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Research Article

Antiperiodic Boundary Value Problems for Second-Order Impulsive Ordinary Differential Equations

Chuanzhi Bai

Department of Mathematics, Huaiyin Teachers College, Huaian, Jiangsu 223300, China

Correspondence should be addressed to Chuanzhi Bai, czbai8@sohu.com

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We consider a second-order ordinary differential equation with antiperiodic boundary conditions and impulses. By using Schaefer's fixed-point theorem, some existence results are obtained.

1. Introduction

Impulsive differential equations, which arise in biology, physics, population dynamics, economics, and so forth, are a basic tool to study evolution processes that are subjected to abrupt in their states (see [1–4]). Many literatures have been published about existence of solutions for first-order and second-order impulsive ordinary differential equations with boundary conditions [5–19], which are important for complementing the theory of impulsive equations. In recent years, the solvability of the antiperiodic boundary value problems of first-order and second-order differential equations were studied by many authors, for example, we refer to [20–32] and the references therein. It should be noted that antiperiodic boundary value problems appear in physics in a variety of situations [33, 34]. Recently, the existence results were extended to antiperiodic boundary value problems for first-order impulsive differential equations [35, 36]. Very recently, Wang and Shen [37] investigated the antiperiodic boundary value problem for a class of second-order differential equations by using Schauder's fixed point theorem and the lower and upper solutions method.

Inspired by [35–37], in this paper, we investigate the antiperiodic boundary value problem for second-order impulsive nonlinear differential equations of the form

$$u''(t) + f\big(t, u(t)\big) = 0, \quad t \in J_0 = J \setminus \{t_1, \dots, t_m\},$$
$$\Delta u(t_k) = I_k\big(u(t_k)\big), \quad k = 1, \dots, m,$$

$$\Delta u'(t_k) = I_k^*(u(t_k)), \quad k = 1, \ldots, m,$$
$$u(0) + u(T) = 0, \qquad u'(0) + u'(T) = 0, \tag{1.1}$$

where $J = [0, T]$, $0 < t_1 < t_2 < \cdots < t_m < T$, $f : [0, T] \times R \to R$ is continuous on $(t, x) \in J_0 \times R$, $f(t_k^+, x) := \lim_{t \to t_k^+} f(t, x)$, $f(t_k^-, x) := \lim_{t \to t_k^-} f(t, x)$ exist, $f(t_k^-, x) = f(t_k, x)$; $\Delta u(t_k) = u(t_k^+) - u(t_k^-)$, $\Delta u'(t_k) = u'(t_k^+) - u'(t_k^-)$; $I_k, I_k^* \in C(R, R)$.

To the best of the authors knowledge, no one has studied the existence of solutions for impulsive antiperiodic boundary value problem (1). The following Schaefer's fixed-point theorem is fundamental in the proof of our main results.

Lemma 1.1 (see [38] (Schaefer)). *Let E be a normed linear space with $H : E \to E$ a compact operator. If the set*

$$S := \{x \in E \mid x = \lambda H x, \ \textit{for some } \lambda \in (0, 1)\} \tag{1.2}$$

is bounded, then H has at least one fixed point.

The paper is formulated as follows. In Section 2, some definitions and lemmas are given. In Section 3, we obtain two new existence theorems by using Schaefer's fixed point theorem. In Section 4, an illustrative example is given to demonstrate the effectiveness of the obtained results.

2. Preliminaries

In order to define the concept of solution for (1), we introduce the following spaces of functions:

$PC(J) = \{u : J \to R : u$ is continuous for any $t \in J_0$, $u(t_k^+)$, $u(t_k^-)$ exist, and $u(t_k^-) = u(t_k)$, $k = 1, \ldots, m\}$,

$PC^1(J) = \{u : J \to R : u$ is continuously differentiable for any $t \in J_0$, $u'(t_k^+)$, $u'(t_k^-)$ exist, and $u'(t_k^-) = u'(t_k)$, $k = 1, \ldots, m\}$.

$PC(J)$ and $PC^1(J)$ are Banach space with the norms

$$\|u\|_{PC} = \sup_{t \in J} |u(t)|,$$
$$\|u\|_{PC^1} = \max\{\|u\|_{PC}, \|u'\|_{PC}\}. \tag{2.1}$$

A solution to the impulsive BVP (1) is a function $u \in PC^1(J) \cap C^2(J_0)$ that satisfies (1) for each $t \in J$.

Consider the following impulsive BVP with $\lambda > 0$

$$-u''(t) + \lambda^2 u(t) = \sigma(t), \quad t \in J_0,$$
$$\Delta u(t_k) = I_k(u(t_k)), \quad k = 1, \ldots, m,$$
$$\Delta u'(t_k) = I_k^*(u(t_k)), \quad k = 1, \ldots, m,$$
$$u(0) + u(T) = 0, \qquad u'(0) + u'(T) = 0, \tag{2.2}$$

where $\sigma \in PC(J)$.

For convenience, we set $I_k = I_k(u(t_k))$, $I_k^* = I_k^*(u(t_k))$.

Lemma 2.1. *$u \in PC^1(J) \cap C^2(J_0)$ is a solution of (2.2) if and only if $u \in PC^1(J)$ is a solution of the impulsive integral equation*

$$u(t) = \int_0^T G(t,s)\sigma(s)ds + \sum_{k=1}^{m} \left[G(t,t_k)\left(-I_k^*\right) + W(t,t_k)I_k\right], \tag{2.3}$$

where

$$\begin{gathered} G(t,s) = \frac{1}{2\lambda} \begin{cases} \dfrac{e^{-\lambda(t-s)}}{1+e^{-\lambda T}} - \dfrac{e^{\lambda(t-s)}}{1+e^{\lambda T}}, & 0 \le s < t \le T, \\ \dfrac{e^{\lambda(T+t-s)}}{1+e^{\lambda T}} - \dfrac{e^{-\lambda(T+t-s)}}{1+e^{-\lambda T}}, & 0 \le t \le s \le T, \end{cases} \\ W(t,s) = \frac{1}{2} \begin{cases} \dfrac{e^{-\lambda(t-s)}}{1+e^{-\lambda T}} + \dfrac{e^{\lambda(t-s)}}{1+e^{\lambda T}}, & 0 \le s < t \le T, \\ -\dfrac{e^{\lambda(T+t-s)}}{1+e^{\lambda T}} - \dfrac{e^{-\lambda(T+t-s)}}{1+e^{-\lambda T}}, & 0 \le t \le s \le T. \end{cases} \end{gathered} \tag{2.4}$$

Proof. If $u \in PC^1(J) \cap C^2(J_0)$is a solution of (2.2), setting

$$v(t) = u'(t) + \lambda u(t), \tag{2.5}$$

then, by the first equation of (2.2) we have

$$v'(t) - \lambda v(t) = -\sigma(t), \quad t \ne t_k. \tag{2.6}$$

Multiplying (2.6) by $e^{-\lambda t}$ and integrating on $[0, t_1)$ and $(t_1, t]$ $(t_1 < t \le t_2)$, respectively, we get

$$\begin{gathered} e^{-\lambda t_1} v\left(t_1^-\right) - v(0) = -\int_0^{t_1} \sigma(s) e^{-\lambda s} ds, \\ e^{-\lambda t} v(t) - e^{-\lambda t_1} v\left(t_1^+\right) = -\int_{t_1}^{t} \sigma(s) e^{-\lambda s} ds, \quad t_1 < t \le t_2. \end{gathered} \tag{2.7}$$

So

$$v(t) = e^{\lambda t}\left[v(0) - \int_0^t e^{-\lambda s}\sigma(s)ds + e^{-\lambda t_1}\Delta v(t_1)\right], \quad t_1 < t \le t_2. \tag{2.8}$$

In the same way, we can obtain that

$$v(t) = e^{\lambda t}\left[v(0) - \int_0^t e^{-\lambda s}\sigma(s)ds + \sum_{0<t_k<t} e^{-\lambda t_k}\left(I_k^* + \lambda I_k\right)\right], \quad t \in J, \tag{2.9}$$

where $v(0) = u'(0) + \lambda u(0)$. Integrating (2.5), we have

$$u(t) = e^{-\lambda t}\left[u(0) + \int_0^t v(s)e^{\lambda s}ds + \sum_{0<t_k<t} e^{\lambda t_k} I_k\right], \quad t \in J. \tag{2.10}$$

By (2.9), we get

$$\begin{aligned}\int_0^t v(s)e^{\lambda s}ds = \frac{1}{2\lambda}\Bigg[& v(0)\left(e^{2\lambda t}-1\right) - \int_0^t \left(e^{2\lambda t} - e^{2\lambda s}\right)\sigma(s)e^{-\lambda s}ds \\ & + \sum_{0<t_k<t}\left(e^{2\lambda t} - e^{2\lambda t_k}\right)e^{-\lambda t_k}\left(I_k^* + \lambda I_k\right)\Bigg].\end{aligned} \tag{2.11}$$

Substituting (2.11) into (2.10), we obtain

$$\begin{aligned}u(t) = \frac{1}{2\lambda}\Bigg[&\left(\lambda u(0) - u'(0)\right)e^{-\lambda t} + \left(u'(0) + \lambda u(0)\right)e^{\lambda t} \\ &+ \int_0^t \left(e^{-\lambda(t-s)} - e^{\lambda(t-s)}\right)\sigma(s)ds + \sum_{0<t_k<t} e^{\lambda(t-t_k)}\left(I_k^* + \lambda I_k\right) \\ &- \sum_{0<t_k<t} e^{-\lambda(t-t_k)}\left(I_k^* - \lambda I_k\right)\Bigg], \quad t \in J,\end{aligned} \tag{2.12}$$

$$\begin{aligned}u'(t) = \frac{1}{2}\Bigg[& -\left(\lambda u(0) - u'(0)\right)e^{-\lambda t} + \left(u'(0) + \lambda u(0)\right)e^{\lambda t} \\ &- \int_0^t \left(e^{-\lambda(t-s)} + e^{\lambda(t-s)}\right)\sigma(s)ds + \sum_{0<t_k<t} e^{\lambda(t-t_k)}\left(I_k^* + \lambda I_k\right) \\ &+ \sum_{0<t_k<t} e^{-\lambda(t-t_k)}\left(I_k^* - \lambda I_k\right)\Bigg], \quad t \in J.\end{aligned} \tag{2.13}$$

In view of $u(0) + u(T) = 0$ and $u'(0) + u'(T) = 0$, we have

$$\begin{aligned}u'(0) + \lambda u(0) &= \frac{1}{1+e^{\lambda T}}\left[\int_0^T e^{\lambda(T-s)}\sigma(s)ds - \sum_{0<t_k<T} e^{\lambda(T-t_k)}\left(I_k^* + \lambda I_k\right)\right], \\ \lambda u(0) - u'(0) &= \frac{1}{1+e^{-\lambda T}}\left[-\int_0^T e^{-\lambda(T-s)}\sigma(s)ds + \sum_{0<t_k<T} e^{-\lambda(T-t_k)}\left(I_k^* - \lambda I_k\right)\right].\end{aligned} \tag{2.14}$$

Substituting (2.14) into (2.12), by routine calculation, we can get (2.3).

Conversely, if u is a solution of (2.3), then direct differentiation of (2.3) gives $-u''(t) = \sigma(t) - \lambda^2 u(t)$, $t \neq t_k$. Moreover, we obtain $\Delta u|_{t=t_k} = I_k(u(t_k))$, $\Delta u'|_{t=t_k} = I_k^*(u(t_k))$, $u(0)+u(T) = 0$ and $u'(0) + u'(T) = 0$. Hence, $u \in PC^1(J) \cap C^2(J_0)$ is a solution of (2.2). □

Remark 2.2. We call $G(t,s)$ above the Green function for the following homogeneous BVP:

$$\begin{gathered} -u''(t)+\lambda^2 u(t)=0, \quad t\in J, \\ u(0)+u(T)=0, \qquad u'(0)+u'(T)=0. \end{gathered} \tag{2.15}$$

Define a mapping $A : PC^1(J) \to PC^1(J)$ by

$$Au(t)=\int_0^T G(t,s)\left[f(s,u(s))+\lambda^2 u(s)\right]ds+\sum_{k=1}^{m}\left[G(t,t_k)(-I_k^*)+W(t,t_k)I_k\right], \quad t\in[0,T]. \tag{2.16}$$

In view of Lemma 2.1, we easily see that u is a fixed point of operator A if and only if u is a solution to the impulsive boundary value problem (1).

It is easy to check that

$$|G(t,s)| \le \frac{e^{\lambda T}-1}{2\lambda(1+e^{\lambda T})}, \qquad |W(t,s)| \le \frac{1}{2}. \tag{2.17}$$

Lemma 2.3. *If $u \in PC^1(J)$ and $u(0)+u(T)=0$, then*

$$\|u\|_{PC} \le \frac{1}{2}\left(\int_0^T |u'(s)|ds+\sum_{k=1}^{m}|\Delta u(t_k)|\right). \tag{2.18}$$

Proof. Since $u \in PC^1(J)$, we have

$$u(t)=u(0)+\sum_{0<t_k<t}\Delta u(t_k)+\int_0^t u'(s)ds. \tag{2.19}$$

Set $t=T$, we obtain from $u(0)+u(T)=0$ that

$$u(0)=-\frac{1}{2}\left(\sum_{k=1}^{m}\Delta u(t_k)+\int_0^T u'(s)ds\right). \tag{2.20}$$

Substituting (2.20) into (2.19), we get

$$\begin{aligned} |u(t)| &= \left|\frac{1}{2}\left(\int_0^t u'(s)ds-\int_t^T u'(s)ds\right)+\frac{1}{2}\left(\sum_{0<t_k<t}\Delta u(t_k)-\sum_{t\le t_k}\Delta u(t_k)\right)\right| \\ &\le \frac{1}{2}\left(\int_0^t |u'(s)|ds+\int_t^T |u'(s)|ds\right)+\frac{1}{2}\left(\sum_{0<t_k<t}|\Delta u(t_k)|+\sum_{t\le t_k}|\Delta u(t_k)|\right) \\ &= \frac{1}{2}\left(\int_0^T |u'(s)|ds+\sum_{k=1}^{m}|\Delta u(t_k)|\right). \end{aligned} \tag{2.21}$$

The proof is complete. □

3. Main results

In this section, we study the existence of solutions for BVP (1). For this purpose we assume that there exist constants $0 < \eta < 1$, functions $a, b, h \in C(J, [0, +\infty))$, and nonnegative constants $\alpha_k, \beta_k, \gamma_k, \delta_k$ $(k = 1, 2, \ldots, m)$ such that

(H_1) $|f(t,u)| \le a(t)|u| + b(t)|u|^\eta + h(t)$, and

(H_2) $|I_k(u)| \le \alpha_k|u| + \beta_k$, $|I_k^*(u)| \le \gamma_k|u| + \delta_k$, $k = 1, \ldots, m$

hold.

Remark 3.1. (H_1) means that the nonlinearity growths at most linearly in u, (H_2) implies that the impulses are (at most) linear.

For convenience, let

$$p_1 = \frac{3}{2}\left(\int_0^T a(t)dt + \lambda^2 T + \sum_{i=1}^m \gamma_i\right),$$
$$p_2 = \frac{3}{2}\int_0^T b(t)dt, \tag{3.1}$$
$$p_3 = \frac{3}{2}\left(\int_0^T h(t)dt + \sum_{i=1}^m \delta_i\right).$$

Theorem 3.2. *Suppose that conditions (H_1) and (H_2) are satisfied. Further assume that*

$$\frac{T}{4}q_1 + \frac{\sqrt{Tq_1}}{2}\sum_{i=1}^m \alpha_i + \frac{m}{4}\sum_{i=1}^m \alpha_i^2 < 1, \tag{3.2}$$

holds, where $q_1 = \int_0^T a(t)dt + \sum_{i=1}^m (p_1\alpha_i + \gamma_i)$ *and* p_1 *as in* (3.1). *Then, BVP* (1) *has at least one solution.*

Proof. It is easy to check by Arzela-Ascoli theorem that the operator A is completely continuous. Assume that u is a solution of the equation

$$u = \mu A u, \quad \mu \in (0,1). \tag{3.3}$$

Then,

$$\begin{aligned} u''(t) &= \mu(Au)''(t) = \mu\left[-f(t,u(t)) - \lambda^2 u(t) + \lambda^2 (Au)(t)\right] \\ &= -\mu f(t,u(t)) - \lambda^2(\mu - 1)u(t), \end{aligned} \tag{3.4}$$

$$-u(t)u''(t) = \mu u(t) f(t,u(t)) + \lambda^2(\mu-1)u^2(t) \le \mu u(t) f(t,u(t)). \tag{3.5}$$

Integrating (3.4) from 0 to T, we get that

$$u'(T) - u'(0) = \int_0^T u''(t)dt + \sum_{i=1}^{m} I_i^* = -\mu\int_0^T f(t,u(t))dt - \lambda^2(\mu-1)\int_0^T u(t)dt + \sum_{i=1}^{m} I_i^*. \tag{3.6}$$

In view of $u'(0) + u'(T) = 0$, we obtain by (3.6) that

$$|u'(0)| \le \frac{1}{2}\int_0^T |f(t,u(t))|dt + \frac{\lambda^2}{2}\int_0^T |u(t)|dt + \frac{1}{2}\sum_{i=1}^{m}|I_i^*|. \tag{3.7}$$

Integrating (3.4) from 0 to t, we obtain that

$$u'(t) - u'(0) = \int_0^t u''(s)ds + \sum_{0<t_i<t} I_i^* = -\mu\int_0^t f(s,u(s))ds - \lambda^2(\mu-1)\int_0^t u(s)ds + \sum_{0<t_i<t} I_i^*. \tag{3.8}$$

From (3.7) and (3.8), we have

$$\begin{aligned}
|u'(t)| &\le |u'(0)| + \int_0^T |f(s,u(s))|ds + \lambda^2\int_0^T |u(s)|ds + \sum_{i=1}^{m}|I_i^*| \\
&\le \frac{3}{2}\int_0^T \left(a(t)|u(t)| + b(t)|u(t)|^\eta + h(t)\right)dt + \frac{3}{2}\lambda^2\int_0^T |u(t)|dt + \frac{3}{2}\sum_{i=1}^{m}(\gamma_i\|u\|_{PC} + \delta_i) \\
&\le \frac{3}{2}\left(\|u\|_{PC}\int_0^T a(t)dt + \|u\|_{PC}^\eta\int_0^T b(t)dt + \int_0^T h(t)dt\right) + \frac{3}{2}\lambda^2 T\|u\|_{PC} + \frac{3}{2}\sum_{i=1}^{m}(\gamma_i\|u\|_{PC} + \delta_i),
\end{aligned} \tag{3.9}$$

that is,

$$\|u'\|_{PC} \le \frac{3}{2}\left(\int_0^T a(t)dt + \lambda^2 T + \sum_{i=1}^{m}\gamma_i\right)\|u\|_{PC} + \frac{3}{2}\int_0^T b(t)dt\|u\|_{PC}^\eta + \frac{3}{2}\int_0^T h(t)dt + \frac{3}{2}\sum_{i=1}^{m}\delta_i. \tag{3.10}$$

Thus,

$$\|u'\|_{PC} \le p_1\|u\|_{PC} + p_2\|u\|_{PC}^\eta + p_3, \tag{3.11}$$

where $p_1,\ p_2,\ p_3$ are as in (3.1). Integrating (3.5) from 0 to T, we get that

$$-\int_0^T u(t)u''(t)dt \le \mu\int_0^T u(t)f(t,u(t))dt. \tag{3.12}$$

In view of $u(0)+u(T)=0$ and $u'(0)+u'(T)=0$, we have

$$\begin{aligned}
\int_0^T u(t)u''(t)dt &= \int_0^T u(t)d\left(u'(t)\right)\\
&= \int_0^{t_1} u(t)d\left(u'(t)\right)+\int_{t_1}^{t_2} u(t)d\left(u'(t)\right)+\cdots+\int_{t_n}^{T} u(t)d\left(u'(t)\right)\\
&= (t)u'(t)|_0^{t_1}-\int_0^{t_1}\left(u'(t)\right)^2dt+u(t)u'(t)|_{t_1}^{t_2}-\int_{t_1}^{t_2}\left(u'(t)\right)^2dt\\
&\quad+\cdots+u(t)u'(t)|_{t_n}^{T}-\int_{t_n}^{T}\left(u'(t)\right)^2dt\\
&= u(t_1-0)u'(t_1-0)-u(0)u'(0)+u(t_2-0)u'(t_2-0)-u(t_1+0)u'(t_1+0)\\
&\quad+\cdots+u(T)u'(T)-u(t_n+0)u'(t_n+0)-\int_0^T\left(u'(t)\right)^2dt\\
&= u(t_1-0)u'(t_1-0)-u(t_1+0)u'(t_1+0)\\
&\quad+\cdots+u(t_n-0)u'(t_n-0)-u(t_n+0)u'(t_n+0)-\int_0^T\left(u'(t)\right)^2dt\\
&= u(t_1-0)u'(t_1-0)-u(t_1-0)u'(t_1+0)+u(t_1-0)u'(t_1+0)\\
&\quad-u(t_1+0)u'(t_1+0)+\cdots+u(t_n-0)u'(t_n-0)-u(t_n-0)u'(t_n+0)\\
&\quad+u(t_n-0)u'(t_n+0)-u(t_n+0)u'(t_n+0)-\int_0^T\left(u'(t)\right)^2dt\\
&= -u(t_1-0)I_1^*-u'(t_1+0)I_1-\cdots-u(t_n-0)I_n^*-u'(t_n+0)I_n-\int_0^T\left(u'(t)\right)^2dt.
\end{aligned} \tag{3.13}$$

Substituting (3.13) into (3.12), we obtain by (H_2), (H_3), and (3.11) that

$$\begin{aligned}
\int_0^T\left(u'(t)\right)^2dt &\le \mu\int_0^T u(t)f\left(t,u(t)\right)dt-u(t_1-0)I_1^*\\
&\quad-u'(t_1+0)I_1-\cdots-u(t_n-0)I_n^*-u'(t_n+0)I_n\\
&\le \mu\int_0^T u(t)f\left(t,u(t)\right)dt+\|u\|_{PC}\sum_{i=1}^m\left|I_i^*\right|+\|u'\|_{PC}\sum_{i=1}^m\left|I_i\right|\\
&\le \int_0^T\left(a(t)u^2(t)+b(t)\left|u(t)\right|^{1+\eta}+h(t)\left|u(t)\right|\right)dt\\
&\quad+\|u\|_{PC}\sum_{i=1}^m\left(\gamma_i\|u\|_{PC}+\delta_i\right)+\|u'\|_{PC}\sum_{i=1}^m\left(\alpha_i\|u\|_{PC}+\beta_i\right)\\
&\le \|u\|_{PC}^2\int_0^T a(t)dt+\|u\|_{PC}^{1+\eta}\int_0^T b(t)dt+\|u\|_{PC}\int_0^T h(t)dt\\
&\quad+\sum_{i=1}^m\gamma_i\|u\|_{PC}^2+\sum_{i=1}^m\delta_i\|u\|_{PC}+\left(p_1\|u\|_{PC}+p_2\|u\|_{PC}^{\eta}+p_3\right)\sum_{i=1}^m\left(\alpha_i\|u\|_{PC}+\beta_i\right).
\end{aligned} \tag{3.14}$$

Thus,

$$\int_0^T (u'(t))^2 dt \le q_1\|u\|_{PC}^2 + q_2\|u\|_{PC}^{1+\eta} + q_3\|u\|_{PC} + q_4\|u\|_{PC}^{\eta} + q_5, \tag{3.15}$$

where

$$\begin{aligned}
q_1 &= \int_0^T a(t)dt + \sum_{i=1}^m (p_1\alpha_i + \gamma_i),\\
q_2 &= \int_0^T b(t)dt + p_2\sum_{i=1}^m \alpha_i,\\
q_3 &= \int_0^T h(t)dt + \sum_{i=1}^m (p_1\beta_i + p_3\alpha_i + \delta_i),\\
q_4 &= p_2\sum_{i=1}^m \beta_i,\\
q_5 &= p_3\sum_{i=1}^m \beta_i.
\end{aligned} \tag{3.16}$$

By Lemma 2.3 and (3.15), we have

$$\begin{aligned}
\|u\|_{PC}^2 &\le \frac{1}{4}\left(\int_0^T |u'(t)|dt\right)^2 + \frac{1}{2}\int_0^T |u'(t)|dt\sum_{i=1}^m |I_i| + \frac{1}{4}\left(\sum_{i=1}^m |I_i|\right)^2\\
&\le \frac{T}{4}\int_0^T (u'(t))^2 dt + \frac{\sqrt{T}}{2}\left(\int_0^T (u'(t))^2 dt\right)^{1/2}\sum_{i=1}^m |I_i| + \frac{m}{4}\sum_{i=1}^m |I_i|^2\\
&\le \frac{T}{4}\left[q_1\|u\|_{PC}^2 + q_2\|u\|_{PC}^{1+\eta} + q_3\|u\|_{PC} + q_4\|u\|_{PC}^{\eta} + q_5\right]\\
&\quad + \frac{\sqrt{T}}{2}\left[q_1\|u\|_{PC}^2 + q_2\|u\|_{PC}^{1+\eta} + q_3\|u\|_{PC} + q_4\|u\|_{PC}^{\eta} + q_5\right]^{1/2}\\
&\quad \times \sum_{i=1}^m (\alpha_i\|u\|_{PC} + \beta_i) + \frac{m}{4}\sum_{i=1}^m (\alpha_i^2\|u\|_{PC}^2 + 2\alpha_i\beta_i\|u\|_{PC} + \beta_i^2)\\
&= \left(\frac{Tq_1}{4} + \frac{\sqrt{Tq_1}}{2}\sum_{i=1}^m \alpha_i + \frac{m}{4}\sum_{i=1}^m \alpha_i^2\right)\|u\|_{PC}^2 + \cdots .
\end{aligned} \tag{3.17}$$

It follows from the above inequality and (3.2) that there exists $M_1 > 0$ such that $\|u\|_{PC} \le M_1$. Hence, we get by (3.11) that

$$\|u'\|_{PC} \le a_1M_1 + a_2M_1^{\eta} + a_3 := M_2. \tag{3.18}$$

Thus, $\|u\|_{PC^1} \leq \max\{M_1, M_2\}$. It follows from Lemma 1.1 that BVP (1) has at least one solution. The proof is complete. □

Theorem 3.3. *Assume that* (H_2) *holds. Suppose that there exist a continuous and nondecreasing function* $\varphi : [0,\infty) \to [0,\infty)$ *and a nonnegative function* $c \in C(J)$ *with*

$$\left|f(t,u) + \lambda^2 u\right| \leq c(t)\varphi(|u|), \quad t \in J,\, u \in R. \tag{3.19}$$

Moreover suppose that

$$\limsup_{u\to\infty} \frac{\varphi(u)}{u} < L \tag{3.20}$$

holds, where

$$L := \frac{1 - \left(\left(e^{\lambda T}-1\right)/2\lambda\left(1+e^{\lambda T}\right)\right)\sum_{i=1}^{m}\gamma_i - (1/2)\sum_{i=1}^{m}\alpha_i}{\left(\left(e^{\lambda T}-1\right)/2\lambda\left(1+e^{\lambda T}\right)\right)\int_0^T c(s)ds} > 0. \tag{3.21}$$

Then, BVP (1) *has at least one solution.*

Proof. From (3.20), there exist $0 < \varepsilon < L$ and $M > 0$ such that

$$\varphi(v) \leq (L-\varepsilon)v, \quad v \geq M. \tag{3.22}$$

Thus, there exists $K > 0$ such that

$$\varphi(v) \leq (L-\varepsilon)v + K, \quad v \geq 0. \tag{3.23}$$

Assume that u is a solution of the equation

$$u = \mu A u, \quad \mu \in (0,1). \tag{3.24}$$

Then, we have by (3.19), (2.17), and (3.23) that

$$\begin{aligned} \left|u(t)\right| &= \mu\left|\int_0^T G(t,s)\left(f(s,u(s)) + \lambda^2 u(s)\right)ds + \sum_{k=1}^{m}\left[G(t,t_k)\left(-I_k^*\right) + W(t,t_k)I_k\right]\right| \\ &\leq \frac{e^{\lambda T}-1}{2\lambda\left(1+e^{\lambda T}\right)}\int_0^T c(s)\varphi(|u|)ds + \frac{e^{\lambda T}-1}{2\lambda\left(1+e^{\lambda T}\right)}\sum_{i=1}^{m}\left(\gamma_i\|u\|_{PC} + \delta_i\right) + \frac{1}{2}\sum_{i=1}^{m}\left(\alpha_i\|u\|_{PC} + \beta_i\right). \end{aligned} \tag{3.25}$$

Thus, we have

$$\begin{aligned}\|u\|_{PC} &\le \frac{e^{\lambda T}-1}{2\lambda(1+e^{\lambda T})}\int_0^T c(s)ds\big((L-\varepsilon)\|u\|_{PC}+K\big)\\&\quad+\left(\frac{e^{\lambda T}-1}{2\lambda(1+e^{\lambda T})}\sum_{i=1}^m\gamma_i+\frac{1}{2}\sum_{i=1}^m\alpha_i\right)\|u\|_{PC}+\frac{e^{\lambda T}-1}{2\lambda(1+e^{\lambda T})}\sum_{i=1}^m\delta_i+\frac{1}{2}\sum_{i=1}^m\beta_i,\end{aligned}\tag{3.26}$$

that is,

$$\varepsilon\frac{e^{\lambda T}-1}{2\lambda(1+e^{\lambda T})}\int_0^T c(s)ds\|u\|_{PC}\le\frac{e^{\lambda T}-1}{2\lambda(1+e^{\lambda T})}K\int_0^T c(s)ds+\frac{e^{\lambda T}-1}{2\lambda(1+e^{\lambda T})}\sum_{i=1}^m\delta_i+\frac{1}{2}\sum_{i=1}^m\beta_i,\tag{3.27}$$

which implies that there exists $M_3>0$ such that $\|u\|_{PC}\le M_3$. By (3.7), (3.8), and (3.23), we get

$$\begin{aligned}|u'(t)| &\le \frac{3}{2}\int_0^T|f(s,u(s))|ds+\frac{3}{2}\lambda^2\int_0^T|u(s)|ds+\frac{3}{2}\sum_{i=1}^m|I_i^*|\\&\le\frac{3}{2}\int_0^T|f(s,u(s))+\lambda^2u(s)|ds+3\lambda^2\int_0^T|u(s)|ds+\frac{3}{2}\sum_{i=1}^m|I_i^*|\\&\le\frac{3}{2}\int_0^T c(s)\psi(|u(s)|)ds+3\lambda^2\int_0^T|u(s)|ds+\frac{3}{2}\sum_{i=1}^m(\gamma_i\|u\|_{PC}+\delta_i)\\&\le\frac{3}{2}\int_0^T c(s)ds(L\|u\|_{PC}+K)+3\lambda^2T\|u\|_{PC}+\frac{3}{2}\sum_{i=1}^m(\gamma_i\|u\|_{PC}+\delta_i),\end{aligned}\tag{3.28}$$

which implies that

$$\begin{aligned}\|u'\|_{PC} &\le\left(\frac{3L}{2}\int_0^T c(s)ds+3\lambda^2T+\frac{3}{2}\sum_{i=1}^m\gamma_i\right)\|u\|_{PC}+\frac{3K}{2}\int_0^T c(s)ds+\frac{3}{2}\sum_{i=1}^m\delta_i\\&\le\frac{3}{2}\left(L\int_0^T c(s)ds+2\lambda^2T+\sum_{i=1}^m\gamma_i\right)M_3+\frac{3K}{2}\int_0^T c(s)ds+\frac{3}{2}\sum_{i=1}^m\delta_i:=M_4.\end{aligned}\tag{3.29}$$

Hence, $\|u\|_{PC^1}\le\max\{M_3,M_4\}$. It follows from Lemma 1.1 that BVP (1) has at least one solution. The proof is complete. □

4. Example

In this section, we give an example to illustrate the effectiveness of our results.

Example 4.1. Consider the problem

$$
\begin{gathered}
u''(t) + \frac{1}{\pi}u(t)\cos^2 t + \frac{1}{2}e^t u^{1/2}(t) + 1 + \tan t = 0, \quad t \in \left[0, \frac{\pi}{2}\right] \setminus \left\{\frac{\pi}{3}\right\}, \\
\Delta u(t_1) = \frac{1}{3}\sin\left(u(t_1)\right) + \frac{1}{4}, \quad \Delta u'(t_1) = \frac{1}{2}u(t_1) + \frac{1}{3}, \quad t_1 = \frac{\pi}{3}, \\
u(0) + u(T) = 0, \qquad u'(0) + u'(T) = 0,
\end{gathered}
\tag{4.1}
$$

Let $f(t,u) = (1/\pi)u\cos^2 t + (1/2)e^t u^{1/2} + 1 + \tan t$, $I_1(u) = (1/3)\sin u + (1/4)$, $I_1^*(u) = (1/2)u + (1/3)$, $T = (\pi/2)$, $J = [0, \pi/2]$. It is easy to show that

$$
\left|f(t,u)\right| \le a(t)|u| + b(t)|u|^{1/2} + h(t),
\tag{4.2}
$$

where $a(t) = (1/\pi)\cos^2 t$, $b(t) = (1/2)e^t$, $h(t) = 1 + \tan t$. And

$$
\left|I_1(u)\right| \le \frac{1}{3}|u| + \frac{1}{4}, \qquad \left|I_1^*(u)\right| \le \frac{1}{2}|u| + \frac{1}{3}.
\tag{4.3}
$$

Thus, (H_1) and (H_2) hold. Obviously, $\alpha_1 = 1/3$, $\beta_1 = 1/4$, $\gamma_1 = 1/2$, $\delta_1 = 1/3$, and $m = 1$. Let $\lambda^2 = 1/4\pi$, we have

$$
\begin{aligned}
p_1 &= \frac{3}{2}\left(\int_0^T a(t)dt + 2\lambda^2 T + \sum_{i=1}^{m}\gamma_i\right) = \frac{3}{2}\left(\frac{1}{4} + \frac{1}{4} + \frac{1}{2}\right) = \frac{3}{2}, \\
q_1 &= \int_0^T a(t)dt + \sum_{i=1}^{m}(p_1\alpha_i + \gamma_i) = \frac{1}{4} + \frac{3}{2}\cdot\frac{1}{3} + \frac{1}{2} = \frac{5}{4}.
\end{aligned}
\tag{4.4}
$$

Therefore,

$$
\frac{Tq_1}{4} + \frac{\sqrt{Tq_1}}{2}\sum_{i=1}^{m}\alpha_i + \frac{m}{4}\sum_{i=1}^{m}\alpha_i^2 = 0.7522 < 1,
\tag{4.5}
$$

which implies that (3.2) holds. So, all the conditions of Theorem 3.2 are satisfied. By Theorem 3.2, antiperiod boundary value problem (4.1) has at least one solution.

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