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Research Article

Existence Result for a Class of Elliptic Systems with Indefinite Weights in R^2

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We obtain the existence of a nontrivial solution for a class of subcritical elliptic systems with indefinite weights in R^2. The proofs base on Trudinger-Moser inequality and a generalized linking theorem introduced by Kryszewski and Szulkin.

1. Introduction

In this paper, we study the existence of a nontrivial solution for the following systems of two semilinear coupled Poisson equations

$$(P)\begin{cases}-\Delta u+u=g(x,v), & x\in R^2,\\ -\Delta v+v=f(x,u), & x\in R^2,\end{cases} \tag{1.1}$$

where $f(x,t)$ and $g(x,t)$ are continuous functions on $R^2\times R$ and have the maximal growth on t which allows to treat problem (P) variationally, Δ is the Laplace operator.

Recently, there exists an extensive bibliography in the study of elliptic problem in R^N [1–6]. As dimensions $N\geq 3$, in 1998, de Figueiredo and Yang [5] considered the following coupled elliptic systems:

$$\begin{aligned}-\Delta u+u&=g(x,v), \quad x\in R^N,\\ -\Delta v+v&=f(x,u), \quad x\in R^N,\end{aligned} \tag{1.2}$$

where f, g are radially symmetric in x and satisfied the following Ambrosetti-Rabinowitz condition:

$$\int_0^t f(x,s)ds \geq c|t|^{2+\delta_1}, \quad \int_0^t g(x,s)ds \geq c|t|^{2+\delta_2}, \quad \forall t \in R, \tag{1.3}$$

and for some $\delta_1 > 0$, $\delta_2 > 0$. They obtained the decay, symmetry, and existence of solutions for problem (1.2). In 2004, Li and Yang [6] proved that problem (1.2) possesses at least a positive solution when the nonlinearities $f(x,t)$ and $g(x,t)$ are "asymptotically linear" at infinity and "superlinear" at zero, that is,

(1) $\lim_{t\to\infty}(f(x,t)/t) = l > 1$, $\lim_{t\to\infty}(g(x,t)/t) = m > 1$, uniformly in $x \in R^N$;

(2) $\lim_{t\to 0}(f(x,t)/t) = \lim_{t\to 0}(g(x,t)/t) = 0$, uniformly with respect to $x \in R^N$.

In 2006, Colin and Frigon [1] studied the following systems of coupled Poission equations with critical growth in unbounded domains:

$$\begin{aligned} -\Delta u &= |v|^{2^*-2}v, \\ -\Delta v &= |u|^{2^*-2}u, \end{aligned} \tag{1.4}$$

where $2^* = 2N/(N-2)$ is critical Sobolev exponent, $u, v \in D_0^{1,2}(\Omega_*)$ and $\Omega_* = R^N \setminus E$ with $E = \bigcup_{a\in Z^N} a + \omega_*$ for a domain containing the origin $\omega_* \subset \overline{\omega}_* \subset B(0,1/2)$. Here, $B(0,1/2)$ denotes the open ball centered at the origin of radius $1/2$. The existence of a nontrivial solution was obtained by using a generalized linking theorem.

As it is well known in dimensions $N \geq 3$, the nonlinearities are required to have polynomial growth at infinity, so that one can define associated functionals in Sobolev spaces. Coming to dimension $N = 2$, much faster growth is allowed for the nonlinearity. In fact, the Trudinger-Moser estimates in $N = 2$ replace the Sobolev embedding theorem used in $N \geq 3$.

In dimension $N = 2$, Adimurth and Yadava [7], de Figueiredo et al. [8] discussed the solvability of problems of the type

$$\begin{aligned} -\Delta u &= f(x,u), \quad x \in \Omega, \\ u &= 0, \quad x \in \partial\Omega, \end{aligned} \tag{1.5}$$

where Ω is some bounded domain in R^2. Shen et al. [9] considered the following nonlinear elliptic problems with critical potential:

$$\begin{aligned} \Delta u - \mu \frac{u}{(|x|\log(R/|x|))^2} &= f(x,u), \quad x \in \Omega \\ u &= 0, \quad x \in \partial\Omega, \end{aligned} \tag{1.6}$$

and obtained some existence results. In the whole space R^2, some authors considered the following single semilinear elliptic equations:

$$-\Delta u + V(x)u = f(x,u), \quad x \in R^2. \tag{1.7}$$

As the potential $V(x)$ and the nonlinearity $f(x,t)$ are asymptotic to a constant function, Cao [10] obtained the existence of a nontrivial solution. As the potential $V(x)$ and the nonlinearity $f(x,t)$ are asymptotically periodic at infinity, Alves et al. [11] proved the existence of at least one positive weak solution.

Our aim in this paper is to establish the existence of a nontrivial solution for problem (P) in subcritical case. To our knowledge, there are no results in the literature establishing the existence of solutions to these problems in the whole space. However, it contains a basic difficulty. Namely, the energy functional associated with problem (P) has strong indefinite quadratic part, so there is not any more mountain pass structure but linking one. Therefore, the proofs of our main results cannot rely on classical min-max results. Combining a generalized linking theorem introduced by Kryszewski and Szulkin [12] and Trudinger-Moser inequality, we prove an existence result for problem (P).

The paper is organized as follows. In Section 2, we recall some results and state our main results. In Section 3, main result is proved.

2. Preliminaries and main results

Consider the Hilbert space [13]

$$H^1(R^2) = \{u \in L^2(R^2), \nabla u \in L^2(R^2)\}, \tag{2.1}$$

and denote the product space $Z = H^1(R^2) \times H^1(R^2)$ endowed with the inner product:

$$\langle (u,v),(\phi,\psi)\rangle = \int_{R^2} (\nabla u \nabla \phi + u\phi)dx + \int_{R^2} (\nabla v \nabla \psi + v\psi)dx, \quad \forall (\phi,\psi) \in Z. \tag{2.2}$$

If we define

$$Z^+ = \{(u,u) \in Z\}, \qquad Z^- = \{(v,-v) \in Z\}. \tag{2.3}$$

It is easy to check that $Z = Z^+ \oplus Z^-$, since

$$(u,v) = \frac{1}{2}(u+v,u+v) + \frac{1}{2}(u-v,v-u). \tag{2.4}$$

Let us denote by P (resp., Q) the projection of Z on to Z^+ (resp., Z^-), we have

$$\begin{aligned}
\frac{1}{2}\left(\|P(u,v)\|^2 - \|Q(u,v)\|^2\right) &= \frac{1}{2}\left\|\frac{1}{2}(u+v,u+v)\right\|^2 - \frac{1}{2}\left\|\frac{1}{2}(u-v,v-u)\right\|^2 \\
&= \frac{1}{4}\bigg(\int_{R^2} (|\nabla u|^2 + |\nabla v|^2 + 2\nabla u \nabla v)dx + \int_{R^2} (|u|^2 + |v|^2 + 2uv)dx \\
&\quad - \int_{R^2} (|\nabla u|^2 + |\nabla v|^2 - 2\nabla u \nabla v)dx - \int_{R^2} (|u|^2 + |v|^2 - 2uv)dx \bigg) \\
&= \int_{R^2} (\nabla u \nabla v + uv)dx.
\end{aligned} \tag{2.5}$$

Now, we define the functional

$$I(u,v) = \int_{R^2} (\nabla u \nabla v + uv)dx - \int_{R^2} \big(F(x,u) + G(x,v)\big)dx$$
$$= \frac{\|P(u,v)\|^2}{2} - \frac{\|Q(u,v)\|^2}{2} - \varphi(u,v), \quad \forall (u,v) \in Z, \tag{2.6}$$

where

$$\varphi(u,v) = \int_{R^2} \big(F(x,u) + G(x,v)\big)dx. \tag{2.7}$$

Let $z_0 \in Z^+ \setminus \{0\}$ and let $R > r > 0$, we define

$$M = \{z = z^- + \lambda z_0 : z^- \in Z^-, \|z\| \le R, \lambda \ge 0\},$$
$$M_0 = \{z = z^- + \lambda z_0 : z^- \in Z^-, \|z\| = R \text{ and } \lambda \ge 0 \text{ or } \|z\| \le R \text{ and } \lambda \ge 0\}, \tag{2.8}$$
$$N = \{z \in Z^+ : \|z\| = r\}.$$

Here, we assume the following condition:

(H1) $f, g \in C(R^2 \times R, R)$;

(H2) $\lim_{t\to 0}(f(x,t)/t) = \lim_{t\to 0}(g(x,t)/t) = 0$ uniformly with respect to $x \in R^2$;

(H3) there exist $\mu > 2$ and $\eta > 0$ such that

$$0 < \mu F(x,t) \le t f(x,t), \quad 0 < \mu G(x,t) \le t g(x,t), \quad \forall |t| \ge \eta. \tag{2.9}$$

Lemma 2.1 (see [12, 14]). *Assume (H1), (H2), and (H3), and suppose*

(1) $I(z) = (1/2)(\|Pz\|^2 - \|Qz\|^2) - \varphi(z)$, *where* $\varphi \in C^1(Z,R)$ *is sequentially lower semicontinuous, bounded below, and* $\nabla\varphi$ *is weakly sequentially continuous;*

(2) *there exist* $z_0 \in Z^+ \setminus \{0\}$, $\alpha > 0$, *and* $R > r > 0$, *such that*

$$\inf_N I(z) \ge \alpha > 0, \qquad \sup_{M_0} I(z) \le 0. \tag{2.10}$$

Then, there exist $c > 0$ and a sequence $(z_n) \subset Z$ such that

$$I(z_n) \longrightarrow c, \quad I'(z_n) \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{2.11}$$

Moreover, $c \ge \alpha$.

Theorem 2.2. *Under the assumptions (H1), (H2), and (H3), if f and g has subcritical growth (see definition below), problem (P) possesses a nontrivial weak solution.*

In the whole space R^2, do Ó and Souto [15] proved a version of Trudinger-Moser inequality, that is,

(i) if $u \in H^1(R^2)$, $\beta > 0$, we have

$$\int_{R^2} \left(\exp\left(\beta |u|^2\right) - 1 \right) dx < +\infty; \tag{2.12}$$

(ii) if $0 < \beta < 4\pi$ and $|u|_{L^2(R^2)} \leq c$, then there exists a constant $c_2 = c_1(c, \beta)$ such that

$$\sup_{|\nabla u|_{L^2(R^2)} \leq 1} \int_{R^2} \left(\exp\left(\beta |u|^2\right) - 1 \right) dx < c_2. \tag{2.13}$$

Definition 2.3. We say $f(x,t)$ has subcritical growth at $+\infty$, if for all $\beta > 0$, there exists a positive constant c_3 such that

$$f(x,t) \leq c_3 \exp\left(\beta t^2\right), \quad \forall (x,t) \in R^2 \times [0, +\infty). \tag{2.14}$$

3. Proof of Theorem 2.2

In this section, we will prove Theorem 2.2. under our assumptions and (2.14), there exist $c_\varepsilon > 0, \beta > 0$ such that

$$\left|F(x,t)\right|, \left|G(x,t)\right| \leq \frac{t^2}{2}\varepsilon + c_\varepsilon \left(\exp\left(\beta t^2\right) - 1 \right), \quad \forall \varepsilon > 0, \forall t \in R. \tag{3.1}$$

Then, we obtain

$$F(x,u), G(x,v) \in L^2(R^2), \quad \forall u, v \in H^1(R^2). \tag{3.2}$$

Therefore, the functional $I(u,v)$ is well defined. Furthermore, using standard arguments, we obtain the functional $I(u,v)$ is C^1 functional in Z and

$$\begin{aligned} I'(u,v)(\phi,\psi) = & \int_{R^2} (\nabla u \nabla \psi + u\psi) dx + \int_{R^2} (\nabla v \nabla \phi + v\phi) dx \\ & - \int_{R^2} (f(x,u)\phi + g(x,v)\psi) dx, \quad \forall (\phi,\psi) \in Z. \end{aligned} \tag{3.3}$$

Consequently, the weak solutions of problem (P) are exactly the critical points of $I(u,v)$ in Z. Now, we prove that the functional $I(u,v)$ satisfied the geometry of Lemma 2.1.

Lemma 3.1. *There exist* $r > 0$ *and* $\alpha > 0$ *such that* $\inf_N I(u,u) \geq \alpha > 0$.

Proof. By (2.14) and assumption (H2), there exists $c_\varepsilon > 0$ such that

$$F(x,t), G(x,t) \leq \frac{t^2}{2}\varepsilon + c_\varepsilon t^3 \left(\exp\left(\beta t^2\right) - 1 \right), \quad \forall t \in R, \tag{3.4}$$

and thus on N, we have

$$\begin{aligned} I(u,u) &\geq \int_{R^2} \left(|\nabla u|^2 + u^2\right) dx - \int_{R^2} \left(\varepsilon u^2 + c_\varepsilon u^3 \left(\exp\left(\beta u^2\right) - 1\right)\right) dx \\ &\geq \int_{R^2} \left(|\nabla u|^2 + u^2\right) dx - \varepsilon \int_{R^2} u^2 dx - c_\varepsilon \left(\int_{R^2} u^6 dx\right)^{1/2} \left(\int_{R^2} \left(\exp\left(\beta u^2\right) - 1\right)^2 dx\right)^{1/2} \\ &\geq \int_{R^2} \left(|\nabla u|^2 + u^2\right) dx - \varepsilon \int_{R^2} u^2 dx - c_\varepsilon \|u\|^3 \left(\int_{R^2} \exp\left(\left(\beta u^2\right) - 1\right) dx\right)^{1/2}. \end{aligned} \tag{3.5}$$

So, by the Sobolev embedding theorem and (2.12), we can choose $r > 0$ sufficiently small, such that

$$I(u,u) \geq \alpha > 0, \quad \text{whenever } \|u\| = r. \tag{3.6}$$

□

Lemma 3.2. *There exist* $(u_0, u_0) \in Z^+ \setminus \{0\}$ *and* $R > r > 0$ *such that* $\sup_{M_0} I \leq 0$.

Proof. (1) By assumption (H3), we have on Z^-

$$I(u,u) = \int_{R^2} \left(|\nabla u|^2 + u^2\right) dx - \int_{R^2} \left(F(x,u) + G(x,-u)\right) dx \leq 0 \tag{3.7}$$

because $F(x,t) \geq 0$, $G(x,t) \geq 0$ for any $(x,t) \in R^2 \times R$.

(2) Assumption (H3) implies that there exist $c_4 > 0$, $c_5 > 0$ such that

$$F(x,t), \quad G(x,t) \geq c_4 t^\mu - c_5, \quad \forall t \in R. \tag{3.8}$$

Now, we choose $(u_0, u_0) \in Z^+ \setminus \{0\}$ such that $\|(u_0, u_0)\| = r$, then

$$\begin{aligned} I\left((-v,v) + \lambda(u_0,u_0)\right) &= \lambda^2 \int_{R^2} \left(|\nabla u_0|^2 + u_0^2\right) dx - \int_{R^2} \left(|\nabla v|^2 + v^2\right) dx \\ &\quad - \int_{R^2} \left(F(\lambda u_0 + v) + G(\lambda u_0 - v)\right) dx \\ &\leq - \int_{R^2} \left(|\nabla u|^2 + u^2\right) dx + c\left(\lambda^2 - \lambda^\mu\right). \end{aligned} \tag{3.9}$$

Because $\mu > 2$, it follows that for $w \in M_0$

$$I(w) \longrightarrow -\infty, \quad \text{whenever } \|w\| \longrightarrow \infty, \tag{3.10}$$

and so, taking $R > r$ large, we get $\sup_{M_0} I \leq 0$. □

Proof of Theorem 2.2. By Lemma 3.1, there exist $r > 0$ and $\alpha > 0$ such that $\inf_N I(u,u) \geq \alpha > 0$. By Lemma 3.2, there exist $(u_0, u_0) \in Z^+ \setminus \{0\}$ and $R > r > 0$ such that $\sup_{M_0} I \leq 0$. Since $Z = Z^+ \oplus Z^-$, we have

$$I(u,v) = \int_{R^2} (\nabla u \nabla v + uv)dx - \int_{R^2} (F(x,u) + G(x,v))dx$$
$$= \frac{\|P(u,v)\|^2}{2} - \frac{\|Q(u,v)\|^2}{2} - \varphi(u,v), \quad \forall (u,v) \in Z. \tag{3.11}$$

From (2.14), (3.1), and assumption (H3), $\varphi(u,v) \in C^1$, $\varphi(u,v) \geq 0$ and $\varphi(u,v)$ is sequentially lower semicontinuous by $Z \subset L^2_{\text{loc}}(R^2) \times L^2_{\text{loc}}(R^2)$ and Fatou's lemma; $\nabla \varphi$ is weakly sequentially continuous. Thus, by Lemma 2.1 there exists a sequence $(u_n, v_n) \subset Z$ such that

$$I(u_n, v_n) \longrightarrow c \geq \alpha, \qquad I'(u_n, v_n) \longrightarrow 0. \tag{3.12}$$

Claim 3.3. There is $c < +\infty$, such that $\|(u_n, v_n)\| \leq c$ for any n. Indeed, from (3.12), we obtain that the sequence $(u_n, v_n) \subset Z$ satisfies

$$I(u_n, v_n) = c + \delta_n, \quad I'(u_n, v_n)(\phi, \psi) = \varepsilon_n \|(u_n, v_n)\|, \quad \text{as } n \longrightarrow \infty, \tag{3.13}$$

where $(\phi, \psi) \in \{u_n, v_n\}$, $\delta_n \to 0$, $\varepsilon_n \to 0$ as $n \to \infty$. Taking $(\phi, \psi) = \{u_n, v_n\}$ in (3.13) and assumption (H3), we have

$$\int_{R^2} (f(x,u_n)u_n + g(x,v_n)v_n)dx$$
$$\leq 2\int_{R^2} (F(x,u_n) + G(x,v_n))dx + 2c + 2\delta_n + \varepsilon_n \|(u_n, v_n)\| \tag{3.14}$$
$$\leq \frac{2}{\mu}\int_{R^2} ((f(x,u_n)u_n + g(x,v_n)v_n))dx + C + 2\delta_n + \varepsilon_n \|(u_n, v_n)\|,$$

where C depends only on c and η in assumption (H3). Since $\mu > 2$, we have $(1 - 2/\mu) > 0$, and thus

$$\left(1 - \frac{2}{\mu}\right)\int_{R^2} ((f(x,u_n)u_n + g(x,v_n)v_n))dx \leq C + 2\delta_n + \varepsilon_n \|(u_n, v_n)\|, \quad \forall n \in N. \tag{3.15}$$

On the other hand, let $(\phi, \psi) = (v_n, 0), (\phi, \psi) = (0, u_n)$ in (3.13), we obtain

$$\|v_n\|^2 - \varepsilon_n \|v_n\| \leq \int_{R^2} f(x,u_n)v_n dx, \qquad \|u_n\|^2 - \varepsilon_n \|u_n\| \leq \int_{R^2} g(x,v_n)u_n dx. \tag{3.16}$$

that is,

$$\|v_n\| \leq \int_{R^2} f(x,u_n)\frac{v_n}{\|v_n\|} dx + \varepsilon_n, \qquad \|u_n\| \leq \int_{R^2} g(x,v_n)\frac{u_n}{\|u_n\|} dx + \varepsilon_n. \tag{3.17}$$

Now, we recall the following inequality (see [7, Lemma 2.4]):

$$mn \leq \begin{cases} (e^{n^2}-1)+m(\log m)^{1/2}, & n \geq 0,\ m \geq e^{1/4}, \\ (e^{n^2}-1)+\dfrac{1}{2}m^2, & n \geq 0,\ 0 \leq m \leq e^{1/4}. \end{cases} \tag{3.18}$$

Let $n = v_n/\|v_n\|$ and $m = f(x,u_n)/c_3$, where c_3 is defined in (2.14), we have

$$\begin{aligned} c_3\int_{R^2}\frac{f(x,u_n)}{c_3}\frac{v_n}{\|v_n\|}dx \leq c_3\int_{R^2}&\left[\exp\left(\frac{v_n}{\|v_n\|}\right)^2-1\right]dx \\ &+c_3\int_{\{x\in R^2, f(x,u_n)/c_3\geq e^{1/4}\}}\frac{f(x,u_n)}{c_3}\left[\log\frac{f(x,u_n)}{c_3}\right]^{1/2}dx \\ &+c_3\int_{\{x\in R^2, f(x,u_n)/c_3\leq e^{1/4}\}}\left(\frac{f(x,u_n)}{c_3}\right)^2dx. \end{aligned} \tag{3.19}$$

By (2.12), we have $\int_{R^2}[\exp(v_n/\|v_n\|)^2-1]dx < +\infty$. By (2.14), we have

$$\left[\log\frac{f(x,t)}{c_3}\right]^{1/2} \leq \beta^{1/2}t. \tag{3.20}$$

Hence, we have

$$c_3\int_{R^2}\frac{f(x,u_n)}{c_3}\frac{v_n}{\|v_n\|}dx \leq c_6+\beta^{1/2}\int_{R^2}f(x,u_n)u_n dx \tag{3.21}$$

for some positive constant c_6. So we have

$$\|v_n\| \leq c_6+\beta^{1/2}\int_{R^2}f(x,u_n)u_n dx+\varepsilon_n. \tag{3.22}$$

Using a similar argument, we obtain

$$\|u_n\| \leq c_7+\beta^{1/2}\int_{R^2}g(x,v_n)v_n dx+\varepsilon_n \tag{3.23}$$

for some positive constant c_7. Combining (3.22) and (3.23), we have

$$\|(u_n,v_n)\| \leq c_8(1+\delta_n+\varepsilon_n\|(u_n,v_n)\|+\varepsilon_n) \tag{3.24}$$

for some positive constant c_8, which implies that $\|(u_n,v_n)\| \leq c$. Thus, for a subsequence still denoted by (u_n,v_n), there is $(u_0,v_0)\in Z$ such that

$$\begin{gathered} (u_n,v_n) \longrightarrow (u_0,v_0) \text{ weakly in } Z, \quad \text{as } n\longrightarrow\infty, \\ (u_n,v_n) \longrightarrow (u_0,v_0) \text{ in } L^s_{\text{loc}}(R^2)\times L^s_{\text{loc}}(R^2) \text{ for } s\geq 1, \quad \text{as } n\longrightarrow\infty, \\ (u_n(x),v_n(x)) \longrightarrow (u_0(x),v_0(x)), \text{ almost every, in } R^2, \quad \text{as } n\longrightarrow\infty. \end{gathered} \tag{3.25}$$

Then, there exists $h(x) \in H^1(R^2)$ such that $|u_n(x)| \le h$, $\forall x \in R^2$, $\forall n \in N$. From (2.12) and (2.14), we have $\int_{R^2}(\exp(\beta h^2(x)) - 1)dx < c$, this implies

$$\int_{R^2} f(x,u_n)\phi dx \longrightarrow \int_{R^2} f(x,u_0)\phi dx, \quad \text{as } n \longrightarrow \infty. \tag{3.26}$$

Similarly, we can obtain

$$\int_{R^2} g(x,v_n)\psi dx \longrightarrow \int_{R^2} g(x,v_0)\psi dx, \quad \text{as } n \longrightarrow \infty. \tag{3.27}$$

From these, we have $I'(u_n,v_n)(\phi,\psi) = 0$, so (u_0,v_0) is weak solution of problem (P).

Claim 3.4. (u_0,v_0) is nontrivial. By contradiction, since $f(x,t)$ has subcritical growth, from (2.14) and Hölder inequality, we have

$$\begin{aligned}\int_{R^2} f(x,u_n)u_n dx &\le c\int_{R^2} u_n(\exp(\beta u_n^2) - 1)dx \\ &\le c'\left(\int_{R^2}|u_n|^{q'}dx\right)^{1/q'}\left(\int_{R^2}(\exp(\beta q u_n^2) - 1)dx\right)^{1/q},\end{aligned} \tag{3.28}$$

where $1/q' + 1/q = 1$. Choosing suitable β and q, we have

$$\int_{R^2}(\exp(\beta q u_n^2) - 1)dx \le c. \tag{3.29}$$

Then, we obtain

$$\int_{R^2} f(x,u_n)u_n dx \le c\left(\int_{R^2}|u_n|^{q'}dx\right)^{1/q'}. \tag{3.30}$$

Since $u_n \to 0$ in $L^{q'}(R^2)$, as $n \to \infty$, this will lead to

$$\int_{R^2} f(x,u_n)u_n dx \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{3.31}$$

Similarly, we have

$$\int_{R^2} g(x,v_n)v_n dx \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{3.32}$$

Using assumption (H3), we obtain

$$\int_{R^2} F(x,u_n)dx \longrightarrow 0, \quad \int_{R^2} G(x,v_n)dx \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{3.33}$$

This together with $I'(u_n,v_n)(u_n,v_n) \to 0$, we have

$$\int_{R^2}(\nabla u_n \nabla v_n + u_n v_n)dx \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{3.34}$$

Thus, we see that

$$I(u_n,v_n) \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{3.35}$$

which is a contradiction to $I(u_n,v_n) \to c \ge \alpha > 0$, as $n \to \infty$.

Consequently, we have a nontrivial critical point of the functional $I(u,v)$ and conclude the proof of Theorem 2.2. □

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