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Research Article

Subsolutions of Elliptic Operators in Divergence Form and Application to Two-Phase Free Boundary Problems

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Let L be a divergence form operator with Lipschitz continuous coefficients in a domain Ω, and let u be a continuous weak solution of $Lu = 0$ in $\{u \neq 0\}$. In this paper, we show that if ϕ satisfies a suitable differential inequality, then $v_\phi(x) = \sup_{B_{\phi(x)}(x)} u$ is a subsolution of $Lu = 0$ away from its zero set. We apply this result to prove $C^{1,\gamma}$ regularity of Lipschitz free boundaries in two-phase problems.

1. Introduction and main results

In the study of the regularity of two-phase elliptic and parabolic problems, a key role is played by certain continuous perturbations of the solution, constructed as supremum of the solution itself over balls of variable radius. The crucial fact is that if the radius satisfies a suitable differential inequality, modulus a small correcting term, the perturbations turn out to be subsolutions of the problem, suitable for comparison purposes.

This kind of subsolutions have been introduced for the first time by Caffarelli in the classical paper [1] in order to prove that, in a general class of two-phase problems for the laplacian, Lipschitz free boundaries are indeed $C^{1,\alpha}$.

This result has been subsequently extended to more general operators: Feldman [2] considers linear anisotropic operators with constant coefficients, Wang [3] a class of concave fully nonlinear operators of the type $F(D^2u)$, and again Feldman [4] fully nonlinear operators, not necessary concave, of the type $F(D^2u, Du)$. In [5], Cerutti et al. consider variable coefficients operators in nondivergence form and Ferrari [6] a class of fully nonlinear operators $F(D^2u, x)$, Hölder continuous in the space variable.

The important case of linear or semilinear operators in divergence form with nonsmooth coefficients (less than $C^{1,\alpha}$, e.g.) is not included in the above results and it is precisely the subject of this paper. Once again, the key point is the construction of the previously mentioned family of subsolutions. Unlike the case of nondivergence or fully nonlinear operators, in the case of divergence form operators, the construction turns out to be rather delicate due to the fact that in this case not only the quadratic part of a function controls in average the action of the operator but also the linear part has an equivalent influence. Here we require Lipschitz continuous coefficients.

To state our first result we introduce the class $\mathcal{L}(\lambda,\Lambda,\omega)$ of elliptic operators

$$L = \operatorname{div}\left(A(x)\nabla\right) \tag{1.1}$$

defined in a domain $\Omega \subset \mathbb{R}^n$, with symmetric and uniformly elliptic matrix, that is,

$$A(x) = A^{\top}(x), \qquad \lambda I \leq A^s(x) \leq \Lambda I \tag{1.2}$$

and modulus of continuity of the coefficients given by

$$\omega(r) = \sup_{|x-y|\leq r} \left|A(x) - A(y)\right|. \tag{1.3}$$

Theorem 1.1. *Let u be a continuous function in Ω. Assume that in $\{u>0\}$ u is a C^2-weak solution of $Lu=0$, $L \in \mathcal{L}(\lambda,\Lambda,\omega)$, $\omega(r) \leq c_0 r$. Let ϕ be a positive C^2-function such that $0 < \phi_{\min} \leq \phi \leq \phi_{\max}$ and*

$$v_\phi(x) = \sup_{B_{\phi(x)}(x)} u = \sup_{|\nu|=1} u(x+\phi(x)\nu) \tag{1.4}$$

is well defined in Ω. There exist positive constants $\mu_0 = \mu_0(n,\lambda,\Lambda)$ and $C = C(n,\lambda,\Lambda)$, such that, if $|\nabla\phi| \leq \mu_0$, $\omega_0 = \omega(\phi_{\max})$, and

$$\phi L\phi \geq C\left(\left|\nabla\phi(x)\right|^2 + \omega_0^2\right), \tag{1.5}$$

then v is a weak subsolution of $Lu=0$ in $\{v>0\}$.

We now introduce the class of free boundary problems we are going to study and the appropriate notion of weak solution.

Let $B_R' = B_R'(0)$ be the ball of radius R in $\mathbb{R}^{n-1}$. In $\mathscr{C}_R = B_R'(0) \times (-R,R)$ we are given a continuous H^1_{loc} function u satisfying the following.

(i)

$$Lu = \operatorname{div}\left(A(x)\nabla u\right) = 0 \tag{1.6}$$

in $\Omega^+(u) = \{x \in \mathscr{C}_R : u(x) > 0\}$, and in $\Omega^-(u) = \{x \in \mathscr{C}_R : u(x) \leq 0\}^0$, in the weak sense.

We call $F(u) \equiv \partial\Omega^+(u) \cap \mathscr{C}_R$ the *free boundary*. We say that a point $x_0 \in F(u)$ is *regular* from the right (left) if there exists a ball B:

$$\begin{gathered} B \subset \Omega^+(u) \quad (\subset \Omega^-(u), \text{ resp.}), \\ B \cap F(u) = \{x_0\}. \end{gathered} \tag{1.7}$$

(ii) Along $F(u)$ the following conditions hold:

(a) if $x_0 \in F(u)$ is regular from the right, then, near x_0,

$$u^+(x) \geq \alpha\langle x - x_0, \nu\rangle^+ - \beta\langle x - x_0, \nu\rangle^- + o(|x - x_0|), \tag{1.8}$$

for some $\alpha > 0$, $\beta \geq 0$ with equality along every nontangential domain in both cases, and

$$\alpha \leq G(\beta); \tag{1.9}$$

(b) if $x_0 \in F(u)$ is regular from the left, then, near x_0,

$$u^+(x) \leq \alpha\langle x - x_0, \nu\rangle^+ - \beta\langle x - x_0, \nu\rangle^- + o(|x - x_0|), \tag{1.10}$$

for some $\alpha \geq 0$, $\beta > 0$ with equality along every nontangential domain in both cases, and

$$\alpha \geq G(\beta). \tag{1.11}$$

The conditions (a) and (b), where ν denotes the unit normal to ∂B at x_0, towards the positive phase, express the free boundary relation $u_\nu^+ = G(u_\nu^-)$ in a weak sense; accordingly, we call u a weak solution of *f.b.p.*

Via an approximation argument it is possible to show that Theorem 1.1 holds for the positive and negative parts of a solution of our f.b.p.

Here are our main results concerning the regularity of Lipschitz free boundaries.

Theorem 1.2. *Let u be a weak solution to f.b.p. in $\mathscr{C}_R = B_R' \times (-R, R)$.*
Suppose that $0 \in F(u)$ and that

(i) *$L \in \mathscr{L}(\lambda, \Lambda, \omega)$;*

(ii) *$\Omega^+(u) = \{(x', x_n) : x_n > f(x')\}$ where f is a Lipschitz continuous function with $Lip(f) \leq l$;*

(iii) *$G = G(z)$ is continuous, strictly increasing and for some $N > 0$, $z^{-N}G(z)$ is decreasing in $(0, +\infty)$.*

Then, on $B_{R/2}'$, f is a $C^{1,\gamma}$ function with $\gamma = \gamma(n, l, N, \lambda, \Lambda, \omega)$.

By using of the monotonicity formula in [7] we can prove the following.

Corollary 1.3. *In f.b.p. let*

$$Lu = \operatorname{div}(A(x, u)\nabla u), \tag{1.12}$$

where L is a uniformly elliptic divergence form operator. Assume (ii) and (iii) in Theorem 1.2 hold and replace (i) with the assumption that A is Lipschitz continuous with respect to x and u. Then the same conclusion holds.

We can allow a dependence on x and ν in the free boundary condition for $G = G(\beta, x, \nu)$ assuming instead of (iii) in Theorem 1.1

(iii′) $G = G(z,\nu,x)$ is continuous strictly increasing in z and, for some $N > 0$ independent of ν and x, $z^{-N}G(z,\nu,x)$ is decreasing in $(0,\infty)$;

(iii″) $\log G$ is Lipschitz continuous with respect to ν, x, uniformly with respect to its first argument $z \in [0,\infty)$.

The proof of Theorem 1.2 goes along well-known guidelines and consists in the following three steps: to improve the Lipschitz constant of the level sets of u far from $F(u)$, to carry this interior gain to the free boundary, to rescale and iterate the first two steps. This procedure gives a geometric decay of the Lipschitz constant of $F(u)$ in dyadic balls that corresponds to a $C^{1,\gamma}$ regularity of $F(u)$ for a suitable γ.

The first step follows with some modifications [5, Sections 2 and 3] and everything works with Hölder continuous coefficients. We will describe the relevant differences in Section 2.

The second step is the crucial one. At difference with [5] we use the particular structure of divergence and the fact that weak sub- (super-) solutions of operators in divergence form with Hölder coefficients can be characterized pointwise, through lower (super) mean properties with respect to a base of regular neighborhoods of a point, involving the L-harmonic measure. Section 3 contains the proof of the main result, Theorem 1.1, and some consequences.

In Section 4 the above results are applied to our free boundary problem, preparing the necessary tools for the final iteration.

The third step can be carried exactly as in [5, Sections 6 and 7], since here the particular form of the operator does not play any role anymore. Actually the linear modulus of continuity allows some simplifications.

2. Monotonicity properties of weak solutions

In this section we assume that $\omega(r) \le c_0 r^a$, $0 < a \le 1$. Let $u \in H^1_{\text{loc}}(\Omega)$ be a weak solution of $Lu = 0$ in Ω, that is,

$$\int_\Omega \langle A(x)\nabla u(x), \nabla\varphi(x)\rangle\, dx = 0, \tag{2.1}$$

for every test function φ supported in Ω. If $L \in \mathscr{L}(\lambda,\Lambda,\omega)$, $u \in C^{1,a}(\Omega)$.

In this section we prove that if the domain Ω is Lipschitz and u vanishes on a relatively open portion $F \subset \partial\Omega$, then, near F, the level sets of u are uniformly Lipschitz surfaces.

Precisely, we consider domains of the form

$$T_s = \{(x',x_n) \in \mathbb{R} : |x'| < s,\ f(x') < x_n < 2ls\}, \tag{2.2}$$

where f is a Lipschitz function with constant l.

Theorem 2.1. *Let u be a positive solution to $Lu = 0$ in T_4, vanishing on $F = \{x_n = f(x')\} \cap \partial T_4$. Then, there exists η such that in*

$$\mathcal{N}_\eta(F) = \{f(x') < x_n < f(x') + \eta\} \cap T_1, \tag{2.3}$$

u is increasing along the directions τ belonging to the cone $\Gamma(e_n,\theta)$, with axis e_n and opening $\theta = (1/2)\cot^{-1} l$. Moreover, in $\mathcal{N}_\eta(F)$,

$$c^{-1}\frac{u(x)}{d_x} \le D_n u(x) \le c\frac{u(x)}{d_x}, \tag{2.4}$$

where $d_x = \operatorname{dist}(x,F)$.

Proof of Theorem 2.1. Let z be the solution of the Dirichlet problem

$$\begin{gathered} \operatorname{div}\left(A(0)\nabla z(x)\right) = 0, \qquad T_2, \\ z = g, \qquad \partial T_2, \end{gathered} \tag{2.5}$$

where g is a smooth function vanishing on $\mathcal{F}$ and equal to 1 at points x with $d_x > 1/10$.

Then, see [1], $D_n z > 0$ in Q_ρ, with $\rho = \rho(n,l)$. By rescaling the problem (if necessary), we may assume $\rho = 3/2$. Since $z(e_n) \ge c > 0$, by Harnack inequality we have that, if $y \in T_1$, $d_y \ge \eta_0$,

$$z(y) \sim c(\eta_0), \qquad D_n z(y) \sim \frac{z(y)}{d_y} \sim c(\eta_0). \tag{2.6}$$

Clearly $z,u \in C^{0,a}(\overline{T}_2)$. □

Lemma 2.2. *For $r > 0$, let w_r be the $C^{1,a}(T_2)$ weak solution to*

$$\begin{gathered} \operatorname{div}\left(A(rx)\nabla w_r(x)\right) = 0, \qquad T_2, \\ w_r = z, \qquad \partial T_2. \end{gathered} \tag{2.7}$$

Then, given $\eta_0 > 0$, there exists $r_0 = r_0(\eta_0)$, such that if $r \le r_0$,

$$D_n w_r(y) \ge 0 \tag{2.8}$$

for every $y \in T_1$, with $d_y \ge \eta_0$.

Proof. Let

$$\operatorname{div}\left(A(rx)\left(\nabla w_r(x) - \nabla z(x)\right)\right) = \operatorname{div}\left(\left(A(rx) - A(0)\right)\nabla z(x)\right). \tag{2.9}$$

For every $\sigma > 0$, let

$$Q_2^\sigma = \{x \in T_2, \operatorname{dist}(x,\partial T_2) > \sigma\}. \tag{2.10}$$

Notice that $h_r = w_r - z \in C^{0,a}(\overline{T}_2)$, and moreover $h_r \in C^{1,a}(Q_2^\sigma)$. Notice that $((A(rx) - A(0))\nabla z(x))_i \in L^\infty(Q_2^\sigma)$, and from standard estimates we have

$$\sup_{Q_2^\sigma} |w_r - z| \le \sup_{\partial Q_2^\sigma} |h_r| + C|Q_2^\sigma|^{1/n}\omega(r)||\nabla z||_{L^\infty(Q_2^\sigma)}. \tag{2.11}$$

Hence

$$\sup_{Q_2^\sigma} |h_r| \le c\left(\sigma^a + \frac{r}{\sigma}\right). \tag{2.12}$$

Choosing $r = \sigma^{1+a}$ we get that for every $y \in Q_2^\sigma$,

$$|h_r(y)| \le cr^\beta, \tag{2.13}$$

where $\beta = a/(1+a)$.

Let $\overline{y} \in T_1$, with $d_{\overline{y}} \ge \eta_0$, $r_0 \le (1/3)\eta_0^{1/(1+a)}$, and $\rho = \eta_0/3$. It follows that

$$\begin{aligned}\rho|D_n h_r(\overline{y})| &\le C(r^a z(\overline{y}) + r\|z\|_{L^\infty(B_\rho(\overline{y}))}) \le C(r^\beta + r)z(\overline{y}) \\ &= C\rho(r^\beta + r)\frac{z(\overline{y})}{\rho} \le C\rho r^\beta D_n z(\overline{y}).\end{aligned} \tag{2.14}$$

Hence if $r \le r_0 = \min\{(2c(\eta_0))^{-1/\beta}, (1/3)\eta_0^{a+1}\}$, we get

$$\frac{1}{2}D_n z(\overline{y}) \le D_n w_r(\overline{y}) \le \frac{3}{2}D_n z(\overline{y}). \tag{2.15}$$

□

The following two lemmas are similar to [5, Lemmas 2 and 3], respectively.

Lemma 2.3. *Let $\eta_0 > 0$ be fixed and w and z as in Lemma 2.2. Then there exist $r_0 = r_0(\eta_0)$ and $t_0 = t_0(\lambda, \Lambda, n) > 1$ such that, if $r \le r_0$,*

$$c^{-1}\frac{w(y)}{d_y} \le D_n w(y) \le c\frac{w(y)}{d_y} \tag{2.16}$$

for every $y \in T_1$, $d_y \ge t_0\eta_0$.

Proof. The right-hand side inequality follows Schauder's estimates and Harnack inequality. Let now $y \in T_1$, with $d_y \ge t_0\eta_0$, t_0 to be chosen. We may assume $y = t\eta_0 e_n$. From the boundary Harnack principle (see, e.g., [8] or [9]) if $\tilde{y} = \eta_0 e_n$, then

$$z(\tilde{y}) \le ct^{-a}z(y) \tag{2.17}$$

and, if $t \ge (2c)^{1/a} \equiv t_0$, then

$$z(\tilde{y}) \le \frac{1}{2}z(y). \tag{2.18}$$

On the other hand, if $d_y \ge t_0\eta_0$ and $r \le r_0(\eta_0)$, from (2.6), (2.13), and (2.15) we have

$$w(y) \le \frac{3}{2}z(y), \qquad D_n w(y) \ge \frac{1}{2}D_n z(y). \tag{2.19}$$

Thus, if $t_0\eta_0 \le d_y \le 10t_0\eta_0$, applying Harnack inequality to $D_n z$, we get

$$\begin{aligned} w(y) &\le \frac{3}{2}z(y) \le 3\big(z(y) - z(\widetilde{y})\big) = 3\int_0^1 \frac{d}{ds} z\big(sy + (1-s)\widetilde{y}\big)\,ds \\ &\le ct_0\eta_0 D_n z(y) \le cD_n z(y) d_y \le cD_n w(y) d_y. \end{aligned} \tag{2.20}$$

Repeating the argument with $\widetilde{y} = 10t_0\eta_0$, we get that (2.18) holds for $10t_0\eta_0 \le d_y \le 20t_0\eta_0$. After a finite number of steps, (2.18) follows for $d_y \ge t_0\eta_0$, $y \in T_1$. □

Lemma 2.4. *Let u be as in Theorem 2.1. Then there exists a positive η, such that for every $x \in T_1$, $d_x \le \eta$,*

$$D_n u(x) \ge 0. \tag{2.21}$$

Moreover, in the same set

$$c^{-1}\frac{u(x)}{d_x} \le D_n u(x) \le c\frac{u(x)}{d_x}. \tag{2.22}$$

Proof. Let t_0 be as in Lemma 2.3, and η_0 small to be chosen later. Set $\eta_1 = 2\eta_0 t_0$. It is enough to show that if $\overline{x} = \eta_1 r e_n$ and $r \le r_0(\eta_0)$, then $D_n u(\overline{x}) \ge 0$. Consider a small box T_{2r} and define $\widetilde{u}(y) = u(ry)$. Then $\widetilde{u}$ satisfies $\operatorname{div}(\widetilde{A}(x)\nabla\widetilde{u}(x)) \equiv \operatorname{div}(A(rx)\nabla\widetilde{u}(x)) = 0$ in T_2, where f is replaced by $f_r(y') = f(ry')/r$.

We will show that $D_n\widetilde{u}(\overline{y}) > 0$, where $\overline{y} = \eta_1 e_n$, by comparing $\widetilde{u}$ with the function w constructed in Lemma 2.2, normalized in order to get $\widetilde{u}(\overline{y}) = w(\overline{y})$. Notice that if we choose $r_0 = r_0(\eta_0)$ according to Lemma 2.3, we have

$$c^{-1}\frac{w(y)}{d_y} \le D_n w(y) \le c\frac{w(y)}{d_y}. \tag{2.23}$$

If $d_y \ge \eta_1$. From the comparison theorem (see [8] or [9]), we know that $\widetilde{u}/w \in C^{0,a}(\overline{T}_{3/2})$ so that in $B_{\eta_0}(\overline{y})$

$$\left|\frac{\widetilde{u}(y)}{w(y)} - 1\right| \le c\eta_0^a, \tag{2.24}$$

which implies

$$\big|\widetilde{u}(y) - w(y)\big| \le c\eta_0^a w(y) \le c\eta_0^a w(\overline{y}). \tag{2.25}$$

Moreover, since $\eta_0 \sim d_{\overline{y}}$,

$$\big|D_n\widetilde{u}(y) - D_n w(y)\big| \le c\eta_0^{a-1} w(\overline{y}) \le c\eta_0^a D_n w(\overline{y}), \tag{2.26}$$

from which we get

$$D_n\widetilde{u}(\overline{y}) \ge \big(1 - c\eta_0^a\big) D_n w(\overline{y}), \tag{2.27}$$

and (2.21) holds if η_0 is sufficiently small. Inequality (2.22) is now a consequence of (2.23) and the fact that $w(\overline{y}) = u(\overline{y})$. □

To complete the proof of Theorem 2.1, it is enough to observe that the above lemmas hold if we replace e_n by any unit vector τ such that the angle between τ and e_n is less than $\theta = 1/2\cot^{-1} l$.

Thus, we obtain a cone $\Gamma(e_n,\theta)$ of monotonicity for u. Applying Theorem 2.1 to the positive and negative parts of the solution u of our free boundary problem, we conclude that in a η-neighborhood of $F(u)$ the function u is increasing along the direction of a cone $\Gamma(e_n,\theta)$. Far from the free boundary, the monotonicity cone can be enlarged improving the Lipschitz constant of the level sets of u.

This is a consequence of the following strong Harnack principle, where the cone $\Gamma'(e_n,\theta)$ is obtained from $\Gamma(e_n,\theta)$ by deleting the "bad" directions, that is, those in a neighborhood of the generatrix opposite to $\nabla u(e_n)$. Precisely, if $\tau \in \Gamma(e_n,\theta)$, denote by ω_τ the solid angle between the planes $\operatorname{span}\{e_n,\nabla\}$ and $\operatorname{span}\{e_n,\tau\}$. Delete from $\Gamma(e_n,\theta)$ the directions τ such that (say) $\omega_\tau \geq (99/100)\pi$ and call $\Gamma'(e_n,\theta)$ the resulting set of directions. If $\tau \in \Gamma'(e_n,\theta)$, then

$$\langle \nabla, \tau \rangle \geq c_3\delta, \tag{2.28}$$

where $\delta = \pi/2 - \theta$. We call δ the *defect angle*.

Lemma 2.5. *Suppose u is a positive solution of $\operatorname{div}(A(rx)\nabla u(x)) = 0$ in T_4 vanishing on $F = \{x_n = f(x')\}$, increasing along every $\tau \in \Gamma(e_n,\theta)$. Assume furthermore that (2.4) holds in T_4. There exist positive r_0 and h, depending only on n, l, and λ, Λ, such that if $r \leq \overline{r}_0$, for every small vector τ, $\tau \in \Gamma'(e_n,\theta/2)$, and for every $x \in B_{1/8}(e_n)$,*

$$\sup_{B_{(1+h\delta)\epsilon}(x)} u(y-\tau) \leq u(x) - C\epsilon\delta u(e_n), \tag{2.29}$$

where $\epsilon = |\tau|\sin(\theta/2)$.

For the proof see [5, Section 3].

Corollary 2.6. *In $B_{1/8}(x_0)$, u is increasing along every $\tau \in \Gamma(\overline{\tau}_1,\overline{\theta}_1)$ with*

$$\begin{gathered} \overline{\delta}_1 \leq \overline{b}\delta, \quad \left(\overline{\delta}_1 = \frac{\pi}{2} - \overline{\theta}_1\right), \\ \left|\overline{\nu}_1 - e_1\right| \leq C\delta, \end{gathered} \tag{2.30}$$

where $\overline{b} = \overline{b}(n,a,l,\lambda,\Lambda) < 1$.

We now apply the above results to the solution of our free boundary problem in a properly chosen neighborhood of the origin. Precisely, set for the moment

$$s = \frac{1}{2}\min\{\overline{r}_0,\eta\}, \tag{2.31}$$

with η as in Theorem 2.1 and $\overline{r}_0$ as in Lemma 2.5. If we define

$$u_s(x) = \frac{u(sx)}{s}, \tag{2.32}$$

then u_s^+ satisfies $L_s u_s^+ \equiv Lu_s(sx) = 0$ in T_4 and falls under the hypothesis of Lemma 2.5. Therefore, rescaling back we get the following result.

Theorem 2.7. *Let u be a solution of our free boundary problem. Then in $B_{s/8}(se_n)$,*

$$\sup_{B_{(1+h\delta)\epsilon}(x)} u(y-\tau) \leq u(x) - c\epsilon\delta u(e_n) \tag{2.33}$$

for every $\tau \in \Gamma'(e_n, \theta/2)$, $|\tau| \ll s$. As a consequence, in $B_{s/8}(se_n)$, u is monotone along every $\tau \in \Gamma(\overline{\nu}_1, \overline{\theta}_1)$, where $\overline{\nu}_1$, $\overline{\theta}_1$ satisfy (2.30).

3. Proof of the main theorem

Before proving Theorem 1.1, we need to introduce some notations and to recall a pointwise characterization of weak subsolutions.

If $\mathbb{O} \subset \Omega$ is an open set, regular for the Dirichlet problem, we denote by $G_{\mathbb{O}} = G_{\mathbb{O}}(x,y)$ the Green function associated with the operator L in $\mathbb{O}$ and by $\omega_{\mathbb{O}}^x$ the L-harmonic measure for L in $\mathbb{O}$. In this way,

$$w(x) = \int_{\mathbb{O}} g\, d\omega_{\mathbb{O}}^x - \int_{\mathbb{O}} G_{\mathbb{O}}(x,y)h(y)\,dy \tag{3.1}$$

is the unique weak solution of $Lu = h$ in $\mathbb{O}$, $h = 0$ on $\partial\mathbb{O}$.

A function $v \in H^1(\Omega)$ is a weak subsolution in Ω if

$$\int_{\Omega} \langle A(x)\nabla u(x), \nabla\phi(x)\rangle\, dx \leq 0 \tag{3.2}$$

for every nonnegative test function φ supported in Ω, while u is a weak supersolution in Ω if $-u$ is a weak subsolution.

We need to recall a pointwise characterization. Indeed, see [10–14] for the details, we say that a function $v : \Omega \to \mathbb{R}$ is L-subharmonic in a set Ω if it is upper semicontinuous in Ω, locally upper bounded in Ω, and

(S) for every $x_0 \in \Omega$ there exists a basis of regular neighborhood $\mathcal{B}_{x_0}$ associated with v such that for every $B \in \mathcal{B}_{x_0}$,

$$v(x_0) \leq \int_{\partial B} v(\sigma)\, d\omega_B^{x_0}. \tag{3.3}$$

A function v is L-superharmonic if $-v$ is L-subharmonic. Thus u is L-harmonic, or simply harmonic, whenever it is both L-subharmonic and L-superharmonic.

With such pointwise characterization, the definition of the Perron-Wiener-Brelot solution of the Dirichlet problem can be stated as usual, see [10] or [11]. The Perron-Wiener-Brelot solution of the Dirichlet problem coincides, in any reasonable case, with the solution of the Dirichlet given by the variational approach. In general, L-subharmonic functions and such subsolutions do not coincide. On the other hand, if v is locally Lipschitz, v is L-subharmonic if and only if v is locally a subsolution.

Precisely, see [12, 13], if f is the trace on $\partial\Omega$ of a function $\tilde{f} \in C(\overline{\Omega}) \cap H^1(\Omega)$, then the weak solution of the Dirichlet problem (even if L has just bounded measurable coefficients)

$$\begin{aligned} Lu &= 0 \quad \text{in } \Omega, \\ u &= f \quad \text{on } \partial\Omega \end{aligned} \tag{3.4}$$

and the parallel Perron-Wiener-Brelot one coincide. Moreover, in [15] Hervé also proved that the same result holds when $\tilde{f}$ is L-subharmonic and $\tilde{f} \in H^1_{\mathrm{loc}}(\Omega)$.

Lemma 3.1. *Let $C > 2$ and ϕ be a C^2 weak solution of*

$$\operatorname{div}(A(x)\nabla\phi(x)) \geq C\frac{|\nabla\phi(x)|2 + \omega_0^2}{\phi(x)} \equiv \Phi(x) \tag{3.5}$$

in Ω, $0 < \phi_{\min} \leq \phi \leq \phi_{\max}$. Then for any $x \in \Omega$ there exists a positive number $\overline{r}(x, \phi_{\max}, \phi_{\min}, C)$ such that, for every $r < \overline{r}(x)$ and every ball $B_r = B_r(x) \subset \Omega$,

$$\int_{\partial B_r}\left[\langle \sigma - x, \nabla\phi(x)\rangle + \frac{1}{2}\langle \mathcal{D}^2\phi(x)(\sigma - x), (\sigma - x)\rangle - \Phi(\sigma)\right]d\omega_B^x(\sigma) \geq 0. \tag{3.6}$$

Proof. From Lemma A.5, for every ball $B_r = B_r(x) \subset \Omega$,

$$\begin{aligned} \nabla\phi(x)\int_{\partial B_r}(\sigma - x)d\omega_B^x(\sigma) + \frac{1}{2}\sum_{i,j=1}^{n} D_{ij}\phi(x)\int_{\partial B_r}(\sigma_i - x_i)(\sigma_j - x)d\omega_B^x(\sigma) \\ \geq \int_{B_r} G_{B_r}(x,y)\Phi(y)dy + o(r^2), \end{aligned} \tag{3.7}$$

the proof follows easily. □

We are now ready for the proof of the main theorem.

Proof of Theorem 1.1. We have

$$v_\phi(x) = u(x + \phi(x)\eta(x)), \tag{3.8}$$

for some $\eta(x)$, where $|\eta(x)| = 1$. To prove that v_ϕ is an L-subsolution we just check condition (S), since by straightforward calculations v_ϕ is locally Lipschitz continuous. In particular we will prove that for every $x \in \Omega^+(v)$ there exists a positive constant $r_0 = r_0(x)$ such that for every ball $B_r \equiv B_r(x) \subset \Omega^+(v)$, $r \leq r_0$, and for every $x_0 \in B_r$,

$$\int_{\partial B_r} v_\phi(\sigma)d\omega_{B_r}^{x_0}(\sigma) \geq v_\phi(x_0). \tag{3.9}$$

Let $\{e_1, \dots, e_n\}$ be an orthonormal basis of $\mathbb{R}^n$ where $e_n = \eta(x)$ and let ξ be the following vectorfield:

$$\xi(h) = e_n + \sum_{i=1}^{n-1}\langle V_i, h\rangle e_i, \tag{3.10}$$

where $\{V_1,\dots,V_{n-1}\}\subset\mathbb{R}^n$ will be chosen later. Let $\nu(h)=\xi(h)/|\xi(h)|$, so that

$$\nu(h)=e_n+\sum_{i=1}^{n-1}\langle V_i,h\rangle e_i-\frac{1}{2}\sum_{i=1}^{n-1}\langle V_i,h\rangle^2 e_n+o(|h|^2). \tag{3.11}$$

Let $x_0\in B_r(x)$ and $h=\sigma-x_0$. Then, letting

$$\phi_0=\phi(x_0),\qquad \nabla\phi_0=\nabla\phi(x_0),\qquad \mathcal{D}^2\phi_0=\mathcal{D}^2\phi(x_0), \tag{3.12}$$

we have

$$\phi(\sigma)=\phi_0+\langle\nabla\phi_0,h\rangle+\frac{1}{2}\langle\mathcal{D}^2\phi_0 h,h\rangle+o(|h|^2) \tag{3.13}$$

as $h\to 0$, uniformly in a neighborhood of x. As a consequence,

$$\sigma+\phi(\sigma)\nu(\sigma-x_0)=y^*+J_1+J_2+J_3, \tag{3.14}$$

where $y^*=x_0+\phi(x_0)e_n$,

$$\begin{gathered}
J_1=\left[\langle\nabla\phi_0,h\rangle e_n+h+\phi_0\sum_{i=1}^{n-1}\langle V_i,h\rangle e_i\right],\\
J_2=\left[\langle\nabla\phi_0,h\rangle\sum_{i=1}^{n-1}\langle V_i,h\rangle e_i+\frac{1}{2}\langle\mathcal{D}^2\phi_0 h,h\rangle e_n-\frac{\phi_0}{2}\sum_{i=1}^{n-1}\langle V_i,h\rangle^2 e_n\right],\\
J_3=o(|h|^2)
\end{gathered} \tag{3.15}$$

uniformly as $h\to 0$.

Let $J=J_1+J_2+J_3$. Then for every $\sigma\in\partial B_r(x_0)$,

$$v(\sigma)\geq u(y^*+J)=u(y^*)+\langle\nabla u(y^*),J\rangle+\frac{1}{2}\langle\mathcal{D}^2u(y^*)J,J\rangle+o(|h|^2), \tag{3.16}$$

as $h\to 0$, uniformly in a neighborhood of y^*. We have

$$\begin{gathered}
\langle\nabla u(y^*),J_1\rangle=|\nabla u(y^*)|(\langle h,e_n\rangle+\langle h,\nabla\phi_0\rangle),\\
\langle\nabla u(y^*),J_2\rangle=|\nabla u(y^*)|\left(-\frac{\phi_0}{2}\sum_{i=1}^{n-1}\langle V_i,h\rangle^2+\frac{1}{2}\langle\mathcal{D}^2\phi_0 h,h\rangle\right).
\end{gathered} \tag{3.17}$$

As a consequence,

$$\begin{aligned}
&\int_{\partial B_r} v(\sigma)d\omega_{B_r}^{x_0}(\sigma) \geq v(x_0)\\
&\quad + |\nabla u(y^*)| \int_{\partial B_r}\left[\langle h,\nabla\phi_0\rangle - \frac{\phi_0}{2}\sum_{i=1}^{n-1}\langle V_i,h\rangle^2 + \frac{1}{2}\langle \mathfrak{D}^2\phi_0 h,h\rangle\right]d\omega_{B_r}^{x_0}\\
&\quad + \int_{\partial B_r}\left[\langle \nabla u(y^*),h\rangle + \frac{1}{2}\langle \mathfrak{D}^2 u(y^*)J,J\rangle + o(|h^2|)\right]d\omega_{B_r}^{x_0} = v(x_0)\\
&\quad + |\nabla u(y^*)| \int_{\partial B_r}\left[\langle h,\nabla\phi_0\rangle - \frac{\phi_0}{2}\sum_{i=1}^{n-1}\langle V_i,h\rangle^2 + \frac{1}{2}\langle \mathfrak{D}^2\phi_0 h,h\rangle\right]d\omega_{B_r}^{x_0}\\
&\quad + \frac{1}{2}\int_{\partial B_r}[\langle \mathfrak{D}^2 u(y^*)J_1,J_1\rangle]d\omega_{B_r}^{x_0} + \nabla u(y^*)\int_{\partial B_r} h\,d\omega_{B_r}^{x_0} + o(r^2),
\end{aligned} \tag{3.18}$$

uniformly with respect to x_0 in a neighborhood of x.

Let

$$\int_{\partial B}[\langle \mathfrak{D}^2 u(y^*)J_1,J_1\rangle]d\omega_{B_r}^{x_0} = \sum_{i,j}^{n} D_{ij}u(y^*)\int_{\partial B_r} a_i a_j\,d\omega_{B_r}^{x_0} \tag{3.19}$$

with

$$a_i = \phi_0\langle V_i,h\rangle + \langle h,e_i\rangle, \quad i = 1,\dots,n, \tag{3.20}$$

where the V_i are still to be chosen, and

$$a_n = \langle \nabla\phi_0,h\rangle + \langle h,e_n\rangle. \tag{3.21}$$

For $i = 1,\dots,n$ and $j = 1,\dots,n$, let

$$d_{ij} = d_{ij}(x_0,x_0) = \int_{\partial B_r} h_i h_j\,d\omega_{B_r(x_0)}^{x_0}, \qquad d_{ij}^* = d_{ij}(y^*,y^*) = \int_{\partial B_r} h_i h_j\,d\omega_{B_r(y^*)}^{y^*} \tag{3.22}$$

be the entries, of the matrix of the moments (see the appendix), respectively, evaluated in x_0 and y^*.

For $i = 1,\dots,n$, and $j = 1,\dots,n-1$, let

$$\begin{gathered}
m_{ij} = \int_{\partial B_r} a_i a_j\,d\omega_{B_r}^{x_0},\\
m_{nn} = \sum_{p,q=1}^{n} D_p\phi_0 D_q\phi_0 d_{pq} + 2\sum_{p=1}^{n} D_p\phi_0 d_{pn} + d_{nn}.
\end{gathered} \tag{3.23}$$

Then

$$m_{ij} = \sum_{p,q=1}^{n} \phi_0^2 V_i^p V_j^q d_{pq} + \phi_0 \sum_{p=1}^{n} V_i^p d_{jp} + \phi_0 \sum_{q=1}^{n} V_j^q d_{iq} + d_{ij}, \tag{3.24}$$

$$\begin{aligned} m_{in} = m_{ni} &= \int_{\partial B_r} \Big(\langle \nabla \phi_0, h \rangle + h_n \Big) \left(\phi_0 \sum_{p=1}^{n} V_i^p h_p + h_i \right) d\omega_{B_r}^{x_0}(\sigma) \\ &= \phi_0 \sum_{p,q=1}^{n} V_i^p D_p \phi_0 d_{pq} + \sum_{p=1}^{n} D_q \phi_0 d_{pi} + \phi_0 \sum_{p=1}^{n} V_i^p d_{pn} + d_{in}. \end{aligned} \tag{3.25}$$

Suppose now we can find $V_1, \dots, V_{n-1}$ and a real number κ_0, such that for every $i = 1, \dots, n-1$ and for every $j = 1, \dots, n$,

$$m_{i,j} = (1+\kappa_0) d_{i,j}^*, \qquad m_{nn} = (1+\kappa_0) d_{nn}^*. \tag{3.26}$$

Then

$$\sum_{i,j=1}^{n-1} D_{ij} u(y^*) m_{ij} + 2 \sum_{i=1}^{n-1} D_{in} u(y^*) m_{in} + D_{nn} u(y^*) m_{nn} = (1+\kappa_0) \sum_{i,j=1}^{n} D_{ij} u(y^*) d_{ij}^*. \tag{3.27}$$

In particular this means that $V_1, \dots, V_{n-1}$, and k_0 must solve the following system, for $i = 1, \dots, n-1$ and $j = 1, \dots, n-1$,

$$\begin{gathered} \phi_0 \sum_{p}^{n} V_i^p d_{jp} + \phi_0 \sum_{q=1}^{n} V_j^q d_{iq} + \phi_0^2 \sum_{p,q=1}^{n} d_{p,q} V_i^p V_j^q = -d_{i,j} + (1+\kappa_0) d_{i,1}^*, \\ \phi_0 \sum_{p=1}^{n} V_i^p d_{pn} + \phi_0 \sum_{p,q=1}^{n} V_i^p D_p \phi_0 d_{pq} = -d_{i,n} + \sum_{p=1}^{n} D_q \phi_0 d_{ip} + (1+\kappa_0) d_{i,n}^*, \\ \sum_{p,q=1}^{n} D_p \phi_0 D_q \phi_0 d_{pq} + 2 \sum_{p=1}^{n} D_q \phi_0 d_{pn} + d_{nn} = (1+\kappa_0) d_{n,n}^*. \end{gathered} \tag{3.28}$$

From the last equations and Lemma A.3, since $d_{nn}^* > c\lambda r^2$, for small r and $|\nabla \phi_0|$, there exists a positive constant $C = C(\lambda, \Lambda)$ such that

$$|\kappa_0| \le C |\nabla \phi_0| + \frac{d_{nn} - d_{nn}^*}{d_{n,n}^*} \le C(|\nabla \phi_0| + \phi_{\max}). \tag{3.29}$$

We now start an iteration process to solve the above system (see [4, 6]).

Let $(\mathcal{V}_i)^0 = 0$, $i = 1,\dots,n-1$, and for $l \geq 0$, define recursively $(\mathcal{V}_i)^{(l+1)}$ as the solution of the system ($i = 1,\dots,n-1$; $j = 1,\dots,n-1$):

$$\phi_0 \sum_{p}^{n} \left(\mathcal{V}_i^p\right)^{(l+1)} d_{jp} + \phi_0 \sum_{q=1}^{n} \left(\mathcal{V}_j^p\right)^{(l)} d_{iq} + \phi_0^2 \sum_{p,q=1}^{n} d_{p,q} \left(\mathcal{V}_i^p\right)^{(l)} \left(\mathcal{V}_j^q\right)^{(l)} = -d_{i,j} + (1+\kappa_0) d_{i,1}^*,$$

$$\phi_0 \sum_{p=1}^{n} \left(\mathcal{V}_i^p\right)^{(l+1)} d_{pn} + \phi_0 \sum_{p,q=1}^{n} \left(\mathcal{V}_j^p\right)^{(l)} D_p \phi_0 d_{pq} = -d_{i,n} + \sum_{p=1}^{n} D_q \phi_0 d_{ip} + (1+\kappa_0) d_{i,n}^*. \tag{3.30}$$

Notice that the sequence is well defined, since the matrix $D(x_0,x_0)$ is nonsingular (Lemma A.3 in the appendix). Moreover, if $|\nabla\phi(x_0)|$ is kept small, denoting by $\mathbf{d}_i$ and $\mathbf{d}_i^*$ the vectors $(d_{i1},\dots,d_{in})$ and $(d_{i1}^*,\dots,d_{in}^*)$, from the estimates in Lemma A.3, we get, by induction,

$$\left|\mathcal{V}_i^{(l+1)}\right| \leq C\left(\frac{|\mathbf{d}_i - \mathbf{d}_i^*|}{r^2\phi_0} + \frac{|\nabla\phi_0|}{\phi}\frac{|\mathbf{d}_i^*|}{r^2} + \phi_0^a\right) \leq C\phi_0^{-1}\left(\phi_0 + |\nabla\phi_0|\right) \tag{3.31}$$

with $C = C(n,\Lambda,\lambda)$. Since the sequences $(\mathcal{V}_i^{(l)})_{l\in\mathbb{N}}$ are bounded for every $i \in \{1,\dots,n-1\}$, there exist subsequences (that we still call) $(\mathcal{V}_i^{(l)})_{l\in\mathbb{N}}$, converging to $\mathcal{V}_i$ with

$$|\mathcal{V}_i| \leq C(n,\Lambda,\lambda)\phi_0^{-1}\left(\phi_0 + |\nabla\phi_0|\right). \tag{3.32}$$

Now, from (3.18), (1.11), and Lemma A.5, we get

$$\begin{aligned}
\int_{\partial B} v(\sigma)\, d\omega_{B_r}^{x_0}(\sigma) \geq{}& v(x_0) \\
&+ |\nabla u(y^*)| \int_{\partial B_r} \left[\langle h,\nabla\phi_0\rangle - \frac{\phi_0}{2}\sum_{i=1}^{n-1}\langle V_i,h\rangle^2 + \frac{1}{2}\langle \mathcal{D}^2\phi_0 h,h\rangle\right] d\omega_{B_r}^{x_0} \\
&+ \int_{\partial B_r} \langle \nabla u(y^*),h\rangle\, d\omega_{B_r}^{x_0} + \frac{1}{2}\int_{\partial B_r}\left[\langle \mathcal{D}^2 u(y^*) J_1,J_1\rangle\right] d\omega_{B_r}^{x_0} + o(r^2) = v(x_0) \\
&+ |\nabla u(y^*)| \int_{\partial B_r} \left[\langle h,\nabla\phi_0\rangle - \frac{\phi_0}{2}\sum_{i=1}^{n-1}\langle V_i,h\rangle^2 + \frac{1}{2}\langle \mathcal{D}^2\phi_0 h,h\rangle\right] d\omega_{B_r}^{x_0} \\
&+ \int_{\partial B_r} \langle \nabla u(y^*),h\rangle\, d\omega_{B_r}^{x_0} + \frac{1+\kappa_0}{2}\int_{\partial B_r(y^*)}\left[\langle \mathcal{D}^2 u(y^*) h^*,h^*\rangle\right] d\omega_{B_r(y^*)}^{y^*} + o(r^2) \\
={}& v(x_0) + \nabla u(y^*)\left[\int_{\partial B_r} h\, d\omega_{B_r}^{x_0} - (1+\kappa_0)\int_{\partial B_r(y^*)} h^*\, d\omega_{B_r(y^*)}^{y^*}\right] \\
&+ |\nabla u(y^*)| \int_{\partial B_r} \left[\langle h,\nabla\phi_0\rangle - \frac{\phi_0}{2}\sum_{i=1}^{n-1}\langle V_i,h\rangle^2 + \frac{1}{2}\langle \mathcal{D}^2\phi_0 h,h\rangle\right] d\omega_{B_r}^{x_0} + o(r^2).
\end{aligned} \tag{3.33}$$

Consider

$$T = \left[\int_{\partial B_r} h\, d\omega_B^{x_0}(\sigma) - (1+\kappa_0)\int_{\partial B_r(y^*)} h\, d\omega_{B_r(y^*)}^{y^*}(\sigma)\right]. \tag{3.34}$$

From Lemma A.3 and (3.29), we get

$$|T| \leq Kr^2. \tag{3.35}$$

Thus, from (3.32), we deduce that

$$\begin{aligned}
&\int_{\partial B_r} v(\sigma)d\omega_B^{x_0}(\sigma) \\
&\geq v(x_0) + |\nabla u(y^*)| \int_{\partial B_r} \left[\langle h, \nabla\phi_0\rangle - \frac{\phi_0}{2}\sum_{i=1}^{n-1}\langle V_i, h\rangle^2 + \frac{1}{2}\langle \mathcal{D}^2\phi_0 h, h\rangle - Kr^2 \right] d\omega_{B_r}^{x_0} \\
&\geq v(x_0) + |\nabla u(y^*)| \int_{\partial B_r} \left[\langle h, \nabla\phi_0\rangle + \frac{1}{2}\langle \mathcal{D}^2\phi_0 h, h\rangle - C\frac{\left(|\nabla\phi_0|^2 + \phi_0^2\right)}{\phi_0} r^2 - Kr^2 \right] d\omega_{B_r}^{x_0}.
\end{aligned} \tag{3.36}$$

From Lemma 2.5, if r is small, and C large depending on x_0 and ϕ_0, we have

$$\int_{\partial B_r} \left[\langle h, \nabla\phi_0\rangle + \frac{1}{2}\langle \mathcal{D}^2\phi_0 h, h\rangle - C\frac{\left(|\nabla\phi_0|^2 + \phi_0^2\right)}{\phi_0} r^2 - Kr^2 \right] d\omega_{B_r}^{x_0} \geq 0, \tag{3.37}$$

so that v_ϕ is a weak L-subsolution in its positivity set. □

Remark 3.2. We emphasize that the construction of the vectors V_i, $i = 1, \dots, n-1$, involves only the Lipschitz continuity of A.

4. Construction of the family of subsolutions and application to the free boundary problem

For the application to our free boundary problem we need a slightly different version of Theorem 1.1. Indeed consider a small vector τ and the function

$$v_\tau(x) = \sup_{B_{\phi(x)}(x)} u(y - \tau) = \sup_{|\nu|=1} u(x - \tau + \phi(x)\nu). \tag{4.1}$$

The proof of Theorem 1.1 holds, with minor changes, also in this case. In particular the following result holds.

Corollary 4.1. *Let u be a continuous function in Ω. Assume that in $\{u > 0\}$ u is a C^2-weak solution of $Lu = 0$, $L \in \mathcal{L}(\lambda, \Lambda, \omega)$. For any vector τ let ϕ be a positive C^2-function such that $0 < \phi_{\min} \leq \phi \leq \phi_{\max}$,*

$$v_\tau(x) = \sup_{B_{\phi(x)}(x)} u(y - \tau) = \sup_{|\nu|=1} u(x - \tau + \phi(x)\nu), \tag{4.2}$$

is well defined in Ω. There exist positive constants ρ_0, $\mu_0 = \mu_0(n, \lambda, \Lambda)$ and $C = C(n, \lambda, \Lambda)$, such that if $|\nabla\phi| \leq \mu_0$, $|\tau| < \rho_0$, $\omega_0 = \omega(\phi_{\max})$, and

$$\phi L\phi \geq C\left(|\nabla\phi(x)|^2 + \omega_0^2\right), \tag{4.3}$$

then v_τ is a weak subsolution of $Lu = 0$ in $\{v_\tau > 0\}$.

Remark 4.2. The key point in Corollary 4.1 is that the estimates (3.29) and (3.32) for the vectors V_i, $i = 1,\dots,n-1$, and k_0 depend only on the distance between the matrices $D(x_0,x_0)$ and $D(y^*,y^*)$.

We now construct a family of radii, with the right properties to be used in the final comparison theorem.

Let $D = B_2(0)\backslash\overline{B}_{1/8}(e_n)$. We may assume with out loss of generality that $A(0) = I$ and that

$$\sup_{B_3} |A(x) - I| \leq \omega_1 \ll 1. \tag{4.4}$$

By a slight modification of [5, Lemma 7] we can construct a family of functions satisfying the properties expressed in the following lemma.

Lemma 4.3. *Let $\widetilde{C} > 0$. There exist positive numbers $c,\overline{\eta},\mu,\overline{\omega} < \overline{\eta}\mu/2$ and a family of functions ϕ_t, $0 \leq t \leq 1$, such that $g_t \in C^2(\overline{D})$ and*

(i) $0 < 1 - \overline{\omega} \leq \phi_t \leq 1 + \mu t$,

(ii) $\phi_t \leq 1 - \omega$ *in* $\overline{B}_2\backslash B_{5/3}$,

(iii) $\phi_t \geq 1 - \overline{\omega} + \overline{\eta}\mu t$ *in* $B_{1/2}$,

(iv) $|\nabla\phi_t| \leq c(\mu t + \overline{\omega})$,

(v)

$$\phi_t L\phi_t \geq \widetilde{C}\left(|\nabla\phi_t|^2 + \omega(\max\phi_t)^2\right). \tag{4.5}$$

We are now in position to prove Theorem 1.2.

Proof of Theorem 1.2. We first observe that Theorem 1.1 (and Corollary 4.1) holds for weak solutions, not necessarily C^2. In fact let $u_j^{\pm}$ be the functions constructed as solutions of the following problems:

$$\begin{aligned} L_j u_j^{\pm} &= 0 \quad \text{in } \Omega^{\pm}(u), \\ u_j^{\pm} &= u_j \quad \text{on } \Omega^{\pm}(u), \end{aligned} \tag{4.6}$$

and set $u_j = u_j^+ - u_j^-$. Then u_j converges locally in $C^{1,a}(\Omega^{\pm}(u))$ to u and it is not difficult to check that (suppressing for clarity the index t)

$$v_j(x) = \sup_{B_{\phi(x)}(x)} u_j \tag{4.7}$$

converges locally in $C^{1,a}(\Omega^{\pm}(u)\cap D)$ to

$$v_\phi(x) = \sup_{B_{\phi(x)}(x)} u. \tag{4.8}$$

From Theorem 1.1, v_j is a weak subsolution for L_j in $\Omega^{\pm}(u_j)\cap D$. But then v_ϕ is a weak L-subsolution in $\Omega^{\pm}(u)\cap D$.

With this result at hand, the proof goes as in [5, Section 7]. Indeed, the particular form of the operator does not play any role anymore. Actually observe that if ϕ_t satisfies inequality (4.5) also $\epsilon\phi_t$ satisfies the same inequality for every $\epsilon > 0$. Therefore, we can simplify the proof given in [5] avoiding, in the iteration process, to go through the improvement of the ϵ-monotonicity and prove directly that in a sequence of dyadic balls $B_{4^{-k}}$ u is monotone along every $\tau \in \Gamma(\nu_k, \theta_k)$ with

$$\delta_{k+1} \le b\delta_k \left(\delta_0 = \delta, \delta_k = \frac{\pi}{2} - \theta_k \right), \qquad |\nu_{k+1} - \nu_k| \le c\delta_k. \tag{4.9}$$

These conditions imply that $F(u)$ is $C^{1,\gamma}$, $\gamma = \gamma(b)$, at the origin. □

Proof of Corollary 1.3. Since $F(u)$ is Lipschitz, u is Hölder continuous in $\mathscr{C}_1$. We only need to show that u is Lipschitz in $\mathscr{C}_{2/3}$ across the free boundary. This follows from a simple application of the monotonicity formula in [16, Lemma 1] and a barrier argument. Precisely, let $x_0 \in \Omega^+(u) \cap \mathscr{C}_{2/3}$, $d_0 = \operatorname{dist}(x_0, F(u))$, and $u(x_0) = \lambda$. From Harnack inequality

$$u(x) \sim \lambda \tag{4.10}$$

in $B_{d_0/2}(x_0)$. Let w be the solution of

$$\operatorname{div}(A(x,u)\nabla w) = 0 \tag{4.11}$$

in $B_{d_0}(x_0) \backslash \overline{B}_{d_0/2}(x_0)$ such that $w = 0$ on $\partial B_{d_0}(x_0)$, $w = \lambda$ on $\partial B_{d_0/2}(x_0)$. By maximum principle

$$u \ge cw \quad \text{in } \overline{B}_{d_0}(x_0) \backslash B_{d_0/2}(x_0) \tag{4.12}$$

and, from the C^a nature of A and $C^{1,a}$ estimates, if $y_0 \in \partial B_{d_0}(x_0) \cap F(u)$,

$$w(x) \ge c\frac{\lambda}{d_0} \langle x - y_0, \nu \rangle^+ \tag{4.13}$$

with $\nu = (x_0 - y_0)/|x_0 - y_0|$. Thus, near y_0, u has the asymptotic behavior

$$u(x) \ge \alpha \langle x - y_0, \nu \rangle^+ - \beta \langle x - y_0, \nu \rangle^- + o(|x - y_0|) \tag{4.14}$$

with

$$c\frac{\lambda}{d_0} \le \alpha \le G(\beta). \tag{4.15}$$

Then, the monotonicity formula gives

$$\frac{\lambda}{d_0} G^{-1}\left(c\frac{\lambda}{d_0} \right) \le C \|u\|^2_{L^\infty(\mathscr{C}_{3/4})} \tag{4.16}$$

so that, from interior estimates,

$$|\nabla u^+(x_0)| G^{-1}\left(|\nabla u^+(x_0)| \right) \le C_1 \|u\|^2_{L^\infty(\mathscr{C}_{3/4})}. \tag{4.17}$$

This gives the Lipschitz continuity of u^+. Similarly, we get

$$G\Big(\,|\nabla u^-(x_0)|\,\Big)\,|\nabla u^-(x_0)|\,\leq C_1\|u\|^2_{L^\infty(\mathscr{C}_{3/4})} \tag{4.18}$$

and the proof is complete. □

Appendix

Auxiliary lemmas

We collect here some estimates on the L-harmonic measure and its moments that are used in the paper. Here $\omega(r)\leq c_0 r^a$, $0<a\leq 1$.

Definition A.1. For any $x_0, x, y\in\Omega$, and $r>0$, $B_r(x_0)\subset\Omega$, let $d_i(x_0,y)$ be, for $i=1,\dots,n$,

$$d_i(x_0,y)=\int_{\partial B_r(x_0)}(\sigma_i-x_{0i})\,d\omega^y_{B_r(x_0)}(\sigma) \tag{A.1}$$

and let $d_{ij}(x_0,y)$ be, for every i, j, $1\leq i, j\leq n$,

$$d_{ij}(x_0,y)=\int_{\partial B_r(x_0)}(\sigma_i-x_{0i})(\sigma_j-x_{0j})\,d\omega^y_{B_r(x_0)}(\sigma). \tag{A.2}$$

We call, respectively, $(d_i(x_0,y))_{1\leq i\leq n}$ the vector of the first moment of the L-harmonic measure in $B_r(x)$, and $D(x_0,y)=(d_{ij}(x_0,y))_{1\leq i,j\leq n}$ the matrix of the second moment of the L-harmonic measure in $B_r(x)$.

Denote by $L_0=\operatorname{div}(A(x_0)\nabla)$ and by $G^0_r=G^0_r(x,y)$ the Green function for L_0 in $B_r=B_r(x_0)$. We have the following.

Lemma A.2. *Let $L_0 w_r=-1$ in $B_r(x_0)$, $w_r=0$ on $\partial B_r(x_0)$. Then*

$$w_r(x_0)=\frac{r^2}{2TrA(x_0)}. \tag{A.3}$$

Proof. Suppose $x_0=0$. Let $g_{ij}(x)=x_ix_j$ and let v_{ij} be the solution of $L_0v_{ij}=0$ in B_r, $v_{ij}=g_{ij}$ on ∂B_r. Since $L_0g_{ij}=2a_{ij}(0)$ and $g_{ij}(0)=0$, we have

$$v_{ij}(0)=2a_{ij}(0)w_r(0). \tag{A.4}$$

On the other hand, $\sum_{i=1}^n v_{ii}(0)=r^2$ and (A.3) follows. □

Lemma A.3. *Let $B_{2r}(x_0)\subset\Omega$. Then:*

(1) *for every $i=1,\dots,n$,*

$$\sup_{B_r(0)}|d(x_0,y)-y_i|\leq C(\lambda,\Lambda,n)r^{1+a}; \tag{A.5}$$

(2) *for every $i,j=1,\dots,n$,*

$$|d_{ij}(x_0,x_0)-2w_r(x_0)a_{ij}(x_0)|\leq Cr^{2+a}, \tag{A.6}$$

where w_r is defined in Lemma A.2.

Proof. Let $x_0 = 0$ and

$$d_i(y) = d_i(0,y) = \int_{\partial B_r} \sigma_i d\omega^y_{B_r}(\sigma). \tag{A.7}$$

Then $d_i(y) - y_i = 0$ on ∂B_r and

$$L_0(d_i(y) - y_i) = \operatorname{div}((A(0) - A(y))e_i) \quad \text{in } B_r. \tag{A.8}$$

From standard estimates, we get

$$||d_i - y_i||_{L^\infty(B_r)} \le Cr||(A(0) - A(y))e_i||_{L^\infty(B_r)} \le Cr^{1+a}. \tag{A.9}$$

Consider now

$$d_{ij}(y) = d_{ij}(0,y) = \int_{\partial B_r} \sigma_i\sigma_j d\omega^y_{B_r}(\sigma). \tag{A.10}$$

If v_{ij} is as in Lemma 2.2, we have $h_{ij} - v_{ij} = 0$ on ∂B_r and

$$L_0(d_{ij}(y) - v_{ij}(y)) = ((A(0) - A(y))\nabla v_{ij}) \quad \text{in } B_r. \tag{A.11}$$

Therefore,

$$||d_{ij} - v_{ij}||_{L^\infty(B_r)} \le Cr||(A(0) - A(y))\nabla v_{ij}||_{L^\infty(B_r)} \le C||\nabla v_{ij}||_{L^s(B_r)} r^{1+a} \le Cr^{2+a}. \tag{A.12}$$

Hence, from Lemma 2.2, we get

$$|d_{ij}(0) - 2a_{ij}(0)w_r(0)| = |d_{ij}(0) - v_{ij}(0)| \le Cr^{2+a}. \tag{A.13}$$

□

Corollary A.4. *For $r \le r_0(n,\lambda,\Lambda,a)$, the matrix $(d_{ij}(0)/r^2)$ is nonsingular.*

Lemma A.5. *Let w be a weak solution of*

$$\operatorname{div}(A(x)\nabla w(x)) = f \tag{A.14}$$

in Ω, where f is continuous. Then for every $x \in B_r(x_0) \subset \Omega$,

$$\nabla w(x) \cdot \int_{\partial B_r(x_0)} (\sigma - x) d\omega^x_{B_r(x_0)}(\sigma) + \int_{B_r(x_0)} G_{B_r(x_0)}(x,y) f(y) dy = R(x_0,x), \tag{A.15}$$

where

$$R(x_0,x) = \int_{\partial B_r(x_0)} \left(\int_0^1 \langle \nabla w(x + s(\sigma - x)) - \nabla w(x), \sigma - x\rangle ds \right) d\omega^x_{B_r(x_0)}(\sigma). \tag{A.16}$$

Moreover, if $u \in C^2(\Omega)$,

$$\nabla w(x) \cdot \int_{\partial B_r(x_0)} (\sigma - x) d\omega^x_{B_r(x_0)}(\sigma) + \frac{1}{2} \int_{\partial B_r(x_0)} \langle D^2 w(x)(\sigma - x), (\sigma - x) \rangle d\omega^x_{B_r(x_0)}(\sigma) + \int_{B_r(x_0)} G_{B_r(x_0)}(x,y) f(y) dy = o(r^2). \tag{A.17}$$

Proof. Let $\operatorname{div}(A(x)\nabla w(x)) = f$, then $w \in C^{1,a}(\Omega)$ and for any $\sigma, x \in \overline{B}_r(x_0) \subset \Omega$,

$$w(\sigma) = w(x) + \int_0^1 \langle \nabla w(x + s(\sigma - x)), \sigma - x \rangle ds = w(x) + \langle \nabla w(x), \sigma - x \rangle + \int_0^1 \langle \nabla w(x + s(\sigma - x)) - \nabla w(x), \sigma - x \rangle ds. \tag{A.18}$$

On the other hand,

$$w(x) = \int_{\partial B_r(x_0)} w(\sigma) d\omega^x_{B_r(x_0)}(\sigma) - \int_{B_r(x_0)} G_{B_r(x_0)}(x,y) f(y) dy, \tag{A.19}$$

hence

$$\nabla w(x) \cdot \int_{\partial B_r(x_0)} (\sigma - x) d\omega^x_{B_r(x_0)}(\sigma) + \int_{B_r(x_0)} G_{B_r(x_0)}(x,y) f(y) dy = R(x_0, x). \tag{A.20}$$

The rest of the proof follows from Taylor expansion. □

Corollary A.6. *Let* $u \in C^2(\Omega)$ *be a weak solution of*

$$\operatorname{div}(A(x)\nabla u(x)) = 0 \tag{A.21}$$

in Ω. *Then*

$$\nabla u(x_0) \cdot \int_{\partial B_r(x_0)} (\sigma - x_0) d\omega^{x_0}_{B_r(x_0)}(\sigma) = O(r^2). \tag{A.22}$$

Proof. It is enough to observe that

$$\int_{\partial B_r(x_0)} (\sigma_i - x_{0i})(\sigma_j - x_{0j}) d\omega^{x_0}_{B_r(x_0)}(\sigma) = O(r^2). \tag{A.23}$$

□

References

[1] L. A. Caffarelli, "A Harnack inequality approach to the regularity of free boundaries. I. Lipschitz free boundaries are $C^{1,\alpha}$," *Revista Matemática Iberoamericana*, vol. 3, no. 2, pp. 139–162, 1987.

[2] M. Feldman, "Regularity for nonisotropic two-phase problems with Lipschitz free boundaries," *Differential and Integral Equations*, vol. 10, no. 6, pp. 1171–1179, 1997.

[3] P.-Y. Wang, "Regularity of free boundaries of two-phase problems for fully nonlinear elliptic equations of second order. I. Lipschitz free boundaries are $C^{1,\alpha}$," *Communications on Pure and Applied Mathematics*, vol. 53, no. 7, pp. 799–810, 2000.

[4] M. Feldman, "Regularity of Lipschitz free boundaries in two-phase problems for fully nonlinear elliptic equations," *Indiana University Mathematics Journal*, vol. 50, no. 3, pp. 1171–1200, 2001.

[5] M. C. Cerutti, F. Ferrari, and S. Salsa, "Two-phase problems for linear elliptic operators with variable coefficients: Lipschitz free boundaries are $C^{1,\gamma}$," *Archive for Rational Mechanics and Analysis*, vol. 171, no. 3, pp. 329–348, 2004.

[6] F. Ferrari, "Two-phase problems for a class of fully nonlinear elliptic operators. Lipschitz free boundaries are $C^{1,\gamma}$," *American Journal of Mathematics*, vol. 128, no. 3, pp. 541–571, 2006.

[7] L. A. Caffarelli, "A Harnack inequality approach to the regularity of free boundaries. II. Flat free boundaries are Lipschitz," *Communications on Pure and Applied Mathematics*, vol. 42, no. 1, pp. 55–78, 1989.

[8] L. A. Caffarelli, E. Fabes, S. Mortola, and S. Salsa, "Boundary behavior of nonnegative solutions of elliptic operators in divergence form," *Indiana University Mathematics Journal*, vol. 30, no. 4, pp. 621–640, 1981.

[9] D. S. Jerison and C. E. Kenig, "Boundary behavior of harmonic functions in nontangentially accessible domains," *Advances in Mathematics*, vol. 46, no. 1, pp. 80–147, 1982.

[10] M. Brelot, *Axiomatique des Fonctions Harmoniques*, Séminaire de Mathématiques Supérieures - Été 1965, Les Presses de l'Université de Montréal, Quebec, 1966.

[11] R.-M. Hervé, "Recherches axiomatiques sur la théorie des fonctions surharmoniques et du potentiel," *Annales de l'Institut Fourier. Université de Grenoble*, vol. 12, pp. 415–571, 1962.

[12] R.-M. Hervé, "Un principe du maximum pour les sous-solutions locales d'une équation uniformément elliptique de la forme $Lu = -\sum_i \frac{\partial}{\partial x_i}(\sum_j a_{ij} \frac{\partial u}{\partial x_j}) = 0$," *Annales de l'Institut Fourier. Université de Grenoble*, vol. 14, no. 2, pp. 493–507, 1964.

[13] R.-M. Hervé and M. Hervé, "Les fonctions surharmoniques associées à un opérateur elliptique du second ordre à coefficients discontinus," *Annales de l'Institut Fourier. Université de Grenoble*, vol. 19, no. 1, pp. 305–359, 1969.

[14] W. Littman, G. Stampacchia, and H. F. Weinberger, "Regular points for elliptic equations with discontinuous coefficients," *Annali della Scuola Normale Superiore di Pisa, Serie III*, vol. 17, pp. 43–77, 1963.

[15] R.-M. Hervé, "Quelques propriétés des fonctions surharmoniques associées à une équation uniformément elliptique de la form $Lu = -\sum_i \frac{\partial}{\partial x_i}(\sum_j a_{ij} \frac{\partial u}{\partial x_j}) = 0$," *Annales de l'Institut Fourier. Université de Grenoble*, vol. 15, no. 2, pp. 215–223, 1965.

[16] L. A. Caffarelli, "A Harnack inequality approach to the regularity of free boundaries. III. Existence theory, compactness, and dependence on X," *Annali della Scuola Normale Superiore di Pisa. Classe di Scienze. Serie IV*, vol. 15, no. 4, pp. 583–602 (1989), 1988.

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