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Research Article

Harnack Inequality for the Schrödinger Problem Relative to Strongly Local Riemannian p-Homogeneous Forms with a Potential in the Kato Class

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We define a notion of Kato class of measures relative to a Riemannian strongly local p-homogeneous Dirichlet form and we prove a Harnack inequality (on balls that are small enough) for the positive solutions to a Schrödinger-type problem relative to the form with a potential in the Kato class.

1. Introduction

In this paper we are interested in a Harnack inequality for the Schrödinger problem relative to strongly local Riemannian p-homogeneous forms with a potential in the Kato class.

The first result in the case of Laplacian has been given by Aizenman and Simon [1]. They proved a Harnack inequality for the corresponding Schrödinger problem with a potential in the Kato measures, by probabilistic methods.

In 1986, Chiarenza et al. [2] gave an analitical proof of the result in the case of elliptic operators with bounded measurable coefficients.

Citti et al. [3] investigated the case of the subelliptic Laplacian and in 1999 Biroli and Mosco [4, 5] extended the result to the case of Riemannian strongly local Dirichlet forms (we recall also that in [6] a Harnack inequality for positive harmonic functions relative to a bilinear strongly local Dirichlet form is proved).

Biroli [7] considered the case $p > 1$ for the subelliptic p-Laplacian, and defining a suitable Kato class for this case, he obtained again Harnack and Hölder inequalities, by methods that use uniform monotonicity properties and a proof by contradiction. The proofs are not easy to be generalized to the case of strongly local Riemannian p-homogeneuous forms essentially due to the absence of any monotonicity property and the absence of a notion of translation or dilation.

In this paper we consider the case of strongly local Riemannian p-homogeneous forms; we define a suitable notion of Kato class of measures. We assume that the potential is a measure in the Kato class and we prove a Harnack inequality (on balls that are small enough for the intrinsic distance). The main difference with respect to [7] is the proof of the L^∞-local estimate. Here it is based on methods from [8, 9] instead of a variant of Moser iteration technique. Finally we will point out that methods of the same type have been also used in [10] in the framework of strongly local Riemannian p-homogeneuous forms. We conclude this introduction remarking our hope to prove similar results also in the parabolic case.

1.1. Assumptions and preliminary results. Firstly we describe the notion of strongly local p-homogeneous Dirichlet form, $p > 1$, as given in [11].

We consider a locally compact separable Hausdorff space X with a metrizable topology and a positive Radon measure m on X such that $\operatorname{supp}[m] = X$. Let $\Phi : L^p(X,m) \to [0,+\infty]$, $p > 1$, be a l.s.c. strictly convex functional with domain D, that is, $D = \{v : \Phi(v) < +\infty\}$, such that $\Phi(0) = 0$. We assume that D is dense in $L^p(X,m)$ and that the following conditions hold.

Assumption (H_1). D is a dense linear subspace of $L^p(X,m)$, which can be endowed with a norm $\|\cdot\|_D$; moreover D has a structure of Banach space with respect to the norm $\|\cdot\|_D$ and the following estimate holds:

$$c_1\|v\|_D^p \le \Phi_1(v) = \Phi(v) + \int_X |v|^p dm \le c_2\|v\|_D^p \tag{1.1}$$

for every $v \in D$, where c_1 and c_2 are positive constants.

Assumption (H_2). We denote by D_0 the closure of $D \cap C_0(X)$ in D (with respect to the norm $\|\cdot\|_D$) and we assume that $D \cap C_0(X)$ is dense in $C_0(X)$ for the uniform convergence on X.

Assumption (H_3). For every $u, v \in D \cap C_0(X)$, we have $u \vee v \in D \cap C_0(X)$, $u \wedge v \in D \cap C_0(X)$ and

$$\Phi(u \vee v) + \Phi(u \wedge v) \le \Phi(u) + \Phi(v). \tag{1.2}$$

Assumptions (H_1), (H_2), and (H_3) allow us to define a capacity relative to the functional Φ (and to the measure space (X,m)). The capacity of an open set O is defined as

$$\operatorname{cap}_\Phi(O) = \inf\{\Phi_1(v);\ v \in D_0,\ v \ge 1 \text{ a.e. on } O\} \tag{1.3}$$

if the set $\{v \in D_0,\ v \ge 1 \text{ a.e. on } O\}$ is not empty, and

$$\operatorname{cap}_\Phi(O) = +\infty \tag{1.4}$$

if the set $\{v \in D_0,\ v \geq 1$ a.e. on $O\}$ is empty. Let E be a subset of X; we define

$$\operatorname{cap}_\Phi(E) = \inf\{\operatorname{cap}(O);\ O \text{ open set with } E \subset O\}. \tag{1.5}$$

We recall that the above-defined capacity is a Choquet capacity [12]. Moreover we can prove that every function in D_0 is quasi-continuous and is defined quasi-everywhere (i.e., a function in $u \in D_0$ is a.e. equal to a function $\tilde{u}$, such that for all $\epsilon > 0$ there exists a set E_ϵ with $\operatorname{cap}(E_\epsilon) \leq \epsilon$ and $\tilde{u}$ continuous on $X - E_\epsilon$) [12].

Assumptions (H_1), (H_2), and (H_3) have a *global* character and are generalizations of the condition defining a bilinear regular Dirichlet form [13]. For bilinear Dirichlet forms the existence of a measure energy density depends only on a locality assumption; in the nonlinear case this does not hold in general and the existence and some easy properties of the measure energy density has to be assumed. We recall now the definition of *strongly local Dirichlet functional* with a homogeneity degree $p > 1$. Let Φ satisfy (H_1), (H_2), and (H_3); we say that Φ is a *strongly local Dirichlet functional* with a homogeneity degree $p > 1$ if the following conditions hold.

Assumption (H_4). Φ has the following representation on $D_0 : \Phi(u) = \int_X \alpha(u)(dx)$, where α is a nonnegative bounded Radon measure depending on $u \in D_0$, which does not charge sets of zero capacity. We say that $\alpha(u)$ is the energy (measure) of our functional. The energy $\alpha(u)$ (of our functional) is convex with respect to u in D_0 in the space of measures, that is, if $u, v \in D_0$ and $t \in [0,1]$ then $\alpha(tu + (1-t)v) \leq t\alpha(u) + (1-t)\alpha(v)$, and it is homogeneous of degree $p > 1$, that is, $\alpha(tu) = |t|^p \alpha(u)$, for all $u \in D_0$, for all $t \in \mathbb{R}$.

Moreover the following closure property holds: if $u_n \to u$ in D_0 and $\alpha(u_n)$ converges to χ in the space of measures, then $\chi \geq \alpha(u)$.

Assumption (H_5). α is of strongly local type, that is, if $u, v \in D_0$ and $u - v =$ constant on an open set A, we have $\alpha(u) = \alpha(v)$ on A.

Assumption (H_6). $\alpha(u)$ is of Markov type; if $\beta \in C^1(\mathbb{R})$ is such that $\beta'(t) \leq 1$ and $\beta(0) = 0$ and $u \in D \cap C_0(X)$, then $\beta(u) \in D \cap C_0(X)$ and $\alpha(\beta(u)) \leq \alpha(u)$ in the space of measures.

Let $\Phi(u) = \int_X \alpha(u)(dx)$ be a strongly local Dirichlet functional with domain D_0. Assume that for every $u, v \in D_0$ we have

$$\lim_{t \to 0} \frac{\alpha(u + tv) - \alpha(u)}{t} = \mu(u, v) \tag{1.6}$$

in the weak* topology of $\mathcal{M}$ (where $\mathcal{M}$ is the space of Radon measures on X) uniformly for u, v in a compact set of D_0, where $\mu(u,v)$ is defined on $D_0 \times D_0$ and is linear in v. We say that $\Psi(u,v) = \int_X \mu(u,v)(dx)$ is a *strongly local p-homogeneous Dirichlet form*. We observe that (H_3) is a consequence of (H_1), (H_2), and (H_4)–(H_6). The strong locality property allows us to define the domain of the form with respect to an open set O, denoted by $D_0[O]$ and the local domain of the form with respect to an open set O, denoted by $D_{\text{loc}}[O]$. We recall that, given an open set O in X, we can define a Choquet capacity $\operatorname{cap}(E; O)$ for a set $E \subset \overline{E} \subset O$ with respect to the open set O. Moreover the sets of zero capacity are the same with respect to O and to X.

We recall now some properties of strongly local (p-homogeneous) Dirichlet forms, which will be used in the following (for the proofs we refer to [11, 12]).

LEMMA 1.1. *Let $\Psi(u,v) = \int_X \mu(u,v)(dx)$ be a strongly local p-homogeneous Dirichlet form. Then the following properties hold.*

(a) *$\mu(u,v)$ is homogeneous of degree $p-1$ in u and linear in v, one has also $\mu(u,u) = p\alpha(u)$.*

(b) *Chain rule: if $u,v \in D_0$ and $g \in C^1(\mathbb{R})$ with $g(0) = 0$ and g' bounded on $\mathbb{R}$, then $g(u)$ and $g(v)$ belong to D_0 and*

$$\begin{gathered} \mu(g(u),v) = |g'(u)|^{p-2}g'(u)\mu(u,v), \\ \mu(u,g(v)) = g'(v)\mu(u,v). \end{gathered} \tag{1.7}$$

Observe that one has also, a chain rule for α,

$$\alpha(g(u)) = |g'(u)|^{p}\alpha(u). \tag{1.8}$$

(c) *Truncation property: for every $u,v \in D_0$,*

$$\begin{gathered} \mu(u^+,v) = 1_{\{u>0\}}\mu(u,v), \\ \mu(u,v^+) = 1_{\{v>0\}}\mu(u,v), \end{gathered} \tag{1.9}$$

where the above relations make sense, since u and v are defined quasi-everywhere.

(d) *for all $a \in \mathbb{R}^+$,*

$$|\mu(u,v)| \le \alpha(u+v) \le 2^{p-1}a^{-p}\alpha(u) + 2^{p-1}a^{p(p-1)}\alpha(v). \tag{1.10}$$

(e) *Leibniz rule with respect to the second argument:*

$$\mu(u,vw) = v\mu(u,w) + w\mu(u,v), \tag{1.11}$$

where $u \in D_0$, $v,w \in D_0 \cap L^\infty(X,m)$.

(f) *For any $f \in L^{p'}(X,\alpha(u))$ and $g \in L^p(X,\alpha(v))$ with $1/p + 1/p' = 1$, fg is integrable with respect to the absolute variation of $\mu(u,v)$ and for all $a \in \mathbb{R}^+$,*

$$|fg|\mu(u,v)||(dx) \le 2^{p-1}a^{-p}|f|^{p'}\alpha(u)(dx) + 2^{p-1}a^{p(p-1)}|g|^{p}\alpha(v)(dx). \tag{1.12}$$

(g) *Properties (e) and (f) give a Leibniz inequality for α, that is, there exists a constant $C > 0$ such that*

$$\alpha(uv) \le C\left[|u|^p\alpha(v) + |v|^p\alpha(u)\right] \tag{1.13}$$

for every $u,v \in D_0 \cap L^\infty(X,m)$.

The properties (a)–(f) are analogous to the corresponding properties of strongly local bilinear Dirichlet forms [6, 13]. Concerning (g) we observe that in the bilinear case the Leibniz rule holds for both the arguments of the form [6, 13], but in the nonlinear case the property holds only for the second argument.

We list now the assumptions that give the relations between the topology, the distance and the measure on X.

Assumption (H_7). A distance d is defined on X, such that $\alpha(d) \leq m$ in the sense of the measures.

(i) The metric topology induced by d is equivalent to the original topology of X.

(ii) Denoting by $B(x,r)$ the ball of center x and radius r (for the distance d), for every fixed compact set K, there exist positive constants c_0 and r_0 such that

$$m(B(x,r)) \leq c_0 m(B(x,s)) \left(\frac{r}{s}\right)^{\nu} \quad \forall x \in K,\ 0 < s < r < r_0. \tag{1.14}$$

We assume without loss of generality $p^2 < \nu$.

Remark 1.2. (a) Assume that

$$d(x,y) = \sup\{\varphi(x) - \varphi(y) : \varphi \in D \cap C_0(X),\ \alpha(\varphi) \leq m \text{ on } X\} \tag{1.15}$$

defines a distance on X, which satisfies (i); then d is in $D_{\text{loc}}[X]$ and $\alpha(d) \leq m$; so we can use the above-defined d as distance on X.

(b) We observe that from (i) and (ii) X has a structure of locally homogeneous space [14]. Moreover the condition, for every fixed compact set K there exist positive constants c_1 and r_0 such that

$$0 < m(B(x,2r)) \leq c_1 m(B(x,r)) \quad \forall x \in K,\ 0 < r < 2r_0, \tag{1.16}$$

$c_1 < 1$, implies (ii) for a suitable ν.

(c) From the properties of d it follows that for any $x \in X$ there exists a function $\phi(\cdot) = \phi(d(x,\cdot))$ such that $\phi \in D_0[B(x,2r)]$, $0 \leq \phi \leq 1$, $\phi = 1$, on $B(x,r)$ and

$$\alpha(\phi) \leq \frac{2}{r^p} m. \tag{1.17}$$

(d) From the assumptions on X and from (ii) the following property follows. For every fixed compact set K, such that the neighborhood of K of radius r_0 (for the distance d) is strictly contained in X, there exists a positive constant c_0', depending on c_0, such that $m(B(x,2r) - B(x,r)) \geq c_0' m(B(x,2r))$ for every $x \in K$ and $0 < r < r_0/2$.

The following assumption describes the functional relations between d, m, and the form.

Assumption (H_8). We assume also that the following scaled *Poincaré inequality* holds. For every fixed compact set K, there exist positive constants c_2, r_1, and $k \geq 1$ such that for every $x \in K$ and every $0 < r < r_1$,

$$\int_{B(x,r)} |u - \overline{u}_{x,r}|^p m(dx) \leq c_2 r^p \int_{B(x,kr)} \mu(u,u)(dx) \tag{1.18}$$

for every $u \in D_{\text{loc}}[B(x,kr)]$, where $\overline{u}_{x,r} = (1/m(B(x,r))) \int_{B(x,r)} u m(dx)$.

A strongly local p-homogeneous Dirichlet form, such that the above assumptions hold, is called a *Riemannian Dirichlet form*.

As proved in [15] the Poincaré inequality implies the following *Sobolev inequality*: for every fixed compact set K, there exist positive constants c_3, r_2, and $k \geq 1$ such that for every $x \in K$ and every $0 < r < r_2$,

$$\begin{aligned}&\left(\frac{1}{m(B(x,r))}\int_{B(x,r)}|u|^{p^*}m(dx)\right)^{1/p^*}\\&\quad\leq c_3\left(\frac{r^p}{m(B(x,r))}\int_{B(x,kr)}\mu(u,u)(dx)+\frac{r^p}{m(B(x,r))}\int_{B(x,r)}|u|^p m(dx)\right)^{1/p}\end{aligned}\tag{1.19}$$

with $p^* = p\nu/(\nu - p)$ and c_3, r_2 depending only on c_0, c_2, r_0, r_1. We observe that we can assume without loss of generality $r_0 = r_1 = r_2$.

Remark 1.3. (a) From (1.18) we can easily deduce by standard methods that

$$\frac{1}{m(B(x,r))}\int_{B(x,r)}|u|^p m(dx) \leq c_2' \frac{r^p}{m(B(x,r)\cap\{u=0\})}\int_{B(x,kr)}\mu(u,u)(dx),\tag{1.20}$$

where c_2' is a positive constant depending only on c_2.

(b) From (a) it follows that for every fixed compact set K, such that the neighborhood of K of radius r_0 is strictly contained in X,

$$\int_{B(x,r)}|u|^p m(dx) \leq c_2^\star r^p \int_{B(x,kr)}\mu(u,u)(dx)\tag{1.21}$$

for every $x \in K$ and $0 < r < r_0/2$, where $c_2^\star$ depends only on c_2' and c_0.

As a consequence of Remark 1.2(d) and of the Poincaré inequality, we have the following estimate on the capacity of a ball.

Proposition 1.4. *For every fixed compact set K, there exist positive constants c_4 and c_5 such that*

$$c_4\frac{m(B(x,r))}{r^p} \leq \operatorname{cap}\left(B(x,r),B(x,2r)\right) \leq c_5\frac{m(B(x,r))}{r^p},\tag{1.22}$$

where $x \in K$ and $0 < 2r < r_0$.

The definition of Kato class of measure was initially given by Kato [16] in the case of Laplacian and extended in [2] to the case of elliptic operators with bounded measurable coefficients.

Kato classes relative to a subelliptic Laplacian were defined in [3], and the case of (bilinear) Riemannian Dirichlet form was considered in [4, 17].

In [7] the Kato class was defined in the case of subelliptic p-Laplacian and in [10] the following definition of Kato class relative to a Riemannian p-homogeneous Dirichlet form has been given.

Definition 1.5. Let σ be a Radon measure. Say that σ is in the Kato space $K(X)$ if

$$\lim_{r\to 0}\eta_\sigma(r)=0, \tag{1.23}$$

where

$$\eta_\sigma(r)=\sup_{x\in X}\int_0^r\left(\frac{|\sigma|(B(x,\rho))}{m(B(x,\rho))}\rho^p\right)^{1/(p-1)}\frac{d\rho}{\rho}. \tag{1.24}$$

Let $\Omega\subset X$ be an open set; $K(\Omega)$ is defined as the space of Radon measures σ on Ω such that the extension of σ by 0 out of Ω is in $K(X)$.

In [10] we have investigated the properties of the space $K(\Omega)$. In particular we have proved that if Ω is a relatively compact open set of diameter $\overline{R}/2$, then

$$\|\sigma\|_{K(\Omega)}:=\eta_\sigma(\overline{R})^{p-1} \tag{1.25}$$

is a norm on $K(\Omega)$ and, as in [18] for the bilinear case, we can prove that $K(\Omega)$ endowed with this norm is a Banach space. Moreover we have proved that $K(\Omega)$ is contained in $D'[\Omega]$, where $D'[\Omega]$ denotes the dual of $D_0[\Omega]$.

1.2. Results. We state now the result that we will prove in the following sections. Let $\Omega\subset X$ be a relatively compact open set. We denote by c_0, c_2, r_0 the constants appearing in (1.14) and (1.18) relative to the compact set $\overline{\Omega}$. We assume that a neighborhood of Ω of radius $\overline{R}/2+r_0$ is strictly contained in X ($\overline{R}=2\operatorname{diam}\Omega$), that $\int_X\mu(u,v)(dx)$ is a Riemannian (p-homogeneous) Dirichlet form, and that $u\in D_{\text{loc}}(\Omega)$ with $\int_\Omega\mu(u,u)(dx)<+\infty$ is a subsolution of the problem

$$\int_\Omega\mu(u,v)+\int_\Omega|u|^{p-2}uv\sigma(dx)=0\quad\text{for every } v\in D_0(\Omega),\ \operatorname{supp}(v)\subset\Omega, \tag{1.26}$$

where $\sigma\in K(\Omega)$, that is,

$$\int_\Omega\mu(u,v)+\int_\Omega|u|^{p-2}uv\sigma(dx)\le 0\quad\text{for every positive } v\in D_0(\Omega),\ \operatorname{supp}(v)\subset\Omega. \tag{1.27}$$

Remark 1.6. We observe that the problems (1.26) and (1.27) make sense, due to a Schechter type inequality, which we will prove in Section 2, giving the continuous embedding of $D_{\text{loc}}[\Omega]$ into $L^p_{\text{loc}}(\Omega,\sigma)$.

Our first result is a local L^∞ estimate for positive subsolutions of (1.26)

Theorem 1.7. *Let $x_0\in\Omega$. For every $q>0$ there exist positive structural constants C_q (depending on q) and R_0 (depending on σ) such that, for every positive local subsolution u of (1.26) and every $r\le R_0$, such that $B_{x_0,2r}\subset\Omega$, one has*

$$\sup_{B(x_0,r/2)}u\le C_q\left(\frac{1}{m(B(x_0,r))}\int_{B(x_0,r)}u^q m(dx)\right)^{1/q}. \tag{1.28}$$

We prove Theorem 1.7 in Section 1 by methods derived from [10]. We use Theorem 1.7 to prove in Section 2 the following Harnack inequality.

Theorem 1.8. *Let $x_o \in \Omega$. There exist positive structural constants $\widetilde{C}$ and R_1 (depending on σ) such that for every positive local solution u of (1.26) and every $r \le R_1$ such that $B_{x_0,2r} \subset \Omega$, one has*

$$\sup_{B(x_0,r)} u \le \widetilde{C} \inf_{B(x_0,r)} u. \tag{1.29}$$

We can assume without loss of generality $R_0 = R_1$. From Theorem 1.8 we obtain the following result on the continuity of harmonic functions (local solutions) for (1.26).

Theorem 1.9. *Every local solution of (1.26) is continuous in Ω. Moreover if $\sigma(B(x,r)) \le c(x)r^{\nu-p+\epsilon}$ for some $\epsilon > 0$ and for a continuous function $c(x)$ (of $x \in \Omega$), then u is locally Hölder continuous in Ω.*

Each of Theorems 1.8 and 1.9 follow from the former. We follow the method of [7] to prove Theorems 1.8 and 1.9. Because of the great generality of the structure we cannot apply the Moser iteration technique to prove Theorem 1.7 which follows from a Schecter-type inequality and an iterative application of a Cacciopoli-type inequality, introduced in [9] in the Euclidean case and extended by [19] to the subelliptic framework and in [10] to our general framework.

2. Proof of Theorem 1.7

Let $\sigma \in K(\Omega)$ and $B = B(x_0,r) \subset\subset \Omega$, we denote

$$\eta_{\mu,B}(r) = \sup_{x\in B} \int_0^r \left(\frac{|\sigma|(B(x,\rho))}{m(B(x,\rho))} \rho^p \right)^{1/(p-1)} \frac{d\rho}{\rho}. \tag{2.1}$$

Consider the problem

$$\int_B \mu(w,v)(dx) = \int_B v\sigma(dx), \tag{2.2}$$

$w \in D_0(B)$, for any $v \in D_0(B)$ with compact support in Ω.

Proposition 2.1. *Let $w \in D_0(B)$ be a subsolution of (2.2). Then there exists a structural constant C depending on σ such that*

$$\sup_B |w| \le C\eta_\sigma(2r). \tag{2.3}$$

Proof. In this proof we denote by C possibly different structural constants.

We observe that $|w|$ is a subsolution of problem (2.2). Then from [10, Theorem 1.1] it follows that

$$\sup_B |w| \le C\left[\left(\frac{1}{m(B)} \int_B |w|^p m(dx) \right)^{1/p} + \eta_{\sigma,B}(2r) \right], \tag{2.4}$$

where C is independent on r. By Poincaré inequality, (2.4) gives

$$\sup_B |w| \le C\left\{\left[\frac{r^p}{m(B)}\int_B \alpha(w)(dx)\right]^{1/p} + \eta_{\sigma,B}(2r)\right\}. \tag{2.5}$$

From (2.2) with $v = w$ and (2.5), we obtain

$$\int_B \alpha(w)(dx) \le C\left[|\sigma|(B)^p\frac{r^p}{m(B)}\right]^{1/p}\left[\int_B \alpha(w)(dx)\right]^{1/p} + C|\sigma|(B)\eta_{\sigma,B}(2r) \tag{2.6}$$

and applying Young inequality to the first term in the right-hand side of (2.6), we have

$$\int_B \alpha(w)(dx) \le C|\sigma|(B)\eta_{\sigma,B}(2r). \tag{2.7}$$

Combining (2.5) and (2.7) gives

$$\sup_B |w| \le C\left\{\left[\frac{|\sigma|(B)}{m(B)}r^p\eta_{\sigma,B}(2r)\right]^{1/p} + \eta_{\sigma,B}(2r)\right\} \tag{2.8}$$

and applying Young inequality,

$$\sup_B |w| \le C\left[\left(\frac{|\sigma|(B)}{m(B)}r^p\right)^{1/(p-1)} + \eta_{\sigma,B}(2r)\right] \le C\eta_{\sigma,B}(2r). \tag{2.9}$$

□

Proposition 2.2 (Schechter's inequality). *Let $x_0 \in \Omega$ and for any $\rho > 0$ denote $B_\rho = B(x_0,\rho)$.*

For any $0 < \epsilon < 1$, there exist some constants $t_0 > 0$ and $C(\epsilon) > 0$ such that $B_{2t_0} \subset\subset \Omega$ and such that if $0 < s < t \le t_0$, then

$$\int_{B_s} |u|^p|\sigma|(dx) \le \epsilon\int_{B_t}\alpha(u)(dx) + \frac{C(\epsilon)}{(t-s)^p}\int_{B_t-B_s}|u|^p m(dx) \tag{2.10}$$

for every $u \in D_{\text{loc}}(B_t)$, where $C(\epsilon)$ depends on ϵ and the structural constants. In particular if $u \in D_0(B_t)$, then

$$\int_{B_s} |u|^p|\mu|(dx) \le \epsilon\int_{B_t}\alpha(u)(dx). \tag{2.11}$$

Proof. Let w be the weak solution of the problem (2.2) in B_{2t} and let φ be a cut-off function between the balls B_s and B_t, where $s < t \le t_0$ and t_0 will be specified at the end of the proof. Set in (2.2) the test function $|u|^p\varphi^p$. We have

$$\begin{aligned}
\int_{B_t}|u|^p\varphi^p|\sigma|(dx) &= \int_{B_t}\mu(w,|u|^p\varphi^p)(dx)\\
&\leq p\int_{B_t}|u|^{p-1}\varphi^p\mu(w,|u|)(dx)+p\int_{B_t}|u|^p\varphi^{p-1}\mu(w,\varphi)(dx)\\
&\leq (p2^{(p+2)})^p\epsilon^{(1-p)}\int_{B_t}|u|^p\varphi^p\alpha(w)(dx)+\frac{\epsilon}{8}\int_{B_t}\varphi^p\alpha(u)(dx)\\
&\quad+(p2^{(p+2)})^p\epsilon^{(1-p)}\int_{B_t}|u|^p\varphi^p\alpha(w)(dx)+\frac{\epsilon}{8}\int_{B_t}|u|^p\alpha(\varphi)(dx)\\
&\leq 2(p2^{(p+2)})^p\epsilon^{(1-p)}\int_{B_t}|u|^p\varphi^p\alpha(w)(dx)\\
&\quad+\frac{\epsilon}{4}\left[\int_{B_t}\varphi^p\alpha(u)(dx)+\int_{B_t}|u|^p\alpha(\varphi)(dx)\right].
\end{aligned}\tag{2.12}$$

Now we estimate the first term in the right-hand side of (2.12). In virtue of Proposition 2.1, we obtain

$$\begin{aligned}
&\int_{B_t}|u|^p\varphi^p\alpha(w)(dx)\\
&\quad=\int_{B_t}\mu(w,w|u|^p\varphi^p)(dx)-p\int_{B_t}w|u|^{p-1}\varphi^{p-1}\mu(w,u\varphi)(dx)\\
&\quad=\int_{B_t}w|u|^p\varphi^p\sigma(dx)-p\int_{B_t}w|u|^{p-1}\varphi^{p-1}\mu(w,u\varphi)(dx)\\
&\quad\leq C\eta_\sigma(2t)\int_{B_t}|u|^p\varphi^p\sigma(dx)\\
&\qquad+\frac{1}{2}\int_{B_t}|u|^p\varphi^p\mu(w,w)(dx)+2^{(p^2+1)}p^{(p-1)}\int_{B_t}w^p\alpha(u\varphi)(dx)\\
&\quad\leq C\eta_\sigma(t)\int_{B_t}|u|^p\varphi^p\mu(dx)+\frac{1}{2}\int_{B_t}|u|^p\varphi^p\alpha(w)(dx)\\
&\qquad+C_p(C\eta_\sigma(t))^p\left[\int_{B_t}\varphi^p\alpha(u)(dx)+\int_{B_t}|u|^p\alpha(\varphi)(dx)\right],
\end{aligned}\tag{2.13}$$

where $C_p>1$ is a constant depending only on p.

Then we have

$$\begin{aligned}
&\int_{B_t}|u|^p\varphi^p\alpha(w)(dx)\\
&\quad\leq 2C\eta_\sigma(t)\int_{B_t}|u|^p\varphi^p\mu(dx)+2C_p(C\eta_\sigma(t))^p\left[\int_{B_t}\varphi^p\alpha(u)(dx)+\int_{B_t}|u|^p\alpha(\varphi)(dx)\right].
\end{aligned}\tag{2.14}$$

Substituting (2.14) in (2.12) and supposing $C\eta_\mu(t) < 1$, we obtain

$$\begin{aligned}
\int_{B_t} |u|^p \varphi^p \sigma(dx) &\\
\leq & 4\big(p2^{(p+2)}\big)^p \epsilon^{(1-p)} C_p C\eta_\sigma(t) \int_{B_t} |u|^p \varphi^p \sigma(dx)\\
&+4\big(p2^{(p+2)}\big)^p \epsilon^{(1-p)} C_p C\eta_\sigma(t) \bigg[\int_{B_t} \varphi^p \alpha(u)(dx) + \int_{B_t} |u|^p \alpha(\varphi)(dx)\bigg]\\
&+\frac{\epsilon}{4}\bigg[\int_{B_t} \varphi^p \alpha(u)(dx) + \int_{B_t} |u|^p \alpha(\varphi)(dx)\bigg].
\end{aligned} \tag{2.15}$$

Let t_0 be such that $4(p2^{(p+2)})^p \epsilon^{(1-p)} C_p C\eta_\sigma(t_0) < \epsilon/4$. Then for $t \geq t_0$, we have

$$\int_{B_t} |u|^p \varphi^p \mu(dx) \leq \epsilon \int_{B_t} \varphi^p \alpha(u)(dx) + C(\epsilon) \int_{B_t} |u|^p \alpha(\varphi)(dx) \tag{2.16}$$

and the proof is concluded. □

Let τ be defined as $1/\tau = (p-1)/q + 1/p$, where $p < q < \nu p/(\nu - p)$. We observe that the infimum of the values of τ is 1 and the supremum of the values of τ is given by

$$\begin{aligned}
\bigg[(p-1)\frac{\nu-p}{\nu p} + \frac{1}{p}\bigg]^{-1} &= \bigg[\frac{p-1}{p} - \frac{p-1}{p}\Big(1 - \frac{\nu-p}{\nu}\Big) + \frac{1}{p}\bigg]^{-1}\\
&= \Big(1 - \frac{p-1}{\nu}\Big)^{-1} = \frac{\nu}{\nu-p+1}.
\end{aligned} \tag{2.17}$$

Then the supremum of the values of τ is less than p^*/p (where $p^* = \nu p/(\nu - p)$) so $u^{\tau p}$ and $u^{(\tau-1)p}$ are integrable for the measure $m(dx)$. We have $\nu > p(p-1)$ so we have that $u^{\tau(p-1)}$ is integrable for the measure $|\sigma|(dx)$.

Proposition 2.3. *Let $x_0 \in \Omega$ and for any $\rho > 0$ denote $B_\rho = B(x_0, \rho)$. Let u be a solution of (1.26) and $B_{4r} \subset \Omega$. Then for any $h > 0$, one has*

$$\int_{B_r} \alpha\big(((u-h)^+)^{\gamma/p}\big)(dx) \leq \frac{C}{r^p} \int_{B_{2r}-B_r} ((u-h)^+)^\gamma (dx) + Ch^\gamma |\sigma|(B_{2r}), \tag{2.18}$$

where $\gamma = \tau(p-1)$.

Proof. We choose in (1.26) the test function

$$v = \varphi^p\big((u-h)^+ + \epsilon\big)^{(\tau-1)(p-1)}, \tag{2.19}$$

where φ is a cut-off function between the balls B_r and B_{2r}. We obtain

$$\int_{B_{2r}} \mu\Big(u, \varphi^p\big((u-h)^+ + \epsilon\big)^{(\tau-1)(p-1)}\Big)(dx) + \int_{B_{2r}} \varphi^p |u|^{p-1}\big((u-h)^+ + \epsilon\big)^{(\tau-1)(p-1)} \sigma(dx) = 0. \tag{2.20}$$

Let us estimate the first term in the left-hand side of (2.20),

$$\begin{aligned}
&\int_{B_{2r}} \mu\Big(u,\varphi^p((u-h)^+ + \epsilon)^{(\tau-1)(p-1)}\Big)(dx)\\
&\quad= \int_{B_{2r}} \varphi^p \mu\Big(u,((u-h)^+ + \epsilon)^{(\tau-1)(p-1)}\Big)(dx)\\
&\qquad+ \int_{B_{2r}} ((u-h)^+ + \epsilon)^{(\tau-1)(p-1)} \mu(u,\varphi^p)(dx)\\
&\quad= p\int_{B_{2r}} \varphi^p(\tau-1)(p-1)((u-h)^+ + \epsilon)^{((\tau-1)(p-1)-1)} \alpha((u-h)^+ + \epsilon)(dx)\\
&\qquad+ \int_{B_{2r}} p\varphi^{p-1}((u-h)^+ + \epsilon)^{(\tau-1)(p-1)} \mu(u,\varphi)(dx)\\
&\quad= p\left(\frac{\gamma}{p}\right)^{-p} \int_{B_{2r}} \varphi^p(\tau-1)(p-1)\alpha\big(((u-h)^+ + \epsilon)^{\gamma/p}\big)(dx)\\
&\qquad+ \int_{B_{2r}} p\varphi^{(p-1)}((u-h)^+ + \epsilon)^{(\tau-1)(p-1)} \mu(u,\varphi)(dx).
\end{aligned} \tag{2.21}$$

From (2.20) and (2.21), we obtain

$$\begin{aligned}
&p\left(\frac{\gamma}{p}\right)^{-p} \int_{B_{2r}} \varphi^p(\tau-1)(p-1)\alpha\big(((u-h)^+ + \epsilon)^{\gamma/p}\big)(dx)\\
&\quad\le \int_{B_{2r}} p\varphi^{(p-1)}((u-h)^+ + \epsilon)^{(\tau-1)(p-1)} \left|\mu(u,\varphi)\right|(dx)\\
&\qquad+ \int_{B_{2r}} \sigma^p |u|^{p-1}((u-h)^+ + \epsilon)^{(\tau-1)(p-1)} |\sigma|(dx).
\end{aligned} \tag{2.22}$$

The right-hand side of (2.22) is uniformly bounded with respect to $0 < \epsilon < 1$, so we can pass to the limit $\epsilon \to 0$ (in the first term of the right-hand side of (2.22), we use the convergence q.e. and the uniform integrability with respect to the capacity) and we obtain

$$\begin{aligned}
&p\left(\frac{\gamma}{p}\right)^{-p} \int_{B_{2r}} \varphi^p(\tau-1)(p-1)\alpha\big(((u-h)^+)^{\gamma/p}\big)(dx)\\
&\quad\le \int_{B_{2r}} p\varphi^{(p-1)}((u-h)^+)^{(\tau-1)(p-1)} \left|\mu(u,\varphi)\right|(dx)\\
&\qquad+ \int_{B_{2r}} \varphi^p |u|^{p-1}((u-h)^+)^{(\tau-1)(p-1)} |\sigma|(dx).
\end{aligned} \tag{2.23}$$

Then

$$
\begin{aligned}
p\left(\frac{\gamma}{p}\right)^{-p} & \int_{B_{2r}} \varphi^p(\tau-1)(p-1)\alpha\left(\left((u-h)^+\right)^{\gamma/p}\right)(dx) \\
& \le \int_{B_{2r}} p\varphi^{(p-1)}\left((u-h)^+\right)^{(\tau-1)(p-1)}\left|\mu\left((u-h)^+,\varphi\right)\right|(dx) \\
& \quad + \int_{B_{2r}} \varphi^p|u|^{p-1}\left((u-h)^+\right)^{(\tau-1)(p-1)}|\sigma|(dx) \\
& \le \left(\frac{\gamma}{p}\right)^{-p}\int_{B_{2r}} p\varphi^{(p-1)}\left((u-h)^+\right)^{(\tau-1)(p-1)-((\tau(p-1)-p)/p)(p-1)}\left|\mu\left(\left((u-h)^+\right)^{\gamma/p},\varphi\right)\right|(dx) \\
& \quad + \int_{B_{2r}} \varphi^p|u|^{p-1}\left((u-h)^+\right)^{(\tau-1)(p-1)}|\sigma|(dx) \\
& \le \left(\frac{\gamma}{p}\right)^{-p}\int_{B_{2r}} p\varphi^{(p-1)}\left((u-h)^+\right)^{\gamma/p}\left|\mu\left(\left((u-h)^+\right)^{\gamma/p},\varphi\right)\right|(dx) \\
& \quad + \int_{B_{2r}} \varphi^p|u|^{p-1}\left((u-h)^+\right)^{(\tau-1)(p-1)}|\sigma|(dx) \\
& \le \frac{C}{r^p}\int_{B_{2r}-B_r}\left((u-h)^+\right)^{\gamma}m(dx) + \chi\int_{B_{2r}} \varphi^p\alpha\left(\left((u-h)^+\right)^{\gamma/p}\right)(dx) \\
& \quad + C\int_{B_{2r}} \varphi^p\left((u-h)^+\right)^{\gamma}|\sigma|(dx) + Ch^{\gamma}|\sigma|(B_{2r}),
\end{aligned}
\tag{2.24}
$$

where $\chi = (\gamma/p)^{-p}(\tau-1)(p-1)/2$, and then from (2.24), we obtain

$$
\begin{aligned}
\int_{B_{2r}} & \alpha\left(\left((u-h)^+\right)^{\gamma/p}\right)(dx) \\
& \le \frac{C}{r^p}\int_{B_{2r}-B_r}\left((u-h)^+\right)^{\gamma}m(dx) + C\int_{B_{2r}} \varphi^p\left((u-h)^+\right)^{\gamma}|\sigma|(dx) + Ch^{\gamma}|\sigma|(B_{2r}).
\end{aligned}
\tag{2.25}
$$

Using the Schecter inequality (2.10) applied to the function $\varphi u^{\gamma/p}$ to estimate the second term in the right-hand side of (2.25) and choosing r small enough, we obtain

$$
\begin{aligned}
\int_{B_r}\alpha\left(\left((u-h)^+\right)^{\gamma/p}\right)(dx) & \le \int_{B_{2r}} \varphi^p\alpha\left(\left((u-h)^+\right)^{\gamma/p}\right)(dx) \\
& \le \frac{C}{r^p}\int_{B_{2r}-B_r}\left((u-h)^+\right)^{\gamma}m(dx) + Ch^{\gamma}|\sigma|(B_{2r})
\end{aligned}
\tag{2.26}
$$

and Proposition 2.3 follows. □

We recall now the following result; see [10, Theorem 1].

Proposition 2.4. *Let ζ be a positive Radon measure in $K(\Omega)$ and let $w \in D[\Omega]$ be a positive subsolution of the problem*

$$
\int_{\Omega}\overline{\mu}(w,v)(dx) = \int_{\Omega} v\zeta(dx)
\tag{2.27}
$$

for any $v \in D_0[\Omega]$ with compact support in Ω. Then $\zeta \in D_0'[\Omega]$ (where $D_0'[\Omega]$ is the dual space of $D_0[\Omega]$) and for any $B_r = B(x_0, r) \subset \Omega$, $r \le R^\star$, one has

$$u(x_0) \le C\left(\frac{1}{|B_r|}\int_{B_r} u^\gamma m(dx)\right)^{1/\gamma} + C\int_0^r \left(\zeta(B_s)\frac{s^p}{|B_s|}\right)^{1/(p-1)}\frac{ds}{s}, \tag{2.28}$$

where $R^\star$ is positive suitably fixed.

Remark 2.5. Let us observe that if u is a positive subsolution of (1.26), then the Radon measure $\zeta = \mu u^{p-1}$ is in $D_o'(\Omega)$ and u is a positive subsolution of the problem (2.27).

Proposition 2.6. *Let u be a positive subsolution of (1.26). Then for q.e. $x_0 \in \Omega$ and for every $B_r = B(x_0, r)$ such that $r \le R_0$, $B_{2r} \subset\subset \Omega$, one has*

$$u(x_0) \le C\left(\frac{1}{m(B_r)}\int_{B_r} u^\gamma m(dx)\right)^{1/\gamma}. \tag{2.29}$$

Proof. Let $r_j = 2^{-j}r$, $j = 0, 1, \ldots$. Define recursively $l_0 = 0$ and

$$\begin{aligned} l_{j+1} &= l_j + \left(\frac{1}{\kappa m(B_{r_j})}\int_{B_{r_j}} \varphi_j^q((u-l_j)^+)^\gamma m(dx)\right)^{1/\gamma}, \\ \delta_j &= l_{j+1} - l_j = \left(\frac{1}{\kappa m(B_{r_j})}\int_{B_{r_j}} \varphi_j^q((u-l_j)^+)^\gamma m(dx)\right)^{1/\gamma}, \end{aligned} \tag{2.30}$$

where κ is the constant of [10, Lemma 2.2], and φ_j is a cut-off function between the balls B_{r_j} and $B_{r_{j+1}}$. We have

$$\delta_j = \frac{1}{2}\delta_{j-1} + C\left[\frac{r_j^p}{m(B_{r_j})}\int_{B_{r_j}} u^{p-1}|\sigma|(dx)\right]^{1/(p-1)}. \tag{2.31}$$

The proof of (2.31) can be found in [10, Proof of Theorem 1] taking into account Remark 2.5. We now estimate the second term in the right-hand side of (2.31). We have

$$\begin{aligned} \frac{r_j^p}{m(B_{r_j})}\int_{B_{r_j}} u^{p-1}|\sigma|(dx) &\le C\frac{r_j^p}{m(B_{r_j})}\left[\int_{B_{r_j}} ((u-l_j)^+)^{p-1}|\sigma|(dx) + l_j^{p-1}|\sigma|(B_{r_j})\right] \\ &\le C\frac{r_j^p}{m(B_{r_j})}|\sigma|(B_{r_j})^{(\tau-1)/\tau}\left[\int_{B_{r_{j-1}}} \varphi_{j-1}^q((u-l_j)^+)^\gamma|\sigma|(dx)\right]^{1/\tau} \\ &\quad + l_j^{p-1}\frac{r_j^p}{m(B_{r_j})}|\sigma|(B_{r_j}) \end{aligned} \tag{2.32}$$

(in virtue of Schecter's inequality (2.11) and taking into account that $\varphi_{j-1} \in D_0(B_{j-1})$)

$$\begin{aligned}
&\leq C\frac{r_j^p}{m(B_{r_j})}|\sigma|(B_{r_j})^{(\tau-1)/\tau}\left[\omega(r_j)\int_{B_{r_{j-1}}}\alpha\left(\varphi_{j-1}^{q/p}\left(\left((u-l_j)^+\right)^{\gamma/p}\right)(dx)\right]^{1/\tau} \\
&\quad +Cl_j^{p-1}\frac{r_j^p}{m(B_{r_j})}|\sigma|(B_{r_j}),
\end{aligned} \tag{2.33}$$

where $\omega(s) = C(p)\eta_\sigma(s)$ as in the proof of Schecter's inequality. Using Proposition 2.3, we continue to estimate as follows:

$$\begin{aligned}
&\leq C\frac{r_j^p}{m(B_{r_j})}|\sigma|(B_{r_j})^{(\tau-1)/\tau}\left[\omega(r_j)r_j^{-p}\int_{B_{r_{j-2}}}\varphi_{j-2}^q\left((u-l_{j-2})^+\right)^\gamma m(dx)\right]^{1/\tau} \\
&\quad +Cl_j^{p-1}\left[\omega(r_j)^{1/\tau}\frac{r_{j-1}^p}{m(B_{r_{j-1}})}|\sigma|(B_{r_{j-1}})+\frac{r_j^p}{m(B_{r_j})}|\sigma|(B_{r_j})\right] \\
&\leq C\frac{r_j^p}{m(B_{r_j})}|\sigma|(B_{r_j})^{(\tau-1)/\tau}\left[\omega(r_j)r_j^{-p}\int_{B_{r_{j-2}}}\varphi_{j-2}^q\left((u-l_{j-2})^+\right)^\gamma m(dx)\right]^{1/\tau} \\
&\quad +Cl_{j-1}^{p-1}\omega(r_j)^{1/\tau}\frac{r_{j-1}^p}{m(B_{r_{j-1}})}|\sigma|(B_{r_{j-1}}) \\
&\quad +C\delta_{j-1}^{p-1}\omega(r_j)^{1/\tau}\frac{r_{j-1}^p}{m(B_{r_{j-1}})}|\sigma|(B_{r_{j-1}})+Cl_j^{p-1}\frac{r_j^p}{m(B_{r_j})}|\sigma|(B_{r_j}).
\end{aligned} \tag{2.34}$$

So

$$\begin{aligned}
\delta_j &\leq \frac{1}{2}\delta_{j-1}+C\left[\frac{r_j^p}{m(B_{r_j})}|\sigma|(B_{r_j})\right]^{(\tau-1)/\gamma}\omega(r_j)^{1/\gamma}\delta_{j-1} \\
&\quad +Cl_j\left[\frac{r_j^p}{m(B_{r_j})}|\sigma|(B_{r_j})\right]^{1/(p-1)}+Cl_{j-1}\left[\frac{r_{j-1}^p}{m(B_{r_{j-1}})}|\sigma|(B_{r_{j-1}})\right]^{1/(p-1)}.
\end{aligned} \tag{2.35}$$

We can assume that r is small enough so we obtain

$$C\left[\frac{r_j^p}{m(B_{r_j})}|\sigma|(B_{r_j})\right]^{(\tau-1)/\gamma}\omega(r_j)^{1/\gamma}\leq C\left[\frac{r_j^p}{m(B_{r_j})}|\sigma|(B_{r_j})\right]^{1/(p-1)}+\omega(r_j)^{1/(p-1)}\leq\frac{1}{4}. \tag{2.36}$$

Then

$$\delta_j \leq \frac{3}{4}\delta_{j-1}+Cl_j\left[\frac{r_j^p}{m(B_{r_j})}|\sigma|(B_{r_j})\right]^{1/(p-1)}+Cl_{j-1}\left[\frac{r_{j-1}^p}{m(B_{r_{j-1}})}|\sigma|(B_{r_{j-1}})\right]^{1/(p-1)}. \tag{2.37}$$

We now sum up the previous relations for $j = 2,\dots,s$ and we obtain

$$\sum_2^s\delta_j \leq \frac{3}{4}\sum_1^{s-1}\delta_j+2C\sum_0^s l_j\left[\frac{r_j^p}{m(B_{r_j})}|\sigma|(B_{r_j})\right]^{1/(p-1)}. \tag{2.38}$$

It follows that

$$\sum_0^s \delta_j \le \delta_0 + \delta_1 + \frac{3}{4}\sum_0^s \delta_j + Cl_s \int_{r_s}^{r} \left[\frac{\rho^p}{m(B_\rho)}|\sigma|(B_\rho)\right]^{1/(p-1)} \frac{d\rho}{\rho} \tag{2.39}$$

and then

$$l_s = \sum_0^s \delta_j \le \delta_0 + \delta_1 + Cl_s \eta_\sigma(r). \tag{2.40}$$

Finally we have

$$l_s \le \delta_0 + \delta_1 \le C\left[\frac{1}{m(B_r)}\int_{B_r} u^\gamma\right]^{1/\gamma}. \tag{2.41}$$

We observe that l_s is a bounded increasing sequence. Then l_s converges to l such that

$$l \le C\left[\frac{1}{m(B_r)}\int_{B_r} u^\gamma\right]^{1/\gamma}. \tag{2.42}$$

It remains to prove that

$$u(x_0) \le Cl \tag{2.43}$$

for q.e. $x_0 \in \Omega$. We observe that for every fixed $\epsilon > 0$ and every $j = 0, 1, \ldots,$

$$\begin{aligned} \epsilon^\gamma \frac{m(\{x \in B_{r_j} : u(x) > l + \epsilon\})}{\kappa m(B_{r_j})} &\le \frac{1}{\kappa m(B_{r_j})}\int_{B_{r_j}} ((u-l)^+)^\gamma m(dx) \\ &\le \frac{1}{\kappa m(B_{r_j})}\int_{B_{r_j}} ((u-l_{j-1})^+)^\gamma m(dx). \end{aligned} \tag{2.44}$$

Then

$$\epsilon \sum_{j=\varnothing}^{+\infty} \left[\frac{m(\{x \in B_{r_j} : u(x) > l + \epsilon\})}{\kappa m(B_{r_j})}\right]^{1/\gamma} \le C\sum_{j=0}^{+\infty} \delta_j < +\infty. \tag{2.45}$$

From the previous relation, we obtain

$$\lim_{j\to+\infty} \frac{m(\{x \in B_{r_j} : u(x) > l + \epsilon\})}{\kappa m(B_{r_j})} = 0 \tag{2.46}$$

for every fixed $\epsilon > 0$. Then, taking into account the quasi-continuity of u for the measure m, we have (2.43) for a.e. $x_o \in \Omega$. We recall that u is also quasi-continuous for the capacity relative to α, then (2.43) holds for q.e. $x_o \in \Omega$. We have so proved Proposition 2.6. □

Corollary 2.7. *Let the hypothesis of Proposition 2.6 hold. Then there exists a positive structural constant C such that if* $1/2 \le s < t \le 1$, $r \le R_0$, $B(x_0, 2r) \subset \Omega$, *then one has*

$$\sup_{B(x_0,sr)} |u| \le \frac{C}{(t-s)^{\nu/\gamma}}\left[\frac{1}{m(B(x_0,tr))}\int_{B(x_0,tr)} u^\gamma m(dx)\right]^{1/\gamma}. \tag{2.47}$$

Proof. Let $x \in B(x_0, sr)$. We have from Proposition 2.6,

$$u(x) \leq C\left[\frac{1}{m(B(x,((t-s)/2)r))}\int_{B(x,((t-s)/2)r)} u^{\gamma} m(dx)\right]^{1/\gamma}. \tag{2.48}$$

Then

$$\sup_{B(x_0,sr)} |u| \leq C \sup_{B(x_0,sr)} \left[\frac{m(B(x_0,tr))}{m(B(x,(t-s)r))}\frac{1}{m(B(x_0,tr))}\int_{B(x_0,tr)} u^{\gamma} m(dx)\right]^{1/\gamma}. \tag{2.49}$$

We observe that $B(x_0,tr) \subset B(x,(t+s)r)$, then

$$\sup_{B(x_0,sr)}\left[\frac{m(B(x_0,tr))}{m(B(x,(t-s)r))}\right]^{1/\gamma} \leq \sup_{B(x_0,sr)}\left[\frac{m(B(x,(t+s)r))}{m(B(x,(t-s)r))}\right]^{1/\gamma} \leq \frac{C}{(t-s)^{\nu/\gamma}} \tag{2.50}$$

and (2.47) is so proved. □

Theorem 1.7 is now an immediate consequence of Corollary 2.7 and [10, Lemma 4.1] (see also [6, Lemma 5.2]).

3. Proof of Theorem 1.8

Assume that $u \geq \epsilon > 0$. We recall that for $x_0 \in \Omega$; we denote $B_s = B(x_0, s)$.

Insert in (1.26) as test function $\varphi^p u^{-p+1} \in D_0[\Omega]$, where φ is a cut-off function between the balls B_r and B_{2r}, where $r \leq R_0$ and $B_{2r} \subset \Omega$. Then

$$\int_{B_{2r}} \mu(u, \varphi^p u^{-p+1})(dx) + \int_{B_{2r}} \varphi^p \sigma(dx) = 0. \tag{3.1}$$

Then

$$-\int_{B_{2r}} \varphi^p \mu(u, u^{-p+1})(dx) \leq p\int_{B_{2r}} \varphi^{p-1}u^{-p+1}|\mu|(u,\varphi)(dx) + \int_{B_{2r}} \varphi^p |\sigma|(dx). \tag{3.2}$$

We have

$$-\int_{B_{2r}} \varphi^p \mu(u,u^{-p+1})(dx) \geq p(p-1)\int_{B_{2r}} \varphi^p u^{-p}\alpha(u)(dx), \tag{3.3}$$

$$\begin{aligned}\int_{B_{2r}} \varphi^{p-1}u^{-p+1}|\mu|(u,\varphi)(dx) &\leq \epsilon\int_{B_{2r}} \varphi^p u^{-p}\alpha(u)(dx) + C(\epsilon)\int_{B_{2r}} \alpha(\varphi)(dx)\\ &\leq \epsilon\int_{B_{2r}} \varphi^p \alpha(\log u)(dx) + C(\epsilon)\frac{m(B_{2r})}{r^p},\end{aligned} \tag{3.4}$$

$$\int_{B_{2r}} \varphi^p |\sigma|(dx) \leq C\left[\frac{|\sigma|(B_{2r})}{m(B_{2r})}(2r)^p\right]\frac{m(B_r)}{r^p}. \tag{3.5}$$

Merging now (3.3)–(3.5) into (3.2) and taking into account the doubling inequality, we obtain

$$\int_{B_r} \alpha(\log u)(dx) \leq C\left[1 + \frac{|\sigma|(B_{2r})}{m(B_{2r})}(2r)^p\right]\frac{m(B_r)}{r^p}, \tag{3.6}$$

where C is a structural constant. Since $\sigma \in K(\Omega)$, the term $(|\sigma|(B_{2r})/m(B_{2r}))(2r)^p$ is bounded. Assume $B_{2kr} \subset \Omega$, using Poincaré inequality, we obtain

$$\frac{1}{m(B_r)} \int_{B_r} |\log u - (\log u)_r|^p m(dx) \leq C. \tag{3.7}$$

As in [6, Proposition 5.7] we deduce from (3.7) that there exists a positive constant $\overline{q}$ such that

$$\left(\frac{1}{m(B_r)} \int_{B_r} u^{\overline{q}} m(dx)\right)\left(\frac{1}{m(B_r)} \int_{B_r} u^{-\overline{q}} m(dx)\right) \leq C. \tag{3.8}$$

We observe now that $1/u$ is a positive subsolution of (1.26). Then taking into account Theorem 1.7 and (3.8), we prove the result as in [20] (concerning the case $\sigma = 0$). Finally we apply the above part of the proof to $u+\epsilon$, and taking $\epsilon \to 0$, we conclude the proof.

4. Proof of Theorem 1.9

Let u be a local solution of (1.26). From Theorem 1.7 applied to u^+ and u^- we obtain that u is locally bounded in Ω. Then $|u|^{p-2}u\mu$ is locally a measure in $K(\Omega)$. So the result follows from [10, Theorem 1.2].

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References

[1] M. Aizenman and B. Simon, "Brownian motion and Harnack inequality for Schrödinger operators," *Communications on Pure and Applied Mathematics*, vol. 35, no. 2, pp. 209–273, 1982.

[2] F. Chiarenza, E. Fabes, and N. Garofalo, "Harnack's inequality for Schrödinger operators and the continuity of solutions," *Proceedings of the American Mathematical Society*, vol. 98, no. 3, pp. 415–425, 1986.

[3] G. Citti, N. Garofalo, and E. Lanconelli, "Harnack's inequality for sum of squares of vector fields plus a potential," *American Journal of Mathematics*, vol. 115, no. 3, pp. 699–734, 1993.

[4] M. Biroli, "Weak Kato measures and Schrödinger problems for a Dirichlet form," *Rendiconti della Accademia Nazionale delle Scienze detta dei XL. Memorie di Matematica e Applicazioni. Serie V. Parte I*, vol. 24, pp. 197–217, 2000.

[5] M. Biroli and U. Mosco, "Sobolev inequalities on homogeneous spaces," *Potential Analysis*, vol. 4, no. 4, pp. 311–324, 1995.

[6] M. Biroli and U. Mosco, "A Saint-Venant type principle for Dirichlet forms on discontinuous media," *Annali di Matematica Pura ed Applicata. Serie Quarta*, vol. 169, no. 1, pp. 125–181, 1995.

[7] M. Biroli, "Nonlinear Kato measures and nonlinear subelliptic Schrödinger problems," *Rendiconti della Accademia Nazionale delle Scienze detta dei XL. Memorie di Matematica e Applicazioni. Serie V. Parte I*, vol. 21, pp. 235–252, 1997.

[8] J. Malý, "Pointwise estimates of nonnegative subsolutions of quasilinear elliptic equations at irregular boundary points," *Commentationes Mathematicae Universitatis Carolinae*, vol. 37, no. 1, pp. 23–42, 1996.

[9] J. Malý and W. P. Ziemer, *Fine Regularity of Solutions of Elliptic Partial Differential Equations*, vol. 51 of *Mathematical Surveys and Monographs*, American Mathematical Society, Rhode Island, 1997.

[10] M. Biroli and S. Marchi, "Oscillation estimates relative to p-homogeneous forms and Kato measures data," to appear in Le Matematiche.

[11] M. Biroli, "Strongly local nonlinear Dirichlet functionals and forms," to appear in Rendiconti della Accademia Nazionale delle Scienze detta dei XL. Memorie di Matematica e Applicazioni.

[12] M. Biroli and P. G. Vernole, "Strongly local nonlinear Dirichlet functionals and forms," *Advances in Mathematical Sciences and Applications*, vol. 15, no. 2, pp. 655–682, 2005.

[13] M. Fukushima, Y. Ōshima, and M. Takeda, *Dirichlet Forms and Symmetric Markov Processes*, vol. 19 of *de Gruyter Studies in Mathematics*, Walter de Gruyter, Berlin, 1994.

[14] R. R. Coifman and G. Weiss, *Analyse Harmonique Non-Commutative sur Certains Espaces Homogènes*, vol. 242 of *Lecture Notes in Mathematics*, Springer, Berlin, 1971.

[15] J. Malý and U. Mosco, "Remarks on measure-valued Lagrangians on homogeneous spaces," *Ricerche di Matematica*, vol. 48, no. suppl., pp. 217–231, 1999.

[16] T. Kato, "Schrödinger operators with singular potentials," *Israel Journal of Mathematics*, vol. 13, pp. 135–148 (1973), 1972.

[17] M. Biroli and U. Mosco, "Kato space for Dirichlet forms," *Potential Analysis*, vol. 10, no. 4, pp. 327–345, 1999.

[18] M. Biroli, "Schrödinger type and relaxed Dirichlet problems for the subelliptic p-Laplacian," *Potential Analysis*, vol. 15, no. 1-2, pp. 1–16, 2001.

[19] M. Biroli and N. A. Tchou, "Nonlinear subelliptic problems with measure data," *Rendiconti della Accademia Nazionale delle Scienze detta dei XL. Memorie di Matematica e Applicazioni. Serie V. Parte I*, vol. 23, pp. 57–82, 1999.

[20] M. Biroli and P. Vernole, "Harnack inequality for harmonic functions relative to a nonlinear p-homogeneous Riemannian Dirichlet form," *Nonlinear Analysis*, vol. 64, no. 1, pp. 51–68, 2006.

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