthermore, on account of (4.19) we have (4.27).

The integration of (4.14) from σ_n to s_n and from ξ_n to t_n, respectively, by virtue of (4.58), for every $n \in N$ yields

$$\begin{aligned} v_n(s_n) &= v_n(\sigma_n) + \frac{\delta_n}{\|u_n\|_C}\int_{\sigma_n}^{s_n} H(u_n)(\xi)d\xi - \int_{\sigma_n}^{s_n} q_n(\xi)\operatorname{sgn} v_n(\xi)d\xi, \\ v_n(t_n) &= v_n(\xi_n) + \frac{\delta_n}{\|u_n\|_C}\int_{\xi_n}^{t_n} H(u_n)(\xi)d\xi + \int_{\xi_n}^{t_n} q_n(\xi)\operatorname{sgn} v_n(\xi)d\xi. \end{aligned} \tag{4.64}$$

From (4.64), in view of (4.11), (4.12), (4.18), (4.55), (4.59), the assumptions $\delta_n \in]0,1]$ and $H \in \mathcal{K}_{ab}^{\alpha\beta}(g_0,g_1,p_0,p_1)$, and the choice of points σ_n, s_n, ξ_n, and t_n, for every $n \in N$ we get

$$\begin{aligned} m_n &\le m_n \int_{\sigma_n}^{s_n} g_0(\xi)d\xi + \mu(m_n,\alpha)M_n^{1-\alpha}\int_{\sigma_n}^{s_n} g_1(\xi)d\xi + \frac{1}{n}, \\ M_n &\le M_n \int_{\xi_n}^{t_n} p_0(\xi)d\xi + \mu(M_n,\beta)m_n^{1-\beta}\int_{\xi_n}^{t_n} p_1(\xi)d\xi + \frac{1}{n}. \end{aligned} \tag{4.65}$$

Then, due to (2.4) and (4.24), from (4.65) as $n \to +\infty$ we obtain

$$\begin{aligned} m_0 &\le m_0 \int_a^{s_0} g_0(\xi)d\xi + \mu(m_0,\alpha)M_0^{1-\alpha}\int_a^{s_0} g_1(\xi)d\xi, \\ M_0 &\le M_0 \int_{s_0}^{b} p_0(\xi)d\xi + \mu(M_0,\beta)m_0^{1-\beta}\int_{s_0}^{b} p_1(\xi)d\xi \end{aligned} \tag{4.66}$$

if (4.62) holds, and

$$\begin{aligned} M_0 &\le M_0 \int_a^{t_0} p_0(\xi)d\xi + \mu(M_0,\beta) m_0^{1-\beta} \int_a^{t_0} p_1(\xi)d\xi, \\ m_0 &\le m_0 \int_{t_0}^b g_0(\xi)d\xi + \mu(m_0,\alpha) M_0^{1-\alpha} \int_{t_0}^b g_1(\xi)d\xi \end{aligned} \tag{4.67}$$

if (4.63) is true. Consequently, the pair (M_0, m_0) is a solution of the system $((4.51)_t)_{s_0}$, respectively, $((4.52)_t)_{t_0}$, with $h_1 \equiv g_0$, $h_2 \equiv g_1$, $h_3 \equiv p_0$, $h_4 \equiv p_1$, $\overline{\alpha} = \alpha$, and $\overline{\beta} = \beta$. However, the inequality (4.27) contradicts the inclusion (4.53). □

Lemma 4.11. *Let the inequalities (3.4), (3.8), and (3.9) be fulfilled. Then the inclusion (4.53) holds.*

Proof. Assume on the contrary that there exists $t_0 \in [a,b]$ such that either the system $((4.51)_t)_{t_0}$ or the system $((4.52)_t)_{t_0}$ with $h_1 \equiv g_0$, $h_2 \equiv g_1$, $h_3 \equiv p_0$, $h_4 \equiv p_1$, $\overline{\alpha} = \alpha$, and $\overline{\beta} = \beta$ has a nontrivial solution.

First suppose that (M_0, m_0) is a nontrivial solution of $((4.51)_t)_{t_0}$. Define functions G_0, G_1, P_0, and P_1 by (4.31). Then, according to the assumptions, (M_0, m_0) satisfies

$$m_0 \le m_0 G_0(t_0) + \mu(m_0,\alpha) M_0^{1-\alpha} G_1(t_0), \tag{4.68}$$

$$M_0 \le M_0 P_0(t_0) + \mu(M_0,\beta) m_0^{1-\beta} P_1(t_0). \tag{4.69}$$

If $M_0 = 0$, then $m_0 > 0$, and from (4.68), in view of (3.4) and (4.31), we get a contradiction $m_0 < m_0$. If $m_0 = 0$, then $M_0 > 0$, and from (4.69), in view of (3.4) and (4.31), we get a contradiction $M_0 < M_0$. Therefore assume that (4.34) holds. In this case, according to (2.4), we have (4.35). Thus, on account of (3.4), (4.31), and (4.34), from (4.68) and (4.69) we obtain

$$\begin{aligned} 0 &< m_0^{1-\alpha}(1 - G_0(t_0)) \le M_0^{1-\alpha} G_1(t_0), \\ 0 &< M_0^{1-\beta}(1 - P_0(t_0)) \le m_0^{1-\beta} P_1(t_0). \end{aligned} \tag{4.70}$$

Now the inequalities (4.70) result in

$$\begin{aligned} &M_0^{1-\beta}(1 - P_0(t_0))(1 - G_0(t_0))^{(1-\beta)/(1-\alpha)} \\ &\quad \le (m_0^{1-\alpha}(1 - G_0(t_0)))^{(1-\beta)/(1-\alpha)} P_1(t_0) \le M_0^{1-\beta} G_1(t_0)^{(1-\beta)/(1-\alpha)} P_1(t_0). \end{aligned} \tag{4.71}$$

However, on account of (4.31) and (4.34), the last inequality contradicts (3.8).

In analogous way it can be shown that assuming (M_0, m_0) to be a nontrivial solution of $((4.52)_t)_{t_0}$ we obtain a contradiction to (3.9). □

5. Proofs

Theorem 3.1 follows from Lemmas 4.3, 4.5, and 4.6. Theorem 3.2 follows from Lemmas 4.8, 4.10, and 4.11.

Proof of Theorem 3.3. From the conditions (3.11) and (3.12) it follows that the conditions (3.2) and (3.3) are satisfied with $c = |h(0)|$ and $q \equiv |q^*|$, and so the assumptions of Theorem 3.1 are fulfilled. Therefore, the problem (2.1), (2.8) has at least one solution. It remains to show that the problem (2.1), (2.8) has no more than one solution.

Let $u, v \in \widetilde{C}([a,b];R)$ be solutions of (2.1), (2.8). Then, in view of (2.8) and (3.11), we have

$$|u(a)| = h(u)\operatorname{sgn} u(a) \leq |h(0)|, \qquad |v(a)| = h(v)\operatorname{sgn} v(a) \leq |h(0)|. \tag{5.1}$$

Put

$$w(t) = u(t) - v(t) \quad \text{for } t \in [a,b]. \tag{5.2}$$

Then, in view of (2.8), (3.11), and (5.2), we obtain

$$|w(a)| = |u(a) - v(a)| = [h(u) - h(v)]\operatorname{sgn}(u(a) - v(a)) \leq 0 \tag{5.3}$$

and, with respect to (3.10), (3.12), and (5.1)–(5.3), w is a solution of the problem

$$w'(t) = H_v(w)(t), \quad w(a) = 0. \tag{5.4}$$

Moreover, on account of the inequalities (3.4)–(3.6), and Lemma 4.6, we have the inclusion (4.8). Therefore, according to the assumption $H_v \in \mathscr{H}_{ab}^{\alpha\beta}(g_0, g_1, p_0, p_1)$, Lemma 4.5, and Definition 4.2, $w \equiv 0$, that is, $u \equiv v$. □

Proof of Theorem 3.4. From the conditions (3.11) and (3.13) it follows that the conditions (3.2) and (3.7) are satisfied with $c = |h(0)|$ and $q \equiv |Q(0)|$. Consequently, the assumptions of Theorem 3.2 are fulfilled. Therefore, the problem (2.1), (2.8) has at least one solution. It remains to show that the problem (2.1), (2.8) has no more than one solution.

Let $u, v \in \widetilde{C}([a,b];R)$ be solutions of (2.1), (2.8). Then, in view of (2.8) and (3.11), the inequalities (5.1) are fulfilled. Define w by (5.2). Then, on account of (2.8), (3.11), and (5.2), (5.3) holds, and, according to (3.10) and (5.1)–(5.3), w is a solution of the problem

$$w'(t) = H_v(w)(t) + Q_v(w)(t), \quad w(a) = 0, \tag{5.5}$$

where

$$Q_v(w)(t) = Q(w+v)(t) - Q(v)(t) \quad \text{for } t \in [a,b]. \tag{5.6}$$

Furthermore, by virtue of (3.13), (5.1), (5.2), and (5.6),

$$Q_v(w)(t)\operatorname{sgn} w(t) = [Q(u)(t) - Q(v)(t)]\operatorname{sgn}(u(t) - v(t)) \leq 0 \quad \text{for } t \in [a,b], \tag{5.7}$$

and, with respect to the inequalities (3.4), (3.8), (3.9), and Lemma 4.11, we have the inclusion (4.53). Therefore, according to the assumption $H_v \in \mathscr{H}_{ab}^{\alpha\beta}(g_0, g_1, p_0, p_1)$, Lemma 4.10, and Definition 4.7, $w \equiv 0$, that is, $u \equiv v$. □

Proof of Corollary 3.6. To prove the corollary it is sufficient to show that the assumptions of Theorem 3.1 are fulfilled.

Define operators H and Q by the equalities

$$H(v)(t) \stackrel{\text{def}}{=} p(t)\max\{v(s) : \tau_1(t) \le s \le \tau_2(t)\} \quad \text{for } t \in [a,b], \tag{5.8}$$

$$Q(v)(t) \stackrel{\text{def}}{=} f(t, v(t), \max\{v(s) : v_1(t) \le s \le v_2(t)\}) \quad \text{for } t \in [a,b], \tag{5.9}$$

and put $\alpha = 0$, $\beta = 0$,

$$g_0 \equiv [p]_+, \qquad p_0 \equiv [p]_+, \qquad g_1 \equiv [p]_-, \qquad p_1 \equiv [p]_-. \tag{5.10}$$

Then $H \in \mathcal{H}_{ab}^{\alpha\beta}(g_0, g_1, p_0, p_1)$, the condition (3.14) yields (3.3), and (3.15) implies (3.4). It remains to verify that for $t \in [a,b]$ the inequalities (3.5) and (3.6) hold.

According to (5.10) and since $\alpha = 0$ and $\beta = 0$, the inequalities (3.5) and (3.6) are equivalent. Assume on the contrary that there exists $t_0 \in [a,b]$ such that

$$\int_a^{t_0} [p(s)]_- ds \left(\int_{t_0}^b [p(s)]_- ds - 1 \right) \ge \left(1 - \int_a^{t_0} [p(s)]_+ ds \right) \left(1 - \int_{t_0}^b [p(s)]_+ ds \right). \tag{5.11}$$

Now, since

$$AB \le \frac{1}{4}(A+B)^2, \tag{5.12}$$

we get

$$\int_a^{t_0} [p(s)]_- ds \left(\int_{t_0}^b [p(s)]_- ds - 1 \right) \le \frac{1}{4} \left(\int_a^b [p(s)]_- ds - 1 \right)^2, \tag{5.13}$$

and, according to (3.15),

$$\left(1 - \int_a^{t_0} [p(s)]_+ ds \right) \left(1 - \int_{t_0}^b [p(s)]_+ ds \right) \ge 1 - \int_a^b [p(s)]_+ ds > 0. \tag{5.14}$$

Thus, in view of (5.13) and (5.14), (5.11) yields $\int_a^b [p(s)]_- ds > 1$ and

$$0 < 1 - \int_a^b [p(s)]_+ ds \le \frac{1}{4} \left(\int_a^b [p(s)]_- ds - 1 \right)^2, \tag{5.15}$$

which contradicts (3.16). □

Proof of Corollary 3.7. To prove the corollary it is sufficient to show that the assumptions of Theorem 3.3 are fulfilled.

Define operator H by (5.8) and functions g_0, g_1, p_0, p_1 by (5.10). Put $\alpha = 0$, $\beta = 0$, and

$$Q(v)(t) \stackrel{\text{def}}{=} q_0(t) \quad \text{for } t \in [a,b],\ v \in C([a,b];R). \tag{5.16}$$

Then obviously, the condition (3.12) with $q^* \equiv q_0$ is fulfilled, and (3.15) implies (3.4). Moreover, by the same arguments as in the proof of Corollary 3.6 one can show that, on account of (3.16), for $t \in [a,b]$ the inequalities (3.5) and (3.6) are satisfied.

It remains to show that for every $z \in \{v \in C([a,b];R) : v(a) = h(v)\}$ the operator H_z defined by (3.10) belongs to the set $\mathcal{H}_{ab}^{\alpha\beta}(g_0,g_1,p_0,p_1)$. Denote for almost all $t \in [a,b]$ by $I(t)$ the segment $[\tau_1(t),\tau_2(t)]$. Then obviously,

$$\begin{aligned} H_z(u)(t) &= [p(t)]_+ \big(\max\{u(s)+z(s) : s \in I(t)\} - \max\{z(s) : s \in I(t)\}\big) \\ &\quad - [p(t)]_- \big(\max\{u(s)+z(s) : s \in I(t)\} - \max\{z(s) : s \in I(t)\}\big) \\ &\le [p(t)]_+ \max\{u(s) : s \in I(t)\} \\ &\quad - [p(t)]_- \big(\min\{-z(s) : s \in I(t)\} - \min\{-u(s)-z(s) : s \in I(t)\}\big) \\ &\le [p(t)]_+ \max\{u(s) : s \in I(t)\} - [p(t)]_- \min\{u(s) : s \in I(t)\} \\ &\le M[p(t)]_+ + m[p(t)]_- \quad \text{for } t \in [a,b],\ u \in C([a,b];R), \end{aligned} \tag{5.17}$$

where

$$M = \max\{[u(t)]_+ : t \in [a,b]\}, \qquad m = \max\{[u(t)]_- : t \in [a,b]\}. \tag{5.18}$$

Analogously we can show that

$$H_z(u)(t) \ge -m[p(t)]_+ - M[p(t)]_- \quad \text{for } t \in [a,b],\ u \in C([a,b];R). \tag{5.19}$$

Consequently, $H_z \in \mathcal{H}_{ab}^{\alpha\beta}(g_0,g_1,p_0,p_1)$. □

Proof of Corollary 3.9. To prove the corollary it is sufficient to show that the assumptions of Theorem 3.2 are fulfilled.

Define operators H and Q by the equalities (5.8) and (5.9), respectively, and functions g_0,g_1,p_0,p_1 by (5.10). Put $\alpha = 0$ and $\beta = 0$. Then $H \in \mathcal{H}_{ab}^{\alpha\beta}(g_0,g_1,p_0,p_1)$, the condition (3.17) implies the condition (3.7), and (3.15) yields (3.4). Furthermore, according to (5.12), we have

$$\int_a^t [p(s)]_- ds \int_t^b [p(s)]_- ds \le \frac{1}{4}\left(\int_a^b [p(s)]_- ds\right)^2 \quad \text{for } t \in [a,b], \tag{5.20}$$

and, with respect to (3.15), for $t \in [a,b]$

$$\left(1 - \int_a^t [p(s)]_+ ds\right)\left(1 - \int_t^b [p(s)]_+ ds\right) \ge 1 - \int_a^b [p(s)]_+ ds > 0. \tag{5.21}$$

Thus, by virtue of (3.18), the inequalities (5.20) and (5.21) imply that the inequalities (3.8) and (3.9) are fulfilled. □

Proof of Corollary 3.10. To prove the corollary it is sufficient to show that the assumptions of Theorem 3.4 are fulfilled.

Define operators H and Q by the equalities (5.8) and (5.9), respectively, and functions g_0,g_1,p_0,p_1 by (5.10). Put $\alpha = 0$ and $\beta = 0$. Then $H \in \mathcal{H}_{ab}^{\alpha\beta}(g_0,g_1,p_0,p_1)$ and the inequalities (3.15) and (3.19) yield (3.4) and (3.13). Moreover, by the same arguments as in the proof of Corollary 3.9 one can show that, on account of (3.15) and (3.18), the inequalities (3.8) and (3.9) hold. Furthermore, in a similar manner as in the proof of Corollary 3.7 it can be shown that for every $z \in \{v \in C([a,b];R) : v(a) = h(v)\}$ the operator H_z defined by (3.10) belongs to the set $\mathcal{H}^{\alpha\beta}_{ab}(g_0, g_1, p_0, p_1)$. □

6. Examples

Remark 6.1. In Example 6.2, assuming the first inequality in (3.4) is not satisfied, there is an operator $H \in \mathcal{K}_{ab}$ constructed in such a way that $H \in \mathcal{H}^{\alpha\beta}_{ab}(g_0, g_1, p_0, p_1)$, but the problem

$$u'(t) = H(u)(t) + \omega(t), \quad u(a) = 0, \tag{6.1}$$

for a suitable $\omega \in L([a,b];R)$, has no solution. Furthermore, in Example 6.3 there is an operator $H \in \mathcal{K}_{ab}$ given such that $H \in \mathcal{H}^{\alpha\beta}_{ab}(g_0, g_1, p_0, p_1)$, the condition (3.4) is fulfilled, and the problem (6.1), with a suitable $\omega \in L([a,b];R)$, has no solution, assuming the inequality (3.5) is violated for some $t \in [a,b]$.

Examples verifying the optimality of the second inequality in (3.4) and the inequality (3.6) can be constructed analogously to Examples 6.2 and 6.3, respectively.

Example 6.2. Let $\alpha, \beta \in [0,1[$, $g_0, g_1, p_0, p_1 \in L([a,b];R_+)$, and let g_0 be such that

$$\int_a^b g_0(s)\,ds \geq 1. \tag{6.2}$$

Choose $t_0 \in]a,b]$ and $\omega \in L([a,b];R)$ such that

$$\int_a^{t_0} g_0(s)\,ds = 1, \qquad \int_a^{t_0} \omega(s)\,ds < 0, \tag{6.3}$$

and for $v \in C([a,b];R)$ put

$$\begin{aligned} H(v)(t) = {} & -g_0(t)[v(t_0)]_- - g_1(t)\mu([v(t)]_-, \alpha)[v(a)]_+^{1-\alpha} \\ & + p_0(t)[v(a)]_+ + p_1(t)\mu([v(t)]_+, \beta)[v(a)]_-^{1-\beta} \quad \text{for } t \in [a,b]. \end{aligned} \tag{6.4}$$

Then, obviously, $H \in \mathcal{H}^{\alpha\beta}_{ab}(g_0, g_1, p_0, p_1)$.

Now we will show that the problem (6.1) has no solution. Suppose on the contrary that there exists a solution u of (6.1). Then the integration of (6.1) from a to t_0, on account of (6.3), yields

$$u(t_0) = -[u(t_0)]_- \int_a^{t_0} g_0(s)\,ds + \int_a^{t_0} \omega(s)\,ds < 0. \tag{6.5}$$

However, the last equality, with respect to (6.3), results in

$$0 = u(t_0)\left(1 - \int_a^{t_0} g_0(s)\,ds\right) = \int_a^{t_0} \omega(s)\,ds < 0, \tag{6.6}$$

a contradiction.

Example 6.3. Let $\alpha,\beta \in [0,1[$, and let $g_0, g_1, p_0, p_1 \in L([a,b];R_+)$ be such that the condition (3.4) is fulfilled, while the inequality (3.5) is violated for some $t \in [a,b]$. Define functions G_0, G_1, P_0, and P_1 by (4.31). Then, since $G_1(a) = 0$ and $P_1(b) = 0$, we have

$$\begin{aligned} &\left(\frac{G_1(a)}{1-G_0(a)}\right)^{(1-\beta)/(1-\alpha)} P_1(a) - \left(\frac{G_1(a)}{1-G_0(a)}\right)^{1/(1-\alpha)} < 1 - P_0(a), \\ &\left(\frac{G_1(b)}{1-G_0(b)}\right)^{(1-\beta)/(1-\alpha)} P_1(b) - \left(\frac{G_1(b)}{1-G_0(b)}\right)^{1/(1-\alpha)} < 1 - P_0(b). \end{aligned} \tag{6.7}$$

Consequently, since we assume that (3.5) is violated for some $t \in [a,b]$, there exists $t_0 \in]a,b[$ such that

$$\left(\frac{G_1(t_0)}{1-G_0(t_0)}\right)^{(1-\beta)/(1-\alpha)} P_1(t_0) - \left(\frac{G_1(t_0)}{1-G_0(t_0)}\right)^{1/(1-\alpha)} = 1 - P_0(t_0). \tag{6.8}$$

Define

$$H(v)(t) \overset{\text{def}}{=} \begin{cases} -g_0(t)[v(t_0)]_- - g_1(t)\mu([v(t_0)]_-,\alpha)[v(b)]_+^{1-\alpha} & \\ +p_0(t)[v(a)]_+ + p_1(t)\mu([v(t)]_+,\beta)[v(a)]_-^{1-\beta} & \text{for } t \in [a,t_0[, \\ -g_0(t)[v(a)]_- - g_1(t)\mu([v(t)]_-,\alpha)[v(a)]_+^{1-\alpha} & \\ +p_0(t)[v(b)]_+ + p_1(t)\mu([v(b)]_+,\beta)[v(t_0)]_-^{1-\beta} & \text{for } t \in [t_0,b]. \end{cases} \tag{6.9}$$

Then, obviously, $H \in \mathcal{H}_{ab}^{\alpha\beta}(g_0, g_1, p_0, p_1)$.

Furthermore, with respect to (3.4), (4.31), and (6.8), we have

$$G_1(t_0) \neq 0. \tag{6.10}$$

Put

$$f(z) = z\left(1 + (c_0 - 1)\left(\frac{z-1}{z}\right)^{1/(1-\alpha)} - c_0\left(\frac{z-1}{z}\right)^{\beta/(1-\alpha)}\right) \quad \text{for } z \in]1,+\infty[, \tag{6.11}$$

where

$$c_0 = P_1(t_0)\left(\frac{1-G_0(t_0)}{G_1(t_0)}\right)^{\beta/(1-\alpha)}. \tag{6.12}$$

It can be easily verified that

$$\gamma \overset{\text{def}}{=} \sup\{f(z) : z \in]1,+\infty[\} < +\infty. \tag{6.13}$$

Choose $\omega \in L([a,b];R)$ such that

$$\int_a^{t_0} \omega(s)\,ds = -(1 - G_0(t_0)), \qquad \int_{t_0}^b \omega(s)\,ds > \max\{\gamma, 1\}. \tag{6.14}$$

We will show that the problem (6.1) has no solution. Suppose on the contrary that there exists a solution u of (6.1). Then the integration of (6.1) from a to t_0 and from t_0 to b, respectively, on account of (4.31), (6.9), and (6.14), yields

$$u(t_0) = -G_0(t_0)[u(t_0)]_- - G_1(t_0)\mu([u(t_0)]_-,\alpha)[u(b)]_+^{1-\alpha} - (1-G_0(t_0)), \tag{6.15}$$

$$u(b) = u(t_0) + P_0(t_0)[u(b)]_+ + P_1(t_0)\mu([u(b)]_+,\beta)[u(t_0)]_-^{1-\beta} + \int_{t_0}^{b} \omega(s)ds. \tag{6.16}$$

Hence $u(t_0) < 0$. Assuming $u(b) \le 0$, according to (6.13) and (6.14), from (6.15) and (6.16) we obtain $u(t_0) = -1$ and

$$u(b) \ge \int_{t_0}^{b} \omega(s)ds - 1 > 0, \tag{6.17}$$

a contradiction. Therefore $u(b) > 0$. For short put

$$x = [u(t_0)]_-, \qquad y = [u(b)]_+. \tag{6.18}$$

According to above-mentioned we have $x > 0$, $y > 0$, and the equalities (6.15) and (6.16) can be rewritten as follows

$$x(1-G_0(t_0)) = G_1(t_0)x^{\alpha}y^{1-\alpha} + 1 - G_0(t_0), \tag{6.19}$$

$$y(1-P_0(t_0)) = P_1(t_0)y^{\beta}x^{1-\beta} - x + \int_{t_0}^{b} \omega(s)ds. \tag{6.20}$$

From (6.19), in view of (6.10), we get $x > 1$ and

$$y = x\left(\frac{x-1}{x}\right)^{1/(1-\alpha)}\left(\frac{1-G_0(t_0)}{G_1(t_0)}\right)^{1/(1-\alpha)}. \tag{6.21}$$

Using the last equality in (6.20) we obtain

$$\begin{aligned} x\left(\frac{x-1}{x}\right)^{1/(1-\alpha)}&\left(\frac{1-G_0(t_0)}{G_1(t_0)}\right)^{1/(1-\alpha)}(1-P_0(t_0)) \\ &+ x - x\left(\frac{x-1}{x}\right)^{\beta/(1-\alpha)}\left(\frac{1-G_0(t_0)}{G_1(t_0)}\right)^{\beta/(1-\alpha)}P_1(t_0) = \int_{t_0}^{b} \omega(s)ds, \end{aligned} \tag{6.22}$$

whence, in view of (6.8), the fact that $x > 1$, and the definition of the function f, we get

$$f(x) = \int_{t_0}^{b} \omega(s)ds, \tag{6.23}$$

which, on account of (6.13), contradicts (6.14).

Remark 6.4. The case when the first inequality in (3.4) is not satisfied is discussed in Example 6.2. In Example 6.5 below, there are given operators $H, Q \in \mathcal{K}_{ab}$ such that $H \in \mathcal{H}_{ab}^{\alpha\beta}(g_0, g_1, p_0, p_1)$, Q satisfies the inequalities (3.7) and (3.13) for $v \in C([a,b];R)$, the condition (3.4) is fulfilled, and the problem

$$u'(t) = H(u)(t) + Q(u)(t), \quad u(a) = 0 \tag{6.24}$$

has no solution, assuming the inequality (3.8) is violated.

Examples verifying the optimality of the second inequality in (3.4), respectively, the inequality (3.9), can be constructed analogously to Examples 6.2 and 6.5, respectively.

Example 6.5. Let $\alpha, \beta \in [0,1[$, and let $g_0, g_1, p_0, p_1 \in L([a,b];R_+)$ be such that the condition (3.4) is fulfilled, while the inequality (3.8) is violated. Define functions G_0, G_1, P_0, and P_1 by (4.31). Then, since $G_1(a) = 0$ and $P_1(b) = 0$, we have

$$\left(\frac{G_1(a)}{1 - G_0(a)}\right)^{(1-\beta)/(1-\alpha)} P_1(a) < 1 - P_0(a), \qquad \left(\frac{G_1(b)}{1 - G_0(b)}\right)^{(1-\beta)/(1-\alpha)} P_1(b) < 1 - P_0(b). \tag{6.25}$$

Consequently, there exists $t_0 \in]a,b[$ such that

$$\left(\frac{G_1(t_0)}{1 - G_0(t_0)}\right)^{(1-\beta)/(1-\alpha)} P_1(t_0) = 1 - P_0(t_0). \tag{6.26}$$

Hence, in view of (3.4), we have (6.10). Choose $t_1 \in]t_0, b[$. Define an operator $\varphi : L([a,b];R) \to L([a,b];R)$ by

$$\varphi(p)(t) \overset{\text{def}}{=} \begin{cases} p(t) & \text{for } t \in [a, t_0[, \\ 0 & \text{for } t \in [t_0, t_1], \\ p\left(\dfrac{b - t_0}{b - t_1}(t - t_1) + t_0\right) & \text{for } t \in]t_1, b]. \end{cases} \tag{6.27}$$

Then

$$\begin{gathered} \int_a^b \varphi(g_0)(s)\,ds < 1, \qquad \int_a^b \varphi(p_0)(s)\,ds < 1, \\ \left(\frac{\int_a^{t_0} \varphi(g_1)(s)\,ds}{1 - \int_a^{t_0} \varphi(g_0)(s)\,ds}\right)^{(1-\beta)/(1-\alpha)} \int_{t_0}^b \varphi(p_1)(s)\,ds = 1 - \int_{t_0}^b \varphi(p_0)(s)\,ds, \\ \varphi(p_0)(t) = 0, \qquad \varphi(p_1)(t) = 0 \quad \text{for } t \in [t_0, t_1]. \end{gathered} \tag{6.28}$$

Therefore, without loss of generality, we can assume that

$$p_0(t) = 0, \quad p_1(t) = 0 \quad \text{for } t \in [t_0, t_1]. \tag{6.29}$$

Define operators H and Q by (6.9) and

$$Q(v)(t) \stackrel{\text{def}}{=} \begin{cases} \omega_1(t) & \text{for } t \in [a, t_0[, \\ -v^3(t) & \text{for } t \in [t_0, t_1], \\ \omega_2(t) & \text{for } t \in]t_1, b], \end{cases} \tag{6.30}$$

where ω_1, $\omega_2 \in L([a,b];R)$ are such that

$$\int_a^{t_0} \omega_1(s)ds = -(1 - G_0(t_0)), \qquad \int_{t_1}^{b} \omega_2(s)ds = \frac{1}{\sqrt{2(t_1 - t_0)}}. \tag{6.31}$$

Obviously, $H \in \mathcal{H}_{ab}^{\alpha\beta}(g_0, g_1, p_0, p_1)$ and Q satisfies (3.7) with

$$q(t, \|v\|_C) = |\omega_1(t)| + |\omega_2(t)| \quad \text{for } t \in [a,b], \tag{6.32}$$

and (3.13), as well.

We will show that the problem (6.24) has no solution. Suppose on the contrary that there exists a solution u of (6.24). Then the integration of (6.24) from a to t_0, in view of (4.31) and (6.30), yileds (6.15), whence we get $u(t_0) < 0$. Further, on account of (6.9), (6.29), and (6.30), we have

$$u(t) = \frac{u(t_0)}{\sqrt{1 + 2u^2(t_0)(t - t_0)}} \quad \text{for } t \in [t_0, t_1]. \tag{6.33}$$

Finally, the integration of (6.24) from t_1 to b, with respect to (4.31) and (6.30), results in

$$u(b) = u(t_1) + P_0(t_0)[u(b)]_+ + P_1(t_0)\mu([u(b)]_+, \beta)[u(t_0)]_-^{1-\beta} + \frac{1}{\sqrt{2(t_1 - t_0)}}. \tag{6.34}$$

From (6.34), according to $u(t_0) < 0$ and (6.33), we get

$$u(b) \geq \frac{u(t_0)}{\sqrt{1 + 2u^2(t_0)(t_1 - t_0)}} + \frac{1}{\sqrt{2(t_1 - t_0)}} > 0. \tag{6.35}$$

For short define numbers x and y by (6.18). According to above-mentioned we have $x > 0$, $y > 0$, and the equalities (6.15) and (6.34), using (6.33), can be rewritten as follows

$$x(1 - G_0(t_0)) = G_1(t_0)x^\alpha y^{1-\alpha} + 1 - G_0(t_0), \tag{6.36}$$

$$y(1 - P_0(t_0)) = -\frac{x}{\sqrt{1 + 2x^2(t_1 - t_0)}} + P_1(t_0)y^\beta x^{1-\beta} + \frac{1}{\sqrt{2(t_1 - t_0)}}. \tag{6.37}$$

From (6.36), in view of (6.10), we get $x > 1$ and (6.21). Using (6.21) in (6.37), by virtue of (6.26), we obtain

$$\begin{aligned} &x\left(\frac{x-1}{x}\right)^{1/(1-\alpha)}\left(\frac{1-G_0(t_0)}{G_1(t_0)}\right)^{\beta/(1-\alpha)} P_1(t_0)\left(1-\left(\frac{x}{x-1}\right)^{(1-\beta)/(1-\alpha)}\right) \\ &\quad = \frac{1}{\sqrt{2(t_1-t_0)}} - \frac{x}{\sqrt{1+2x^2(t_1-t_0)}}. \end{aligned} \tag{6.38}$$

However, since $x > 1$, we have

$$\begin{gathered} x\left(\frac{x-1}{x}\right)^{1/(1-\alpha)}\left(\frac{1-G_0(t_0)}{G_1(t_0)}\right)^{\beta/(1-\alpha)} P_1(t_0)\left(1-\left(\frac{x}{x-1}\right)^{(1-\beta)/(1-\alpha)}\right) < 0, \\ \frac{1}{\sqrt{2(t_1-t_0)}} - \frac{x}{\sqrt{1+2x^2(t_1-t_0)}} > 0. \end{gathered} \tag{6.39}$$

The last two inequalities contradict (6.38).

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