# PERIODIC SOLUTIONS OF SECOND-ORDER NONAUTONOMOUS DYNAMICAL SYSTEMS

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We study the existence of periodic solutions for second-order nonautonomous dynamical systems. We give four sets of hypotheses which guarantee the existence of solutions. We were able to weaken the hypotheses considerably from those used previously for such systems. We employ a new saddle point theorem using linking methods.



## 1. Introduction

We consider the following problem. One wishes to solve

$$-x''(t) = \nabla_x V(t,x(t)), \tag{1.1}$$

where

$$x(t) = (x_1(t),\dots,x_n(t)) \tag{1.2}$$

is a map from $I = [0,T]$ to $\mathbb{R}^n$ such that each component $x_j(t)$ is a periodic function in $H^1$ with period $T$, and the function $V(t,x) = V(t,x_1,\dots,x_n)$ is continuous from $\mathbb{R}^{n+1}$ to $\mathbb{R}$ with

$$\nabla_x V(t,x) = \left(\frac{\partial V}{\partial x_1},\dots,\frac{\partial V}{\partial x_n}\right) \in C(\mathbb{R}^{n+1},\mathbb{R}^n). \tag{1.3}$$

Here $H^1$ represents the Hilbert space of periodic functions in $L^2(I)$ with generalized derivatives in $L^2(I)$. The scalar product is given by

$$(u,v)_{H^1} = (u',v') + (u,v). \tag{1.4}$$

For each $x \in \mathbb{R}^n$, the function $V(t,x)$ is periodic in $t$ with period $T$.

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We will study this problem under the following assumptions:

(1)

$$V(t,x) \geq 0, \quad t \in I,\ x \in \mathbb{R}^n; \tag{1.5}$$

(2) there are constants $m > 0$, $\alpha \leq 6m^2/T^2$ such that

$$V(t,x) \leq \alpha, \quad |x| \leq m,\ t \in I,\ x \in \mathbb{R}^n; \tag{1.6}$$

(3) there is a constant $\mu > 2$ such that

$$\frac{H_\mu(t,x)}{|x|^2} \leq W(t) \in L^1(I), \quad |x| \geq C,\ t \in I,\ x \in \mathbb{R}^n, \tag{1.7}$$

$$\limsup_{|x|\to\infty} \frac{H_\mu(t,x)}{|x|^2} \leq 0, \tag{1.8}$$

where

$$H_\mu(t,x) = \mu V(t,x) - \nabla_x V(t,x) \cdot x; \tag{1.9}$$

(4) there is a subset $e \subset I$ of positive measure such that

$$\liminf_{|x|\to\infty} \frac{V(t,x)}{|x|^2} > 0, \quad t \in e. \tag{1.10}$$

We have the following theorem.

Theorem 1.1. *Under the above hypotheses, the system (1.1) has a solution.*

As a variant of Theorem 1.1, we have the following one.

Theorem 1.2. *The conclusion in Theorem 1.1 is the same if Hypothesis (2) is replaced by (2′) there is a constant $q > 2$ such that*

$$V(t,x) \leq C(|x|^q + 1), \quad t \in I,\ x \in \mathbb{R}^n, \tag{1.11}$$

*and there are constants $m > 0$, $\alpha < 2\pi^2/T^2$ such that*

$$V(t,x) \leq \alpha |x|^2, \quad |x| \leq m,\ t \in I,\ x \in \mathbb{R}^n. \tag{1.12}$$

We also have the following theorem.

Theorem 1.3. *The conclusions of Theorems 1.1 and 1.2 hold if Hypothesis (3) is replaced by (3′) there is a constant $\mu < 2$ such that*

$$\begin{gathered} \frac{H_\mu(t,x)}{|x|^2} \geq -W(t) \in L^1(I), \quad |x| \geq C,\ t \in I,\ x \in \mathbb{R}^n, \\ \liminf_{|x|\to\infty} \frac{H_\mu(t,x)}{|x|^2} \geq 0. \end{gathered} \tag{1.13}$$

And we have the following theorem.

Theorem 1.4. *The conclusion of Theorem 1.1 holds if Hypothesis (1) is replaced by*
*(1′)*

$$0 \leq V(t,x) \leq C\big(|x|^2 + 1\big), \quad t \in I,\ x \in \mathbb{R}^n \tag{1.14}$$

*and Hypothesis (3) by*
*(3′′) the function given by*

$$H(t,x) = 2V(t,x) - \nabla_x V(t,x) \cdot x \tag{1.15}$$

*satisfies*

$$\begin{gathered} H(t,x) \leq W(t) \in L^1(I), \quad |x| \geq C,\ t \in I,\ x \in \mathbb{R}^n, \\ H(t,x) \longrightarrow -\infty, \quad |x| \longrightarrow \infty,\ t \in I,\ x \in \mathbb{R}^n. \end{gathered} \tag{1.16}$$

The periodic nonautonomous problem

$$x''(t) = \nabla_x V\big(t,x(t)\big) \tag{1.17}$$

has an extensive history in the case of singular systems (cf., e.g., Ambrosetti-Coti Zelati [1]). The first to consider it for potentials satisfying (1.3) were Berger and Schechter [3]. We proved the existence of solutions to (1.17) under the condition that

$$V(t,x) \longrightarrow \infty \quad \text{as } |x| \longrightarrow \infty \tag{1.18}$$

uniformly for a.e. $t \in I$. Subsequently, Willem [16], Mawhin [6], Mawhin and Willem [8], Tang [11, 12], Tang and Wu [13–15], Wu and Tang [17] and others proved existence under various conditions (cf. the references given in these publications).

The periodic problem (1.1) was studied by Mawhin and Willem [7, 8], Long [5], Tang and Wu [13–15] and others (cf. the refernces quoted in them). Ben-Naoum et al. [2] and Nirenberg (cf. Ekeland and Ghoussoub [4]) proved the existence of nonconstant solutions.

We will prove Theorems 1.1–1.4 in the next section. We use a linking method of critical point theory (cf. [9, 10]). These methods allow us to improve the previous results.

## 2. Proofs of the theorems

We now give the proof of Theorem 1.1.

*Proof.* Let $X$ be the set of vector functions $x(t)$ given by (1.2) and described above. It is a Hilbert space with norm satisfying

$$\|x\|_X^2 = \sum_{j=1}^{n} \big\|x_j\big\|_{H^1}^2. \tag{2.1}$$

We also write

$$\|x\|^2 = \sum_{j=1}^{n} \|x_j\|^2, \tag{2.2}$$

where $\|\cdot\|$ is the $L^2(I)$ norm.

Let

$$N = \{x(t) \in X : x_j(t) \equiv \text{ constant}, \ 1 \leq j \leq n\} \tag{2.3}$$

and $M = N^{\perp}$. The dimension of $N$ is $n$, and $X = M \oplus N$. Proof of the following lemma can be found in [7].

Lemma 2.1. *If $x \in M$, then*

$$\|x\|_\infty^2 \leq \frac{T}{12}\|x'\|^2, \qquad \|x\| \leq \frac{T}{2\pi}\|x'\|. \tag{2.4}$$

We define

$$G(x) = \|x'\|^2 - 2\int_I V(t, x(t))\,dt, \quad x \in X. \tag{2.5}$$

For each $x \in X$ write $x = v + w$, where $v \in N$, $w \in M$. For convenience, we will use the following equivalent norm for $X$:

$$\|x\|_X^2 = \|w'\|^2 + \|v\|^2. \tag{2.6}$$

If $x \in M$ and

$$\|x'\|^2 = \rho^2 = \frac{12}{T}m^2, \tag{2.7}$$

then Lemma 2.1 implies that $\|x\|_\infty \leq m$, and we have by Hypothesis (2) that $V(t,x) \leq \alpha$. Hence,

$$G(x) \geq \|x'\|^2 - 2\int_{|x(t)|<m} \alpha\,dt \geq \rho^2 - 2\alpha T \geq 0. \tag{2.8}$$

We also note that Hypothesis (1) implies

$$G(v) \leq 0, \quad v \in N. \tag{2.9}$$

Take

$$A = \partial B_\rho \cap M, \qquad \rho^2 = \frac{12}{T}m^2, \qquad B = N, \tag{2.10}$$

where

$$B_\sigma = \{x \in X : \|x\|_X < \sigma\}. \tag{2.11}$$

By [9, Theorem 1.1], $A$ links $B$. (For background material on linking theory, cf. [10].) Moreover, by (2.8) and (2.9), we have

$$\sup_A[-G] \le 0 \le \inf_B[-G]. \tag{2.12}$$

Hence, we may apply [9, Theorem 1.1] to conclude that there is a sequence $\{x^{(k)}\} \subset X$ such that

$$G(x^{(k)}) = \left\|[x^{(k)}]'\right\|^2 - 2\int_I V(t,x^{(k)}(t))\,dt \longrightarrow c \le 0, \tag{2.13}$$

$$\frac{(G'(x^{(k)}),z)}{2} = ([x^{(k)}]',z') - \int_I \nabla_x V(t,x^{(k)}(t)) \cdot z(t)\,dt \longrightarrow 0, \quad z \in X, \tag{2.14}$$

$$\frac{(G'(x^{(k)}),x^{(k)})}{2} = \left\|[x^{(k)}]'\right\|^2 - \int_I \nabla_x V(t,x^{(k)}(t)) \cdot x^{(k)}(t)\,dt \longrightarrow 0. \tag{2.15}$$

If

$$\rho_k = \|x^{(k)}\|_X \le C, \tag{2.16}$$

then there is a renamed subsequence such that $x^{(k)}$ converges to a limit $x \in X$ weakly in $X$ and uniformly on $I$. From (2.14) we see that

$$\frac{(G'(x),z)}{2} = (x',z') - \int_I \nabla_x V(t,x(t)) \cdot z(t)\,dt = 0, \quad z \in X, \tag{2.17}$$

from which we conclude easily that $x$ is a solution of (1.1).

If

$$\rho_k = \|x^{(k)}\|_X \longrightarrow \infty, \tag{2.18}$$

let $\tilde{x}^{(k)} = x^{(k)}/\rho_k$. Then, $\|\tilde{x}^{(k)}\|_X = 1$. Let $\tilde{x}^{(k)} = \tilde{w}^{(k)} + \tilde{v}^{(k)}$, where $\tilde{w}^{(k)} \in M$ and $\tilde{v}^{(k)} \in N$. There is a renamed subsequence such that $\|[\tilde{x}^{(k)}]'\| \to r$ and $\|\tilde{x}^{(k)}\| \to \tau$, where $r^2 + \tau^2 = 1$. From (2.13) and (2.15) we obtain

$$\begin{gathered} \|[\tilde{x}^{(k)}]'\|^2 - \frac{2\int_I V(t,x^{(k)}(t))\,dt}{\rho_k^2} \longrightarrow 0, \\ \|[\tilde{x}^{(k)}]'\|^2 - \frac{\int_I \nabla_x V(t,x^{(k)}(t)) \cdot x^{(k)}(t)\,dt}{\rho_k^2} \longrightarrow 0. \end{gathered} \tag{2.19}$$

Thus,

$$\frac{2\int_I V(t,x^{(k)}(t))\,dt}{\rho_k^2} \longrightarrow r^2, \tag{2.20}$$

$$\frac{\int_I \nabla_x V(t,x^{(k)}(t)) \cdot x^{(k)}(t)\,dt}{\rho_k^2} \longrightarrow r^2. \tag{2.21}$$

Hence, by (1.9),

$$\frac{\int_I H_\mu(t,x^{(k)}(t))\,dt}{\rho_k^2} \longrightarrow \left(\frac{\mu}{2}-1\right)r^2. \tag{2.22}$$

Note that

$$|\widetilde{x}^{(k)}(t)| \le C\|\widetilde{x}^{(k)}\|_X = C. \tag{2.23}$$

If

$$|x^{(k)}(t)| \longrightarrow \infty, \tag{2.24}$$

then by (1.8)

$$\limsup \frac{H_\mu(t,x^{(k)}(t))}{\rho_k^2} = \limsup \frac{H_\mu(t,x^{(k)}(t))}{|x^{(k)}(t)|^2}|\widetilde{x}^{(k)}(t)|^2 \le 0. \tag{2.25}$$

If

$$|x^{(k)}(t)| \le C, \tag{2.26}$$

then

$$\frac{H_\mu(t,x^{(k)}(t))}{\rho_k^2} \longrightarrow 0. \tag{2.27}$$

Hence,

$$\limsup \frac{\int_I H_\mu(t,x^{(k)}(t))\,dt}{\rho_k^2} \le 0. \tag{2.28}$$

Hence by (2.22)

$$\left(\frac{\mu}{2}-1\right)r^2 \le 0. \tag{2.29}$$

If $r \neq 0$, this contradicts the fact that $\mu > 2$. If $r = 0$, then $\widetilde{w}^{(k)} \to 0$ uniformly in $I$ by Lemma 2.1. Moreover, $T|\widetilde{v}^{(k)}|^2 = \|\widetilde{v}^{(k)}\|^2 \to 1$. Hence, there is a renamed subsequence such that $\widetilde{v}^{(k)} \to \widetilde{v}$ in $N$ with $|\widetilde{v}|^2 = 1/T$. Hence, $\widetilde{x}^{(k)} \to \widetilde{v}$ uniformly in $I$. Consequently, $|x^{(k)}| \to \infty$ uniformly in $I$. Thus, by Hypothesis (4),

$$\liminf \frac{\int_I V(t,x^{(k)}(t))\,dt}{\rho_k^2} \ge \int_e \liminf \frac{V(t,x^{(k)}(t))}{|x^{(k)}(t)|^2}|\widetilde{x}^{(k)}(t)|^2 dt > 0. \tag{2.30}$$

This contradicts (2.20). Hence the $\rho_k$ are bounded, and the proof is complete. □

The proof of Theorem 1.2 is similar to that of Theorem 1.1 with the exception of the inequality (2.8) resulting from Hypothesis (2). In its place we reason as follows: if $x \in M$, we have by Hypothesis (2′),

$$\begin{aligned} G(x) &\geq \|x'\|^2 - 2\int_{|x|<m} \alpha |x(t)|^2 dt - 2C\int_{|x(t)|>m} (|x(t)|^q + 1) dt \\ &\geq \|x'\|^2 - 2\alpha\|x\|^2 - 2C(1+m^{-q})\int_{|x(t)|>m} |x(t)|^q dt \\ &\geq \|x'\|^2\left(1 - \left[\frac{2\alpha T^2}{4\pi^2}\right]\right) - C'\int_{|x(t)|>m} |x(t)|^q dt \\ &\geq \left(1 - \left[\frac{\alpha T^2}{2\pi^2}\right]\right)\|x\|_X^2 - C''\int_I \|x\|_X^q dt \\ &\geq \left(1 - \left[\frac{\alpha T^2}{2\pi^2}\right]\right)\|x\|_X^2 - C'''\|x\|_X^q \\ &= \left(1 - \left[\frac{\alpha T^2}{2\pi^2}\right] - C'''\|x\|_X^{q-2}\right)\|x\|_X^2 \end{aligned} \tag{2.31}$$

by Lemma 2.1. Hence, we have the following lemma.

Lemma 2.2.

$$G(x) \geq \varepsilon\|x\|_X^2, \quad \|x\|_X \leq \rho,\ x \in M \tag{2.32}$$

*for $\rho > 0$ sufficiently small, where $\varepsilon < 1 - [\alpha T^2/2\pi^2]$.*

The remainder of the proof is essentially the same.

In proving Theorem 1.3 we follow the proof of Theorem 1.1 until we reach (2.20). Then we reason as follows. If

$$|x^{(k)}(t)| \longrightarrow \infty, \tag{2.33}$$

then

$$\liminf \frac{H_\mu(t, x^{(k)}(t))}{\rho_k^2} = \liminf \frac{H_\mu(t, x^{(k)}(t))}{|x^{(k)}(t)|^2} |\tilde{x}^{(k)}(t)|^2 \geq 0. \tag{2.34}$$

If

$$|x^{(k)}(t)| \leq C, \tag{2.35}$$

then by Hypothesis (3′),

$$\frac{H_\mu(t, x^{(k)}(t))}{\rho_k^2} \longrightarrow 0. \tag{2.36}$$

Hence,

$$\liminf \frac{\int_I H_\mu(t, x^{(k)}(t))\,dt}{\rho_k^2} \geq 0. \tag{2.37}$$

Thus by (2.22)

$$\left(\frac{\mu}{2}-1\right)r^2 \geq 0. \tag{2.38}$$

If $r \neq 0$, this contradicts the fact that $\mu < 2$. If $r = 0$, then $\tilde{w}^{(k)} \to 0$ uniformly in $I$ by Lemma 2.1. Moreover, $T|\tilde{v}^{(k)}|^2 = \|\tilde{v}^{(k)}\|^2 \to 1$. Hence, there is a renamed subsequence such that $\tilde{v}^{(k)} \to \tilde{v}$ in $N$ with $|\tilde{v}|^2 = 1/T$. Hence, $\tilde{x}^{(k)} \to \tilde{v}$ uniformly in $I$. Consequently, $|x^{(k)}| \to \infty$ uniformly in $I$. Thus, by Hypothesis (4),

$$\liminf \frac{\int_I V(t,x^{(k)}(t))\,dt}{\rho_k^2} \geq \int_e \liminf \frac{V(t,x^{(k)}(t))}{|x^{(k)}(t)|^2}\,|\tilde{x}^{(k)}(t)|^2\,dt > 0. \tag{2.39}$$

This contradicts (2.20). Hence the $\rho_k$ are bounded, and the proof is complete.

In proving Theorem 1.4, we follow the proof of Theorem 1.1 until (2.20). Assume first that $r > 0$. Note that (2.13) and (2.15) imply that

$$\int_I H(t,x^{(k)}(t))\,dt \longrightarrow -c. \tag{2.40}$$

On the other hand, by Hypothesis (1′), we have

$$\begin{aligned} 0 &\longleftarrow \|[\tilde{x}^{(k)}]'\|^2 - 2\int_I \frac{V(t,x^{(k)}(t))\,dt}{\rho_k^2} \\ &\geq \|[\tilde{x}^{(k)}]'\|^2 - 2C\int_I \left(|\tilde{x}^{(k)}(t)|^2 + \rho_k^{-2}\right)dt \\ &\longrightarrow r^2 - 2C\int_I |\tilde{x}(t)|^2\,dt. \end{aligned} \tag{2.41}$$

Hence, $\tilde{x}(t) \not\equiv 0$. Let $\Omega_0 \subset I$ be the set on which $\tilde{x}(t) \neq 0$. The measure of $\Omega_0$ is positive. Moreover, $|x^{(k)}(t)| \to \infty$ as $k \to \infty$ for $t \in \Omega_0$. Thus,

$$\int_I H(t,x^{(k)}(t))\,dt \leq \int_{\Omega_0} H(t,x^{(k)}(t))\,dt + \int_{I\setminus\Omega_0} W(t)\,dt \longrightarrow -\infty \tag{2.42}$$

by Hypothesis (3″). But this contradicts (2.40). If $r = 0$, then $\tilde{w}^{(k)} \to 0$ uniformly in $I$ by Lemma 2.1. Moreover, $T|\tilde{v}^{(k)}|^2 = \|\tilde{v}^{(k)}\|^2 \to 1$. Thus, there is a renamed subsequence such that $\tilde{v}^{(k)} \to \tilde{v}$ in $N$ with $|\tilde{v}|^2 = 1/T$. Hence, $\tilde{x}^{(k)}(t) \to \tilde{v}$ uniformly in $I$. Consequently, $|x^{(k)}(t)| \to \infty$ uniformly in $I$. Thus, by Hypothesis (4),

$$\liminf \frac{\int_I V(t,x^{(k)}(t))\,dt}{\rho_k^2} \geq \int_e \liminf \frac{V(t,x^{(k)}(t))}{|x^{(k)}(t)|^2}\,|\tilde{x}^{(k)}(t)|^2\,dt > 0. \tag{2.43}$$

This contradicts (2.20). Hence the $\rho_k$ are bounded, and the proof is complete.

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