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Research Article

An Approximation Approach to Eigenvalue Intervals for Singular Boundary Value Problems with Sign Changing and Superlinear Nonlinearities

Haisheng Lü,[1] Ravi P. Agarwal,[2,3] and Donal O'Regan[4]

[1] *Department of Applied Mathematics, Hohai University, Nanjing 210098, China*
[2] *Department of Mathematical Sciences, Florida Institute of Technology, Melbourne, FL 32901-6975, USA*
[3] *KFUPM Chair Professor, Mathematics and Statistics Department, King Fahd University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia*
[4] *Department of Mathematics, National University of Ireland, Galway, Ireland*

Correspondence should be addressed to Haishen Lü, haishen2001@yahoo.com.cn

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This paper studies the eigenvalue interval for the singular boundary value problem $-u'' = g(t,u) + \lambda h(t,u)$, $t \in (0,1)$, $u(0) = 0 = u(1)$, where $g + h$ may be singular at $u = 0$, $t = 0, 1$, and may change sign and be superlinear at $u = +\infty$. The approach is based on an approximation method together with the theory of upper and lower solutions.

1. Introduction

The singular boundary value problems of the form

$$-u'' = f(t,u), \quad t \in (0,1),$$
$$u(0) = 0 = u(1) \tag{1.1}$$

occurs in several problems in applied mathematics, see [1–6] and their references. In many papers, a critical condition is that

$$f(t,r) \geq 0 \quad \text{for } (t,r) \in (0,1) \times (0,\infty) \tag{1.2}$$

or there exists a constant $L > 0$ such that for any compact set $K \subset (0,1)$, there is $\varepsilon = \varepsilon_K > 0$ such that

$$f(t,r) \geq L \quad \forall t \in K,\ r \in (0,\varepsilon], \qquad \lim_{r\to\infty} \frac{f(t,r)}{r} = 0 \quad \forall t \in (0,1). \tag{1.3}$$

We refer the reader to [1–4]. In the case, when $f(t,r)$ may change sign in a neighborhood of $r = 0$ and $\limsup_{r\to+\infty}(f(t,r)/r) = +\infty$ for $t \in (0,1)$, very few existence results are available in literature [1].

In this paper we study positive solutions of the second boundary value problem

$$-u'' = g(t,u) + \lambda h(t,u), \quad t \in (0,1), \qquad u(0) = 0 = u(1); \tag{1.4}$$

here $g : (0,1) \times (0,\infty) \to R$ and $h : (0,1) \times [0,\infty) \to (0,\infty)$ are continuous, so as a result, our nonlinearity may be singular at $t = 0,1$ and $u = 0$. Also our nonlinearity may change sign and be superlinear at $u = +\infty$. Our main existence results (Theorems 1.1, 1.2 and 1.4) are new (see Remark 1.5, Examples 3.1 and 3.2).

A function u is a solution of the boundary value problem (1.4) if $u : [0,1] \to R, u$ satisfies the differential equation (1.4) on $(0,1)$ and the stated boundary data.

Let $C[0,1]$ denote the class of maps u continuous on $[0,1]$, with norm $|u|_\infty = \max_{t\in[0,1]}|u(t)|$. We put $\min\{a,b\} = a \wedge b$; $\max\{a,b\} = a \vee b$. Given $\alpha,\beta \in C[0,1]$, $\alpha \leq \beta$, let

$$D_\alpha^\beta = \{v \mid v \in C[0,1],\ \alpha \leq v \leq \beta\}. \tag{1.5}$$

Let

$$M = \left\{h \in C(0,1) : \int_0^1 |h(s)|ds < \infty \text{ with } \lim_{t\to 0^+} t|h(t)| < \infty,\ \lim_{t\to 1^-}(1-t)|h(t)| < \infty\right\}. \tag{1.6}$$

In this paper, we suppose the following conditions hold:

$(G1)$ suppose there exist $g_i : (0,1) \times (0,\infty) \to (0,\infty)$ $(i = 1,2)$ continuous functions such that

$$g_i(t,\cdot) \text{ is strictly decreasing for } t \in (0,1), \qquad g_1(\cdot, r\phi_1(\cdot)),\quad g_2(\cdot,r) \in M \quad \forall r > 0, \qquad -g_1(t,r) \leq g(t,r) \leq g_2(t,r) \quad \text{for } (t,r) \in (0,1) \times (0,\infty), \tag{1.7}$$

where ϕ_1 is defined in Lemma 2.1;

$(H1)$ there exist $h_i : (0,1) \times [0,\infty) \to [0,\infty)$ $(i = 1,2)$ continuous functions such that

$$\begin{gathered} h_i(t,\cdot) \text{ is increasing for } t \in (0,1), \\ h_1(\cdot,r), h_2(\cdot,r) \in M \quad \text{for } r > 0, \\ h_1(t,r) \le h(t,r) \le h_2(t,r) \quad \text{for } (t,r) \in (0,1) \times [0,\infty); \end{gathered} \tag{1.8}$$

$(H2)$ there exists $\overline{r} > 0$ such that $h_1(t,\overline{r}) > 0$ for $t \in (0,1)$.

The main results of the paper are the following.

Theorem 1.1. *Suppose* $(G1)$, $(H1)$, $(H2)$ *and the following conditions hold:*

$(G2)$ *for all* $r_2 > r_1 > 0$, *there exists* $\gamma(\cdot) \in M$ *such that* $g_2(\cdot,r) + \gamma(\cdot)r$ *is increasing in* (r_1,r_2):

$(H3)$

$$\lim_{r\to\infty} \frac{h_1(t,r)}{r} = 0 \quad \forall t \in (0,1); \tag{1.9}$$

$(H4)$ there exists a sequence $\{R_j\}_{j=1}^{\infty}$ such that $\lim_{j\to\infty} R_j = \infty$ and

$$\lim_{j\to\infty} \frac{h_2(s, R_j + a_1)}{R_j} = 0, \tag{1.10}$$

where $a_1 = 1 + \int_0^1 g_2(s,1)ds$.

Then there exists $\lambda_1^* > 0$ such that for every $\lambda \ge \lambda_1^*$, (1.4) has at least one positive solution $u \in C[0,1] \cap C^1(0,1)$ and $u > 0$ for $t \in (0,1)$.

Theorem 1.2. *Suppose* $(G1)$, $(H1)$, $(H2)$ *and the following conditions hold:*

$(G3)$ *for all* $r_2 > r_1 > 0$ *there exists* $\gamma(\cdot) \in M$ *such that* $g(t,r) + \gamma(t)r$ *is increasing in* (r_1,r_2);

$(G4)$ *there exists* $c_1 > 0$ *such that*

$$0 \le g(t,r), \quad t \in (0,1), \ 0 < r < c_1; \tag{1.11}$$

$(G5)$ *there exists* $c_2 \in (0,c_1)$, $0 < \beta < 1$ *such that for all* $r \in (0,c_2)$

$$\int_0^1 t(1-t)\overline{g}_1(t, rl(t))dt \ge r\pi, \tag{1.12}$$

where

$$\overline{g}_m(t,r) = \min\left\{ g(t,r), \frac{m}{r^\beta} \right\} \quad \text{for } m \ge 1, \tag{1.13}$$

and $l(t) = \min\{t, 1-t\}$ *for* $t \in [0,1]$.

Then there exists $\lambda_2^* > 0$ such that

(i) if $0 < \lambda < \lambda_2^*$, (1.4) has at least one solution $u \in C[0,1] \cap C^1(0,1)$ and $u > 0$ for $t \in (0,1)$;

(ii) if $\lambda > \lambda_2^*$, (1.4) has no solutions.

Remark 1.3. Notice that $\overline{g}_m(t,r)$ satisfies $(G1)$, $(G3)$, $(G4)$ and for fixed $m \geq 1$,

$$\int_0^1 t(1-t)\overline{g}_m(t,rl(t))dt \geq r\pi \quad \text{for } r \in (0,c_2),$$
$$g(t,r) \geq \overline{g}_m(t,r) \geq \overline{g}_1(t,r) \quad \text{for } t \in (0,1),\ r \in (0,\infty). \tag{1.14}$$

Theorem 1.4. *Suppose* $(G1)$, $(H1)$, $(H2)$ *and the following conditions hold:*

$(G6)$ *there exists* $\tau \geq \tau_1$ *such that*

$$\lim_{r\to 0^+} \frac{\tau r + g^-(t,r)}{h(t,r)} = 0, \tag{1.15}$$

where τ_1 *is defined in Lemma 2.1 and* $g^+(t,r) = \max\{0, g(t,r)\}, g^-(t,r) = \max\{0, -g(t,r)\}$;

$(H5)$ *for all* $r_2 > r_1 > 0$,*there exists* $\gamma(\cdot) \in M$ *such that* $h(t,r) + \gamma(t)r$ *is increasing in* (r_1,r_2).

Then there exists $\lambda_3^* > 0$ such that

(i) if $0 < \lambda < \lambda_3^*$, (1.4) has at least one solution $u \in C[0,1] \cap C^1(0,1)$ and $u > 0$ for $t \in (0,1)$;

(ii) if $\lambda > \lambda_3^*$, (1.4) has no solutions.

Remark 1.5. In [5, 6], the authors consider the boundary value problem (1.4) under the conditions

$$\lim_{r\to\infty} \frac{h_2(t,r)}{r} = 0. \tag{1.16}$$

In Section 3, we give two examples (see Examples 3.1 and 3.2) which satisfy the conditions in Theorem 1.1 or Theorem 1.2 but they do not satisfy the conditions in [1–5].

2. Proof of Main Results

2.1. Some Lemmas

Lemma 2.1. *Consider the following eigenvalue problem*

$$-u'' = \tau u(t), \quad t \in (0,1),$$
$$u(0) = u(1) = 0. \tag{2.1}$$

Then the eigenvalues are

$$\tau_m = (m\pi)^2 \quad \text{for } m = 1, 2, \ldots, \tag{2.2}$$

and the corresponding eigenfunctions are

$$\phi_m(t) = \sin m\pi t \quad \text{for } m = 1, 2, \ldots. \tag{2.3}$$

Let $G(t,s)$ be the Green's function for the BVP:

$$\begin{aligned} -u'' &= 0 \quad \text{for } t \in (0,1), \\ u(0) &= u(1) = 0. \end{aligned} \tag{2.4}$$

Then

$$G(t,s) = \begin{cases} s(1-t), & 0 \le s < t \le 1, \\ t(1-s), & 0 \le t < s \le 1. \end{cases} \tag{2.5}$$

Also for all $(t,s) \in [0,1] \times [0,1]$, define

$$N(t,s) = \begin{cases} \dfrac{G(t,s)}{\phi_1(t)} & \text{if } t \ne 0, 1, \\ \dfrac{1-s}{\pi} & \text{if } t = 0, \\ \dfrac{s}{\pi} & \text{if } t = 1. \end{cases} \tag{2.6}$$

It follows easily that

$$\begin{aligned} 0 < G(t,s) \le t(1-t) \quad & \text{for } (t,s) \in (0,1) \times (0,1), \\ \frac{s(1-s)}{2\pi} \le N(t,s) \le \frac{1}{2} \quad & \text{for } (t,s) \in (0,1) \times (0,1). \end{aligned} \tag{2.7}$$

Define the operator $A, B : M \to C[0,1]$ by

$$\begin{aligned} Ax(t) &= \int_0^1 G(t,s) x(s) ds, \\ Bx(t) &= \int_0^1 N(t,s) x(s) ds. \end{aligned} \tag{2.8}$$

The following four results can be found in [5] (notice $\lim_{r \to \infty}(h_2(t,r)/r) = 0$ is not needed in the proofs there).

Lemma 2.2. *Suppose* (G1) *and* (H1) *hold. Let* $n_0 \in N$. *Assume that for every* $n > n_0$, *there exist* $a_n, \delta_n, \delta \in M$ *such that*

$$0 \leq a_n(t), \quad |\delta_n(t)| \leq \delta(t), \quad \lim_{n\to\infty} \delta_n(t) = 0, \quad \text{for } t \in (0,1) \tag{2.9}$$

and there exist $\overline{u}, \overline{u}_n, \widehat{u}_n, \widehat{u} \in C[0,1]$ *such that*

$$0 < \overline{u}(t) \leq \overline{u}_n(t) \leq \widehat{u}_n(t) \leq \widehat{u}(t) \quad \text{for } t \in (0,1), \tag{2.10}$$

and $\widehat{u}(0) = \widehat{u}(1) = 0$. *If*

$$\begin{aligned} &-\overline{u}_n''(t) + a_n(t)\overline{u}_n(t) \\ &\quad \leq g\left(t, \frac{1}{n} + v\right) + \lambda h(t,v) + \delta_n(t) + a_n(t)v(t) \quad \text{for } t \in (0,1), \\ &-\widehat{u}_n''(t) + a_n(t)\widehat{u}_n(t) \\ &\quad \geq g\left(t, \frac{1}{n} + v\right) + \lambda h(t,v) + \delta_n(t) + a_n(t)v(t) \quad \text{for } t \in (0,1), \end{aligned} \tag{2.11}$$

where $\lambda \geq 0$ *and* $v \in D_{\overline{u}_n}^{\widehat{u}_n}$, *then* (1.4) *has a solution* $u \in C[0,1] \cap C^1(0,1)$ *such that* $\overline{u}(t) \leq u(t) \leq \widehat{u}(t)$ *for* $t \in [0,1]$.

Lemma 2.3. *Let* $\psi : (0,1) \times (0,\infty) \to (0,\infty)$ *be a continuous function with*

$$\begin{gathered} \psi(t,\cdot) \text{ is strictly decreasing}, \\ \psi(\cdot, r) \in M \quad \forall r > 0. \end{gathered} \tag{2.12}$$

Then the problem

$$\begin{gathered} -\omega''(t) = \psi\left(t, \omega(t) + \frac{1}{n}\right) \quad \text{for } t \in (0,1), \\ \omega(0) = \omega(1) = 0 \end{gathered} \tag{2.13}$$

has a solution $\omega_n \in C[0,1]$ *such that*

$$\omega_n(t) \leq \omega_{n+1}(t) \leq 1 + \omega_1(t) \leq 1 + \int_0^1 \psi(s,1)ds \quad \text{for } t \in [0,1],\ n \in N. \tag{2.14}$$

If we let $\omega(t) = \lim_{n\to\infty}\omega_n(t)$ *for* $t \in [0,1]$, *then*

$$\begin{gathered} \omega \in C[0,1], \quad \omega(t) > 0 \quad \text{for } t \in (0,1), \\ -\omega''(t) = \psi(t, \omega(t)) \quad \text{for } t \in (0,1), \\ \omega(0) = \omega(1) = 0. \end{gathered} \tag{2.15}$$

Next we consider the boundary value problem

$$-u'' + a(t)u(t) = f(t), \quad t \in (0,1),$$
$$u(0) = 0 = u(1), \tag{2.16}$$

where $a, f \in M$, $a(t) \geq 0$ for $t \in (0,1)$.

Lemma 2.4. *The following statements hold:*

(i) *for any* $f \in M$, (2.16) *is uniquely solvable and*

$$u + A(au) = A(f); \tag{2.17}$$

(ii) *if* $f(t) \geq 0$ *for* $t \in (0,1)$, *then the solution of* (2.16) *is nonnegative.*

Corollary 2.5. *Let* $\Phi : M \to C[0,1] \cap C^1(0,1)$ *be the operator such that* $\Phi(f)$ *is the solution of* (2.16). *Then we have*

(i) *if* $f_1(t) \leq f_2(t)$ *for* $t \in (0,1)$, *then* $\Phi(f_1)(t) \leq \Phi(f_2)(t)$ *for* $t \in [0,1]$;

(ii) *let* $E \subset M$ *and* $\beta \in M$. *If* $|f(t)| \leq \beta(t)$, $t \in (0,1)$ *for all* $f \in E$, *then* $\Phi(E)$ *is relatively compact with respect to the topology of* $C[0,1]$.

Lemma 2.6 (see [2]). *Let* $f \in M$, $f \geq 0$, $f \not\equiv 0$, $u \in C[0,1] \cap C^1(0,1)$ *satisfy*

$$-u'' = f \quad \text{in } (0,1),$$
$$u(0) = u(1) = 0. \tag{2.18}$$

Then there exist $m = m(f) > 0$, $M = M(f) > 0$ *such that*

$$ml(t) \leq u(t) \leq Ml(t) \quad \text{for } t \in [0,1]. \tag{2.19}$$

2.2. The Proof of Theorem 1.1

Claim 1 (see [5]). There exists $\lambda_1^* > 0$, $c > 0$, independent of λ, such that for all $\lambda \geq \lambda_1^*$ there exist $R_\lambda > c$, $\overline{u} \in C([0,1])$, with $c\phi_1(t) \leq \overline{u}(t) \leq R_\lambda \phi_1(t)$ and

$$-\overline{u}''(t) = -g_1(t, \overline{u}(t)) + \lambda h_1(t, \overline{u}(t)), \quad \text{for } t \in (0,1),$$
$$\overline{u}(0) = \overline{u}(1) = 0, \tag{2.20}$$

with

$$g_1(\cdot,\overline{u}(\cdot)), h_1(\cdot,\overline{u}(\cdot)) \in M. \tag{2.21}$$

Let $\lambda_1^* > 0$, $c > 0$ and $\overline{u} \in C[0,1]$ be defined in Claim 1. Define

$$\psi(t,r) = g_2(t,r) \quad \text{for } t \in (0,1). \tag{2.22}$$

From (G1) notice that ψ satisfies the assumptions of Lemma 2.3, so there exist $\omega, \omega_n \in C[0,1]$, $\omega_n(t) > 0$, $\omega(t) > 0$ for $t \in (0,1)$ such that

$$\begin{gathered}
-\omega_n''(t) = g_2\left(t, \frac{1}{n} + \omega_n\right) \quad \text{for } t \in (0,1), \\
\omega_n(0) = \omega_n(1) = 0, \\
\omega_n(t) \le \omega_{n+1}(t) \le 1 + \omega_1(t) \le a_1 \quad \text{for } t \in [0,1],\ n \in N, \\
\omega(t) = \lim_{n\to\infty} \omega_n(t) \quad \text{for } t \in [0,1], \\
-\omega''(t) = g_2(t,\omega(t)) \quad \text{for } t \in (0,1), \\
\omega(0) = \omega(1) = 0,
\end{gathered} \tag{2.23}$$

where $a_1 = 1 + \int_0^1 g_2(s,1)ds$.

Let $\lambda \ge \lambda_1^*$, $n \in N$ be fixed. We consider the following boundary value problem:

$$\begin{gathered}
-v''(t) = \lambda h_2(t, v + \omega_n) + \lambda h_1(t,\overline{u}) \quad \text{for } t \in (0,1), \\
v(0) = v(1) = 0.
\end{gathered} \tag{2.24}$$

By $(H4)$, there exist $\{R_j\}_{j=1}^{\infty}$ such that $\lim_{j\to\infty} R_j = \infty$ and

$$\lim_{j\to\infty} \frac{h_2(t, R_j + a_1)}{R_j} = 0 \quad \text{for } t \in (0,1), \tag{2.25}$$

so

$$\lim_{j\to\infty} \frac{\lambda h_2(t, R_j + a_1) + \lambda h_1(t,\overline{u}(t))}{R_j} = 0 \quad \text{for } t \in (0,1). \tag{2.26}$$

There exists $j_0 \in N$ such that

$$\lambda h_2\left(t, R_{j_0} + a_1\right) + \lambda h_1(t, \overline{u}(t)) \le R_{j_0}. \tag{2.27}$$

If $v \in C[0,1]$ and $0 \le v(t) \le R_{j_0}\phi_1(t)$ for $t \in [0,1]$, then

$$\begin{aligned}
&\int_0^1 N(t,s)[\lambda h_2(s, v(s) + \omega_n(s)) + \lambda h_1(s, \overline{u})]ds \\
&\quad \le \int_0^1 N(t,s)[\lambda h_2(s, v(s) + a_1) + \lambda h_1(s, \overline{u})]ds \\
&\quad \le \int_0^1 N(t,s)\left[\lambda h_2\left(s, R_{j_0}\phi_1(s) + a_1\right) + \lambda h_1(s, \overline{u})\right]ds \\
&\quad \le \int_0^1 N(t,s)\left[\lambda h_2\left(s, R_{j_0} + a_1\right) + \lambda h_1(s, \overline{u})\right]ds \\
&\quad \le \frac{R_{j_0}}{2}, \quad \text{for } t \in (0,1),
\end{aligned} \tag{2.28}$$

and so

$$0 \le \int_0^1 G(t,s)[\lambda h_2(s, v(s) + \omega_n(s)) + \lambda h_1(s, \overline{u}(s))]ds \le R_{j_0}\phi_1(t) \quad \text{for } t \in [0,1]. \tag{2.29}$$

Let $\Phi : C[0,1] \to C[0,1]$ be the operator defined by

$$(\Phi v)(t) := \int_0^1 G(t,s)[\lambda h_2(s, v(s) + \omega_n(s)) + \lambda h_1(s, \overline{u}(s))]ds \quad \text{for } v \in C[0,1],\ t \in [0,1]. \tag{2.30}$$

It is easy to see that Φ is a continuous and completely continuous operator. Also if $0 \le v(t) \le R_{j_0}\phi_1(t)$ for $t \in [0,1]$, then $0 \le \Phi(v)(t) \le R_{j_0}\phi_1(t)$ for $t \in [0,1]$, so Schauder's fixed point theorem guarantees that there exists $\widetilde{v} \in [0, R_{j_0}\phi_1]$ such that $\Phi(\widetilde{v}) = \widetilde{v}$, that is,

$$\begin{gathered}
-\widetilde{v}''(t) = \lambda h_2(t, \widetilde{v}(t) + \omega_n(s)) + \lambda h_1(t, \overline{u}(t)), \\
\widetilde{v}(1) = \widetilde{v}(1) = 0.
\end{gathered} \tag{2.31}$$

Let

$$\widehat{u}_n(t) = \omega_n(t) + \widetilde{v}_n(t) \quad \text{for } t \in [0,1]. \tag{2.32}$$

Then $\widehat{u}_n \in C[0,1]$, $\widehat{u}_n(1) = \widehat{u}_n(1) = 0$, and

$$-\widehat{u}_n''(t) = -\omega_n''(t) - \widetilde{v}_n''(t) = g_2\left(t, \frac{1}{n} + \omega_n\right) + \lambda h_2(t, \omega_n + \widetilde{v}_n) + \lambda h_1(t, \overline{u}) \geq g_2\left(t, \frac{1}{n} + \widehat{u}_n\right) + \lambda h_1(t, \overline{u}) + \lambda h_2(t, \widehat{u}_n) \quad \text{for } t \in (0,1). \tag{2.33}$$

Let

$$\widehat{u}(t) = \omega(t) + R_{j_0}\phi_1(t) \quad \text{for } t \in [0,1], \tag{2.34}$$

so

$$0 \leq \widehat{u}_n(t) \leq \widehat{u}(t) \quad \text{for } t \in [0,1]. \tag{2.35}$$

From Claim 1, we obtain

$$-\overline{u}''(t) = -g_1(t, \overline{u}) + \lambda h_1(t, \overline{u}) \leq \lambda h_1(t, \overline{u}) \leq \lambda h_1(t, \overline{u}) + g_2\left(t, \frac{1}{n} + \widehat{u}_n\right) + \lambda h_2(t, \widehat{u}_n) \leq -\widehat{u}_n''(t) \quad \text{for } t \in (0,1), \tag{2.36}$$

that is,

$$-(\overline{u} - \widehat{u}_n)''(t) \leq 0 \quad \text{for } t \in (0,1). \tag{2.37}$$

A standard argument yields

$$\overline{u}(t) \leq \widehat{u}_n(t) \quad \text{for } t \in [0,1]. \tag{2.38}$$

From (G2), there exists $\gamma \in M$ such that $r \to g_2(t, 1/n + r) + \gamma(t)r$ is increasing on $(0, |\widehat{u}|_\infty)$. Let $\overline{u}_n = \overline{u}$. From (2.35) and (2.38), we have

$$0 < \overline{u}(t) \leq \overline{u}_n(t) \leq \widehat{u}_n(t) \leq \widehat{u}(t) \quad \text{for } t \in (0,1). \tag{2.39}$$

Also for $v \in D_{\overline{u}_n}^{\widehat{u}_n}$ we have

$$
\begin{aligned}
-\overline{u}_n''(t) &+ \gamma(t)\overline{u}_n(t) \\
&= -g_1(t,\overline{u}_n) + \lambda h_1(t,\overline{u}_n) + \gamma(t)\overline{u}_n(t) \\
&\le -g_1(t,v) + \lambda h_1(t,v) + \gamma(t)v(t) \\
&\le -g_1\left(t,\frac{1}{n}+v\right) + \lambda h_1(t,v) + \gamma(t)v(t) \\
&\le g\left(t,\frac{1}{n}+v\right) + \lambda h(t,v) + \gamma(t)v(t) \quad \text{for } t \in (0,1), \\
-\widehat{u}_n''(t) &+ \gamma(t)\widehat{u}_n(t) \\
&\ge g_2\left(t,\frac{1}{n}+\widehat{u}_n\right) + \lambda h_1(t,\overline{u}) + \lambda h_2(t,\widehat{u}_n) + \gamma(t)\widehat{u}_n(t) \\
&\ge g_2\left(t,\frac{1}{n}+\widehat{u}_n\right) + \gamma(t)\widehat{u}_n(t) + \lambda h_2(t,\widehat{u}_n) \\
&\ge g_2\left(t,\frac{1}{n}+v\right) + \gamma(t)v(t) + \lambda h_2(t,v(t)) \\
&\ge g\left(t,\frac{1}{n}+v\right) + \lambda h(t,v) + \gamma(t)v(t) \quad \text{for } t \in (0,1).
\end{aligned}
\tag{2.40}
$$

Now Lemma 2.2 with $\delta_n \equiv 0$, $n \in N$ guarantees that there exists a solution $u \in C[0,1]$ to (1.4) with

$$\overline{u}(t) \le u(t) \le \widehat{u}(t) \quad \text{for } t \in [0,1]. \tag{2.41}$$

2.3. The Proof of Theorem 1.2

Let

$$\Lambda = \left\{\lambda \in R \mid (1.4) \text{ has at least one positive solution}\right\}. \tag{2.42}$$

Claim 2. Let

$$\lambda^* = \frac{1}{\max_{t\in[0,1]}\int_0^1 N(t,s)h_2(s,a_2+\phi_1)\,ds} > 0; \tag{2.43}$$

here

$$a_2 = 1 + \frac{1}{4}\int_0^1 \left[g_2(t,1) + \frac{1}{\left(\underline{\underline{u}}(t)\right)^{\beta}} + e(t) \right] dt,$$

$$\underline{\underline{u}}(t) = c_2 l(t) \quad \text{for } t \in [0,1], \tag{2.44}$$

$$e(t) = \begin{cases} \sup\limits_{r\in[c_1,1+c_2/2]} g^{-}(t,r), & \text{if } c_1 < 1 + \dfrac{c_2}{2}, \\ 0, & \text{if } c_1 \geq 1 + \dfrac{c_2}{2}. \end{cases}$$

Then $(0, \lambda^*) \in \Lambda$.

Proof of Claim 2. Let $n \geq 1$ be fixed. Lemma 2.8 [6] implies that there exists $\alpha_{n,1} \in C[0,1]$ such that

$$\underline{\underline{u}}(t) \leq \alpha_{n,1}(t) \leq \overline{\overline{u}}(t), \tag{2.45}$$

$$-\alpha''_{n,1}(t) = \overline{g}_1\left(t, \frac{1}{n} + \alpha_{n,1}(t)\right) + \delta_n(t) \quad \text{for } t \in (0,1),$$
$$\alpha_{n,1}(0) = \alpha_{n,1}(1) = 0, \tag{2.46}$$

where $\overline{g}_1$ is defined in $(G5)$, and

$$\delta_n(t) = \overline{g}_1\left(t, \underline{\underline{u}}(t)\right) - \overline{g}_1\left(t, \frac{1}{n} + \underline{\underline{u}}(t)\right), \tag{2.47}$$

$$\overline{\overline{u}}(t) = \overline{c} l(t) \quad \text{for } t \in [0,1],$$
$$\overline{c} = \max\left\{ c_1, \pi \sup_{t\in(0,1)} \left[2B\left(\frac{1}{\left(\underline{\underline{u}}(\cdot)\right)^{\beta}} \right)(t) + B(e)(t) \right] \right\}. \tag{2.48}$$

which does not depend on n.

On the other hand, let

$$\varphi(t,r) = g_2(t,r) + \frac{1}{\underline{\underline{u}}(t)^{\beta}} + e(t). \tag{2.49}$$

From $(G1)$ notice ψ satisfies the assumptions of Lemma 2.3, so there exist $\omega, \omega_n \in C[0,1]$ such that

$$\begin{gathered}
-\omega_n''(t) = g_2\left(t, \frac{1}{n} + \omega_n\right) + \frac{1}{\underline{\underline{u}}(t)^\beta} + e(t) \quad \text{for } t \in (0,1), \\
\omega_n(0) = \omega_n(1) = 0, \\
\omega_n(t) \le \omega_{n+1}(t) \le 1 + \omega_1(t) \le a_2 \quad \text{for } t \in [0,1],\ n \in N, \\
\omega(t) = \lim_{n \to \infty} \omega_n(t) \quad \text{for } t \in [0,1], \\
-\omega''(t) = g_2(t, \omega(t)) + \frac{1}{\underline{\underline{u}}(t)^\beta} + e(t) \quad \text{for } t \in (0,1), \\
\omega(0) = \omega(1) = 0.
\end{gathered} \tag{2.50}$$

Next we consider the boundary value problem

$$\begin{gathered}
-\widetilde{v}_n''(t) = \lambda h_2(t, \omega_n + \widetilde{v}_n) \quad \text{for } t \in (0,1), \\
\widetilde{v}_n(0) = \widetilde{v}_n(1) = 0,
\end{gathered} \tag{2.51}$$

where $\lambda \in (0, \lambda^*)$.

Let $\Phi : C[0,1] \to C[0,1]$ be the operator defined by

$$(\Phi v)(t) := \lambda \int_0^1 G(t,s) h_2(s, \omega_n + v) ds \quad \text{for } v \in C[0,1],\ t \in [0,1]. \tag{2.52}$$

It is easy to see that Φ is a continuous and completely continuous operator. Also, if $0 \le v(t) \le \phi_1(t)$ for $t \in [0,1]$, then

$$\begin{aligned}
0 \le \Phi(v)(t) &= \lambda \int_0^1 G(t,s) h_2(s, \omega_n + v) ds \\
&\le \lambda^* \int_0^1 G(t,s) h_2\left(s, a_2 + \phi_1\right) ds \\
&= \frac{\phi_1(t) \int_0^1 N(t,s) h_2\left(s, a_2 + \phi_1\right) ds}{\max_{t \in [0,1]} \int_0^1 N(t,s) h_2\left(s, a_2 + \phi_1\right) ds} \\
&\le \phi_1(t) \quad \text{for } t \in [0,1].
\end{aligned} \tag{2.53}$$

Thus Schauder fixed point theorem guarantees that there exists $\widetilde{v}_n \in [0, \phi_1]$ such that $\Phi(\widetilde{v}_n) = \widetilde{v}_n$, that is,

$$\begin{gathered} -\widetilde{v}_n''(t) = \lambda h_2(t, \omega_n + \widetilde{v}_n), \\ \widetilde{v}_n(0) = \widetilde{v}_n(1) = 0. \end{gathered} \tag{2.54}$$

Let

$$\widehat{u}_n(t) = \omega_n(t) + \widetilde{v}_n(t), \quad \widehat{u}(t) = \omega(t) + \phi_1(t) \quad \text{for } t \in [0,1]. \tag{2.55}$$

Then $\widehat{u}_n, \widehat{u} \in C[0,1]$, $\widehat{u}_n(0) = \widehat{u}_n(1) = 0$, $\widehat{u}(0) = \widehat{u}(1) = 0$,

$$0 \le \widehat{u}_n(t) \le \widehat{u}(t) \quad \text{for } t \in [0,1], \tag{2.56}$$

$$\begin{aligned} -\widehat{u}_n''(t) &= -\omega_n''(t) - \widetilde{v}_n(t) \\ &= g_2\left(t, \frac{1}{n} + \omega_n\right) + \frac{1}{\underline{\underline{u}}(t)^\beta} + e(t) + \lambda h_2(t, \omega_n + \widetilde{v}_n) \\ &\ge g_2\left(t, \frac{1}{n} + \widehat{u}_n\right) + \frac{1}{\underline{\underline{u}}(t)^\beta} + e(t) + \lambda h_2(t, \widehat{u}_n) \quad \text{for } t \in (0,1),\ \lambda \in (0, \lambda^*). \end{aligned} \tag{2.57}$$

Now let us consider the problem

$$\begin{gathered} -u''(t) = g\left(t, \frac{1}{n} + u\right) + \lambda h(t, u) + \delta_n(t) \quad \text{for } t \in (0,1),\ \lambda \in (0, \lambda^*) \\ u(0) = u(1) = 0, \end{gathered} \tag{2.58}$$

where δ_n is defined in (2.47).

We will prove $\alpha_{n,1}$ is a lower solution of (2.58) and $\widehat{u}_n$ is an upper solution of (2.58). Now (2.46) and the positivity of $h(t,s)$ implies that

$$\begin{aligned} -\alpha_{n,1}''(t) &= \overline{g}_1\left(t, \frac{1}{n} + \alpha_{n,1}(t)\right) + \delta_n(t) \\ &\le g\left(t, \frac{1}{n} + \alpha_{n,1}(t)\right) + \lambda h(t, \alpha_{n,1}(t)) + \delta_n(t), \end{aligned} \tag{2.59}$$

so $\alpha_{n,1}$ is a lower solution of (2.58). On the other hand, from the definition of $\overline{g}_1$ and $\underline{\underline{u}}$, we have

$$\begin{aligned}\overline{g}_1\left(t,\underline{\underline{u}}\right) &= \min\left\{g\left(t,\underline{\underline{u}}\right), \frac{1}{\underline{\underline{u}}(t)^{\beta}}\right\} \leq \frac{1}{\underline{\underline{u}}(t)^{\beta}} \quad \text{for } t \in (0,1),\\ -\overline{g}_1\left(t,\frac{1}{n}+\underline{\underline{u}}\right) &= -\min\left\{g^{+}\left(t,\frac{1}{n}+\underline{\underline{u}}\right), \frac{1}{\left(1/n+\underline{\underline{u}}\right)^{\beta}}\right\} + g^{-}\left(t,\frac{1}{n}+\underline{\underline{u}}\right)\\ &\leq g^{-}\left(t,\frac{1}{n}+\underline{\underline{u}}\right)\\ &\leq e(t) \quad \text{for } t \in (0,1),\end{aligned} \tag{2.60}$$

so

$$\delta_n(t) \leq \frac{1}{\underline{\underline{u}}(t)^{\beta}} + e(t) \quad \text{for } t \in (0,1). \tag{2.61}$$

Consequently, we have

$$\begin{aligned}-\widehat{u}_n''(t) &\geq g_2\left(t,\frac{1}{n}+\widehat{u}_n\right) + \frac{1}{\underline{\underline{u}}(t)^{\beta}} + e(t) + \lambda h_2(t,\widehat{u}_n)\\ &\geq g\left(t,\frac{1}{n}+\widehat{u}_n\right) + \frac{1}{\underline{\underline{u}}(t)^{\beta}} + e(t) + \lambda h(t,\widehat{u}_n)\\ &\geq g\left(t,\frac{1}{n}+\widehat{u}_n\right) + \lambda h(t,\widehat{u}_n) + \delta_n(t),\end{aligned} \tag{2.62}$$

so $\widehat{u}_n$ is an upper solution of (2.58). We next prove that

$$\alpha_{n,1}(t) \leq \widehat{u}_n(t) \quad \text{for } t \in [0,1]. \tag{2.63}$$

Suppose (2.63) is not true. Let $y(t) = \alpha_{n,1}(t) - \widehat{u}_n(t)$ and let $\sigma \in (0,1)$ be the point where $y(t)$ attains its maximum over $(0,1)$. We have

$$y(\sigma) > 0, \qquad y''(\sigma) \leq 0. \tag{2.64}$$

On the other hand, since $\alpha_{n,1}(\sigma) > \widehat{u}_n(\sigma)$, we have

$$\begin{aligned}
-\alpha''_{n,1}(\sigma) &= \overline{g}_1\left(\sigma, \frac{1}{n} + \alpha_{n,1}(\sigma)\right) + \delta_n(\sigma) \\
&\le g\left(\sigma, \frac{1}{n} + \alpha_{n,1}(\sigma)\right) + \delta_n(\sigma) \\
&\le g\left(\sigma, \frac{1}{n} + \alpha_{n,1}(\sigma)\right) + \frac{1}{\underline{\underline{u}}(\sigma)^{\beta}} + e(\sigma) \\
&\le g_2\left(\sigma, \frac{1}{n} + \alpha_{n,1}(\sigma)\right) + \frac{1}{\underline{\underline{u}}(\sigma)^{\beta}} + e(\sigma) \\
&< g_2\left(\sigma, \frac{1}{n} + \widehat{u}_n(\sigma)\right) + \frac{1}{\underline{\underline{u}}(\sigma)^{\beta}} + e(\sigma) + \lambda h_2(\sigma, \widehat{u}_n(\sigma)) \\
&\le -\widehat{u}''_n(\sigma),
\end{aligned} \tag{2.65}$$

so

$$y''(\sigma) = \alpha''_{n,1}(\sigma) - \widehat{u}''_n(\sigma) > 0, \tag{2.66}$$

and this is a contradiction.

From (G3), there exists $\gamma \in M$ such that $r \to g(t, 1/n + r) + \gamma(t)r$ is increasing in $(0, |\widehat{u}|_\infty)$. Let $\overline{u}(t) \equiv \underline{\underline{u}}(t)$, $\overline{u}_n(t) = \alpha_{n,1}(t)$. From (2.45), (2.56), and (2.63), we have

$$0 < \overline{u}(t) \le \overline{u}_n(t) \le \widehat{u}_n(t) \le \widehat{u}(t) \quad \text{for } t \in (0,1). \tag{2.67}$$

Also for $v \in D_{\overline{u}_n}^{\widehat{u}_n}$, we have

$$\begin{aligned}
&- \overline{u}''_n(t) + \gamma(t)\overline{u}_n(t) \\
&\quad \le g\left(t, \frac{1}{n} + \overline{u}_n\right) + \gamma(t)\overline{u}_n + \delta_n(t) \\
&\quad \le g\left(t, \frac{1}{n} + v\right) + \gamma(t)v + \delta_n(t) + \lambda h(t, v) \\
&\quad \le g\left(t, \frac{1}{n} + \widehat{u}_n\right) + \gamma(t)\widehat{u}_n + \delta_n(t) + \lambda h_2(t, \widehat{u}_n) \\
&\quad \le -\widehat{u}''_n(t) + \gamma(t)\widehat{u}_n(t).
\end{aligned} \tag{2.68}$$

On the other hand, by (2.61)

$$|\delta_n(t)| \le \frac{1}{\overline{u}(t)^\beta} + e(t) \equiv \delta(t),$$
$$\lim_{n\to\infty} \delta_n(t) = 0 \quad \text{for } t \in (0,1). \tag{2.69}$$

Now Lemma 2.2 guarantees that there exists a solution $u \in C[0,1] \cap C^1(0,1)$ to (1.4) with

$$\overline{u}(t) \le u(t) \le \hat{u}(t) \quad \text{for } t \in [0,1]. \tag{2.70}$$

Thus (1.4) has a solution for $\lambda \in (0,\lambda^*)$ so Claim 2 holds. In particular, $\Lambda \neq \emptyset$ and $\sup \Lambda > 0$. □

Claim 3. If $\lambda \in \Lambda$, then $(0,\lambda] \in \Lambda$.

Proof of Claim 3.

Step 1. We may assume that $\lambda > 0$. Let χ be a positive solution of (1.4), that is,

$$-\chi'' = g(t,\chi) + \lambda h(t,\chi), \quad t \in (0,1),$$
$$\chi(0) = 0 = \chi(1). \tag{2.71}$$

We prove that there exists $\rho > 0$ such that

$$\chi(t) \ge \rho l(t) \quad \text{for } t \in [0,1]. \tag{2.72}$$

By $(G4)$, $g(t,r) \ge 0$ for $t \in (0,1)$, $r \in (0,c_1]$. From the continuity of χ and $\chi(0) = 0 = \chi(1)$, it follows that there is $0 < \delta < 1/2$ such that

$$0 \le \chi(t) < c_1 \quad \text{for } t \in [0,\delta] \cup [1-\delta,1]. \tag{2.73}$$

Then

$$-\chi'' \ge \lambda h(t,\chi) \quad \text{for } t \in [0,\delta] \cup [1-\delta,1]. \tag{2.74}$$

Let $v \in C^1(0,\delta) \cap C[0,\delta]$ so that

$$-v''(t) = h(t,\chi) \quad \text{for } t \in (0,\delta),$$
$$v(0) = v(\delta) = 0. \tag{2.75}$$

It follows that $\lambda v(t) \le \chi(t)$ for $t \in [0,\delta]$. Lemma 2.6 implies that there exists $m > 0$ so that

$$m \inf\{t, \delta - t\} \le v(t) \quad \text{for } t \in [0,\delta]. \tag{2.76}$$

The same reason implies that

$$m \inf\{t + \delta - 1, 1 - t\} \leq v(t) \quad \text{for } t \in [1 - \delta, 1]. \tag{2.77}$$

It follows that

$$m\lambda l(t) \leq \chi(t) \quad \text{for } t \in \left[0, \frac{\delta}{2}\right] \cup \left[\frac{1-\delta}{2}, 1\right]. \tag{2.78}$$

Moreover,

$$\inf\left\{ \frac{\chi(t)}{l(t)} : t \in \left(0, \frac{\delta}{2}\right) \cup \left(\frac{1-\delta}{2}, 1\right) \right\} > 0. \tag{2.79}$$

On the other hand, we easily have

$$\inf\left\{ \frac{\chi(t)}{l(t)} : t \in \left[\frac{\delta}{2}, \frac{1-\delta}{2}\right] \right\} > 0, \tag{2.80}$$

so

$$\inf\left\{ \frac{\chi(t)}{l(t)} : t \in (0,1) \right\} = \rho > 0, \tag{2.81}$$

and thus

$$\chi(t) \geq \rho l(t) \quad \text{for } t \in [0,1]. \tag{2.82}$$

Step 2. Let $\underline{\underline{r}} = \rho \wedge c_2$ and $\underline{\underline{u}}(t) = \underline{\underline{r}} l(t)$. Then

$$\underline{\underline{u}}(t) \leq A\left(\overline{g}_m\left(\cdot, \frac{1}{n} + \underline{\underline{u}} \wedge \chi\right) + \delta_n\right)(t) \quad \text{for } t \in [0,1],\ m, n \geq 1, \tag{2.83}$$

where

$$\delta_n(t) = \overline{g}_1\left(t, \underline{\underline{u}} \wedge \chi\right) - \overline{g}_1\left(t, \frac{1}{n} + \underline{\underline{u}} \wedge \chi\right). \tag{2.84}$$

Notice

$$\underline{\underline{u}}(t) \leq \chi(t), \quad \underline{\underline{u}}(t) \leq c_2 l(t) \quad \text{for } t \in [0,1]. \tag{2.85}$$

From (G5), we have

$$A\left(\overline{g}_1\left(\cdot,\underline{\underline{u}}\wedge\chi\right)\right)(t)=\int_0^1 G(t,s)\overline{g}_1\left(s,\underline{\underline{u}}\right)ds$$
$$=\phi_1(t)\int_0^1 N(t,s)\overline{g}_1\left(s,\underline{\underline{u}}\right)ds$$
$$\geq\frac{\phi_1(t)}{2\pi}\int_0^1 s(1-s)\overline{g}_1\left(s,\underline{\underline{r}}l(s)\right)ds$$
$$\geq\frac{\underline{\underline{r}}\phi_1(t)}{2}\geq\underline{\underline{u}}(t)\quad\text{for } t\in[0,1],\tag{2.86}$$

so

$$A\left(\overline{g}_m\left(\cdot,\frac{1}{n}+\underline{\underline{u}}\wedge\chi\right)+\delta_n\right)(t)$$
$$=\int_0^1 G(t,s)\left[\overline{g}_m\left(s,\frac{1}{n}+\underline{\underline{u}}\wedge\chi\right)-\overline{g}_1\left(s,\frac{1}{n}+\underline{\underline{u}}\wedge\chi\right)+\overline{g}_1\left(s,\underline{\underline{u}}\wedge\chi\right)\right]ds$$
$$\geq\int_0^1 G(t,s)\overline{g}_1\left(s,\underline{\underline{u}}\wedge\chi\right)ds$$
$$\geq\underline{\underline{u}}(t)\quad\text{for } t\in[0,1].\tag{2.87}$$

Step 3. Let $0<\mu<\lambda$. For each $m\geq 1$, there exists $\overline{\overline{r}}_m>\underline{\underline{r}}$, independent of n. Let

$$\overline{\overline{u}}_m(t)=\overline{\overline{r}}_m l(t)\quad\text{for } t\in[0,1].\tag{2.88}$$

Then

$$A\left(\overline{g}_m\left(\cdot,\frac{1}{n}+v\wedge\chi\right)+\delta_n+\mu h_2(\cdot,v\wedge\chi)\right)(t)\leq\overline{\overline{u}}_m(t)\quad\text{for } t\in[0,1],\ v\in D_{\underline{\underline{u}}}^{\overline{\overline{u}}_m},\ n\geq 1.\tag{2.89}$$

Let $v\in C[0,1]\cap C^1(0,1)$ such that

$$-v''=\lambda h_2(t,\chi)\quad\text{for } t\in(0,1),$$
$$v(0)=v(1)=0.\tag{2.90}$$

By Lemma 2.6, there exists $M>0$ such that

$$v(t)\leq Ml(t)\quad\text{for } t\in[0,1].\tag{2.91}$$

Let

$$\overline{\overline{r}}_m > \max\left\{ M + \pi \sup_{t\in(0,1)} B\left(\frac{m}{\left(\underline{\underline{u}} \wedge \chi\right)^{\beta}} + \frac{1}{\underline{\underline{u}}^{\beta}} + e \right)(t), \underline{\underline{r}} \right\}. \tag{2.92}$$

Note $\underline{\underline{u}} \leq \overline{\overline{u}}$ since $\overline{\overline{r}}_m > \underline{\underline{r}}$. Let $v \geq \underline{\underline{u}}$ and notice (note $g^{-}(\cdot, r) = 0$ if $0 < r < c_1$ from (G4))

$$\begin{aligned}
&A\left(\overline{g}_m\left(\cdot, \frac{1}{n} + v \wedge \chi\right) + \delta_n\right)(t) \\
&\quad = \int_0^1 G(t,s)\left[\overline{g}_m\left(s, \frac{1}{n} + v \wedge \chi\right) - \overline{g}_1\left(s, \frac{1}{n} + \underline{\underline{u}} \wedge \chi\right) + \overline{g}_1\left(s, \underline{\underline{u}} \wedge \chi\right)\right] ds \\
&\quad \leq \int_0^1 G(t,s)\left[\frac{m}{(1/n + v \wedge \chi)^{\beta}} + g^{-}\left(s, \frac{1}{n} + \underline{\underline{u}}\right) + \overline{g}_1\left(s, \underline{\underline{u}}\right)\right] ds \\
&\quad \leq \int_0^1 G(t,s)\left[\frac{m}{(v \wedge \chi)^{\beta}} + \frac{1}{\underline{\underline{u}}^{\beta}} + e\right] ds \\
&\quad \leq \int_0^1 G(t,s)\left[\frac{m}{\left(\underline{\underline{u}} \wedge \chi\right)^{\beta}} + \frac{1}{\underline{\underline{u}}^{\beta}} + e\right] ds \\
&\quad = \phi_1(t)\left[B\left(\frac{m}{\left(\underline{\underline{u}} \wedge \chi\right)^{\beta}} + \frac{1}{\underline{\underline{u}}^{\beta}} + e\right)\right](t) \\
&\quad \leq \pi\left[B\left(\frac{m}{\left(\underline{\underline{u}} \wedge \chi\right)^{\beta}} + \frac{1}{\underline{\underline{u}}^{\beta}} + e\right)\right](t) \cdot l(t) \quad \text{for } t \in [0,1].
\end{aligned} \tag{2.93}$$

On the other hand,

$$\begin{aligned}
A(\mu h(\cdot, v \wedge \chi))(t) &= \mu \int_0^1 G(t,s) h(s, v \wedge \chi) ds \\
&\leq \lambda \int_0^1 G(t,s) h_2(s, \chi) ds \\
&= v(t) \leq M l(t) \quad \text{for } t \in [0,1],
\end{aligned} \tag{2.94}$$

so

$$
\begin{aligned}
A&\left(\overline{g}_m\left(\cdot,\frac{1}{n}+v\wedge\chi\right)+\delta_n+\mu h(\cdot,v\wedge\chi)\right)(t)\\
&\le A\left(\overline{g}_m\left(\cdot,\frac{1}{n}+v\wedge\chi\right)+\delta_n\right)(t)+A(\mu h(\cdot,v\wedge\chi))(t)\\
&\le \pi\left[B\left(\frac{m}{\left(\underline{\underline{u}}\wedge\chi\right)^{\beta}}+\frac{1}{\underline{\underline{u}}^{\beta}}+e\right)\right](t)\cdot l(t)+Ml(t)\\
&\le \overline{\overline{u}}_m(t)\quad \text{for } t\in[0,1],\ v\in\left[\underline{\underline{u}},\overline{\overline{u}}_m\right],\ n\ge 1.
\end{aligned}
\tag{2.95}
$$

Step 4. Let $0<\mu<\lambda$. Let $n,m\ge 1$ be fixed. There exists $\beta_{n,m}\in C[0,1]$ such that

$$
\begin{gathered}
\underline{\underline{u}}(t)\le\beta_{n,m}(t)\le\overline{\overline{u}}_m(t),\\
-\beta''_{n,m}(t)=\overline{g}_m\left(t,\frac{1}{n}+\beta_{n,m}\wedge\chi\right)+\mu h(t,\beta_{n,m}\wedge\chi)+\delta_n(t)\quad\text{for } t\in(0,1),\\
\beta_{n,m}(0)=\beta_{n,m}(1)=0.
\end{gathered}
\tag{2.96}
$$

Let $n,m>1$ be fixed. From Remark 1.3, there exist $\gamma_n\in M$, $\gamma_n\ge 0$ such that $\overline{g}_m(t,r)+\gamma_n(t)r$ is increasing in $(1/n,1/n+\overline{\overline{r}}_m/2)$. We easily prove that

$$
\overline{g}_m(t,r\wedge\chi)+\gamma_n(t)r \text{ is increasing in } \left(\frac{1}{n},\frac{1}{n}+\frac{\overline{\overline{r}}_m}{2}\right).
\tag{2.97}
$$

Let $\overline{\gamma}(t)=\gamma_n$. We have $\overline{g}_m(t,1/n+r\wedge\chi)+\overline{\gamma}(t)r$ is increasing in $(0,\overline{\overline{r}}_m/2)$. From (2.83) and (2.89), we have for fixed $v\in C[0,1]$, $\underline{\underline{u}}(t)\le v(t)\le\overline{\overline{u}}_m(t)$ that

$$
\begin{aligned}
\underline{\underline{u}}(t)+A\left(\overline{\gamma}\underline{\underline{u}}\right)(t)&\le A\left(\overline{g}_m\left(\cdot,\frac{1}{n}+\underline{\underline{u}}\wedge\chi\right)+\delta_n\right)(t)+A\left(\overline{\gamma}\underline{\underline{u}}\right)(t)\\
&\le A\left(\overline{g}_m\left(\cdot,\frac{1}{n}+\underline{\underline{u}}\wedge\chi\right)+\overline{\gamma}\underline{\underline{u}}+\delta_n+\mu h(\cdot,v\wedge\chi)\right)(t)\\
&\le A\left(\overline{g}_m\left(\cdot,\frac{1}{n}+v\wedge\chi\right)+\overline{\gamma}v+\delta_n+\mu h(\cdot,v\wedge\chi)\right)(t)\\
&\le \overline{\overline{u}}_m(t)+A\left(\overline{\gamma}\,\overline{\overline{u}}_m\right)(t).
\end{aligned}
\tag{2.98}
$$

Fix $v \in C[0,1]$ with $\underline{\underline{u}}(t) \leq v(t) \leq \overline{\overline{u}}_m(t)$. From Lemma 2.4, there exists $\Psi(v) \in C[0,1]$ such that

$$\begin{aligned}
&-\Psi''(v)(t) + \overline{\gamma}(t)\Psi(v)(t) \\
&\quad = \overline{g}_m\left(t, \frac{1}{n} + v \wedge \chi\right) + \overline{\gamma}(t)v(t) + \delta_n(t) + \mu h(t, v \wedge \chi) \quad \text{for } t \in (0,1) \\
&\Psi(v)(0) = \Psi(v)(1) = 0.
\end{aligned} \tag{2.99}$$

Then

$$\Psi(v)(t) + A(\overline{\gamma}\Psi(v))(t) = A\left(\overline{g}_m\left(\cdot, \frac{1}{n} + v \wedge \chi\right) + \overline{\gamma}v + \delta_n + \mu h(\cdot, v \wedge \chi)\right)(t) \quad \text{for } t \in (0,1), \tag{2.100}$$

so (2.98) implies that

$$\begin{aligned}
\underline{\underline{u}}(t) + A\left(\overline{\gamma}\underline{\underline{u}}\right)(t) &\leq \Psi(v)(t) + A(\overline{\gamma}\Psi(v))(t) \\
&\leq \overline{\overline{u}}_m(t) + A\left(\overline{\gamma}\, \overline{\overline{u}}_m\right)(t) \quad \text{for } t \in (0,1).
\end{aligned} \tag{2.101}$$

From Corollary 2.5, we have

$$\underline{\underline{u}}(t) \leq \Psi(v)(t) \leq \overline{\overline{u}}_m(t) \quad \text{for } t \in [0,1]. \tag{2.102}$$

Also,

$$\begin{aligned}
&\left|\overline{g}_m\left(t, \frac{1}{n} + v \wedge \chi\right) + \overline{\gamma}v + \delta_n + \mu h(t, v \wedge \chi)\right| \\
&\quad \leq g_1\left(t, \frac{\phi_1(t)}{n}\right) + g_2\left(t, \frac{1}{n}\right) + \overline{\gamma}\left|\overline{\overline{u}}_m\right|_\infty + |\delta_n(t)| + \lambda h_2(t, |\chi|_\infty) \\
&\quad \equiv \beta(t) \in M \quad \text{for } t \in (0,1).
\end{aligned} \tag{2.103}$$

Now $\Psi : D_{\underline{\underline{u}}}^{\overline{\overline{u}}_m} \to D_{\underline{\underline{u}}}^{\overline{\overline{u}}_m}$ is compact, so Schauder's fixed point theorem implies that there exists $\beta_{n,m} \in C[0,1]$ such that $\underline{\underline{u}}(t) \leq \beta_{n,m}(t) \leq \overline{\overline{u}}_m(t)$ and $\Psi(\beta_{n,m})(t) = \beta_{n,m}(t)$ for $t \in (0,1)$:

$$\begin{aligned}
&-\beta''_{n,m}(t) = \overline{g}_m\left(t, \frac{1}{n} + \beta_{n,m} \wedge \chi\right) + \mu h(t, \beta_{n,m} \wedge \chi) + \delta_n(t) \quad \text{for } t \in (0,1), \\
&\beta_{n,m}(0) = \beta_{n,m}(1) = 0, \\
&\left|\overline{g}_m\left(t, \frac{1}{n} + \beta_{n,m} \wedge \chi\right) + \mu h(t, \beta_{n,m} \wedge \chi) + \delta_n(t)\right| \leq 3g_2\left(t, \underline{\underline{u}} \wedge \chi\right) + \lambda h_2(t, \chi).
\end{aligned} \tag{2.104}$$

Let $m \geq 1$ be fixed. We consider the sequence $\{\beta_{n,m}\}_{n=1}^{\infty}$. Fix $n_0 \in \{2, 3, \ldots\}$. Let us look at the interval $[1/2^{n_0+1}, 1 - 1/2^{n_0+1}]$. The mean value theorem implies that there exists $\tau \in (1/2^{m_0+1}, 1 - 1/2^{m_0+1})$ with $|\beta'_{n,m}(\tau)| \leq (8/3)\sup_{t\in[0,1]}\overline{\overline{u}}_m(t)$. As a result

$$\{\beta_{n,m}(t)\}_{n=n_0+1}^{\infty} \text{ is bounded, equicontinuous family on } \left[\frac{1}{2^{n_0+1}}, 1 - \frac{1}{2^{n_0+1}}\right]. \tag{2.105}$$

The Arzela-Ascoli theorem guarantees the existence of subsequence N_{n_0} of integers and a function $z_{n_0,m} \in [1/2^{n_0+1}, 1 - 1/2^{n_0+1}]$ with $\beta_{n,m}$ converging uniformly to $z_{n_0,m}$ on $[1/2^{n_0+1}, 1 - 1/2^{n_0+1}]$ as $n \to \infty$ through N_{n_0}. Similarly,

$$\{\beta_{n,m}\}_{n=n_0+1}^{\infty} \text{ is bounded, equicontinuous family on } \left[\frac{1}{2^{n_0+2}}, 1 - \frac{1}{2^{n_0+2}}\right], \tag{2.106}$$

so there is a subsequence N_{n_0+1} of N_{n_0} and a function $z_{n_0+1,m} \in C[1/2^{n_0+2}, 1 - 1/2^{n_0+2}]$ with $\beta_{n,m}$ converging uniformly to $z_{n_0+1,m}$ on $[1/2^{n_0+2}, 1 - 1/2^{n_0+2}]$ as $n \to \infty$ through N_{n_0+1}. Note $z_{n_0+1,m} = z_{n_0,m}$ on $[1/2^{n_0+1}, 1 - 1/2^{n_0+1}]$ since $N_{n_0+1} \subseteq N_{n_0}$. Proceed inductively to obtain subsequences of integers $N_{n_0} \supseteq N_{n_0+1} \supseteq \cdots \supseteq N_k \supseteq \cdots$ and functions $z_{k,m} \in C[1/2^{k+1}, 1 - 1/2^{k+1}]$ with $\beta_{n,m}$ converging uniformly to $z_{k,m}$ on $[1/2^{k+1}, 1-1/2^{k+1}]$ as $n \to \infty$ through N_k, and $z_{k,m} = z_{k-1,m}$ on $[1/2^k, 1 - 1/2^k]$.

Define a function $u_m : [0,1] \to [0,\infty)$ by $u_m(t) = z_{k,m}(t)$ on $[1/2^{k+1}, 1 - 1/2^{k+1}]$ and $u_m(0) = u_m(1) = 0$. Notice u_m is well defined and $\underline{\underline{u}}(t) \leq u_m(t) \leq \overline{\overline{u}}_m(t)$ for $t \in (0,1)$. Next, fix $t \in (0,1)$ (without loss of generality assume $t \neq 1/2$) and let $n^* \in \{n_0, n_0 + 1, \ldots\}$ be such that $1/2^{n^*+1} < t < 1 - 1/2^{n^*+1}$. Let $N^*_{n^*} = \{i \in N_n : i \geq n^*\}$. Now $\beta_{n,m}, n \in N^*_{n^*}$ satisfies the integral equation

$$\begin{aligned}\beta_{n,m}(t) &= \beta_{n,m}\left(\frac{1}{2}\right) + \beta'_{n,m}\left(\frac{1}{2}\right)\left(t - \frac{1}{2}\right) \\ &\quad + \int_{1/2}^{t} (s-t)\left(\overline{g}_m\left(s, \frac{1}{n} + \beta_{n,m} \wedge \chi\right) + \mu h(s, \beta_{n,m} \wedge \chi) + \delta_n(s)\right) ds,\end{aligned} \tag{2.107}$$

for $t \in [1/2^{n+1}, 1 - 1/2^{n+1}]$. Notice (take $t = 2/3$ say) that $\{\beta_{n,m}(1/2)\}, n \in N^*_{n^*}$, is a bounded sequence since $\underline{\underline{u}}(t) \leq \beta_{n,m}(t) \leq \overline{\overline{u}}_m(t)$ for $t \in [0,1]$. Thus $\{\beta_{n,m}(1/2)\}_{n\in N^*_{n^*}}$ has a convergent subsequence; for convenience we will let $\{\beta_{n,m}(1/2)\}_{n\in N^*_{n^*}}$ denote this subsequence also, and let $\tau \in R$ be its limit. Now for the above fixed t, and let $n \to \infty$ through $N^*_{n^*}$ to obtain

$$\begin{gathered} g_m\left(t, \frac{1}{n} + \beta_{n,m} \wedge \chi\right) \longrightarrow g_m(t, z_{k,m} \wedge \chi), \\ h(t, \beta_{n,m} \wedge \chi) \longrightarrow h(t, z_{k,m} \wedge \chi), \\ \delta_n \longrightarrow 0. \end{gathered} \tag{2.108}$$

As a result,

$$z_{k,m}(t) = z_{k,m}\left(\frac{1}{2}\right) + \tau\left(t - \frac{1}{2}\right) + \int_{1/2}^{t} (s-t)\left(\overline{g}_m\left(s, z_{k,m} \wedge \chi\right) + \mu h\left(s, z_{k,m} \wedge \chi\right)\right) ds, \quad (2.109)$$

that is,

$$u_m(t) = u_m\left(\frac{1}{2}\right) + \tau\left(t - \frac{1}{2}\right) + \int_{1/2}^{t} (s-t)\left(\overline{g}_m\left(s, u_m \wedge \chi\right) + \mu h\left(s, u_m \wedge \chi\right)\right) ds. \quad (2.110)$$

We can do this argument for each $t \in (0,1)$ and so

$$-u_m''(t) = \overline{g}_m(t, u_m \wedge \chi) + \mu h(t, u_m \wedge \chi) \quad \text{for } t \in (0,1). \quad (2.111)$$

It remains to show that u_m is continuous at 0 and 1.

Let $\varepsilon > 0$ be given. Since $\overline{\overline{u}}_m \in C[0,1]$ there exists $\delta > 0$ with $\overline{\overline{u}}_m(t) < \varepsilon/2$ for $t \in [0,\delta]$. As a result $\underline{\underline{u}}(t) \le \beta_{n,m}(t) \le \overline{\overline{u}}_m(t) < \varepsilon/2$ for $t \in [0,\delta]$. Consequently, $\underline{\underline{u}}(t) \le u_m(t) \le \varepsilon/2 < \varepsilon$ for $t \in [0,\delta]$ and so u_m is continuous at 0. Similarly, u_m is continuous at 1. As a result $u_m \in C[0,1]$ and

$$\begin{gathered} -u_m''(t) = \overline{g}_m(t, u_m \wedge \chi) + \mu h(t, u_m \wedge \chi) \quad \text{for } t \in (0,1), \\ u_m(0) = u_m(1) = 0. \end{gathered} \quad (2.112)$$

Next we prove

$$u_m(t) \le \chi(t) \quad \text{for } t \in [0,1]. \quad (2.113)$$

Suppose (2.113) is not true. Let $y(t) = u_m(t) - \chi(t)$ and $\sigma \in (0,1)$ be the point where $y(t)$ attains its maximum over $(0,1)$. We have

$$y(\sigma) > 0, \qquad y''(\sigma) \le 0. \quad (2.114)$$

On the other hand, since $u_m(\sigma) > \chi(\sigma)$, we have

$$\begin{aligned} y''(\sigma) &= u_m''(\sigma) - \chi''(\sigma) \\ &= -\overline{g}_m(\sigma, u_m \wedge \chi) - \mu h(\sigma, u_m \wedge \chi) + g(\sigma, \chi) + \lambda h(\sigma, \chi) \\ &= -\overline{g}_m(\sigma, \chi(\sigma)) - \mu h(\sigma, \chi(\sigma)) + g(\sigma, \chi(\sigma)) + \lambda h(\sigma, \chi(\sigma)) \\ &\ge (\lambda - \sigma) h(\sigma, \chi(\sigma)) > 0. \end{aligned} \quad (2.115)$$

This is a contradiction, so (2.113) is true.

Thus we have

$$-u_m'' = g_m(t,u_m) + \mu h(t,u_m),$$
$$u_m(0) = u_m(1) = 0, \tag{2.116}$$
$$\underline{\underline{u}}(t) \le u_m(t) \le \chi(t) \quad \text{for } t \in [0,1].$$

By the same reason as above, we obtain subsequences of integers $N_{m_0} \supseteq N_{m_0+1} \supseteq \cdots \supseteq N_k \supseteq \cdots$ and functions $z_m \in C[1/2^{k+1}, 1-1/2^{k+1}]$ with u_m converging uniformly to z_k on $[1/2^{k+1}, 1-1/2^{k+1}]$ as $m \to \infty$ through N_k, and $z_k = z_{k-1}$ on $[1/2^k, 1-1/2^k]$.

Define a function $u : [0,1] \to [0,\infty)$ by $u(t) = z_k(t)$ on $[1/2^{k+1}, 1-1/2^{k+1}]$ and $u(0) = u(1) = 0$. Notice u is well defined and $\underline{\underline{u}}(t) \le u(t) \le \chi(t)$ for $t \in (0,1)$. Next fix $t \in (0,1)$ (without loss of generality assume $t \ne 1/2$) and let $m^* \in \{m_0, m_0+1, \ldots\}$ be such that $1/2^{m^*+1} < t < 1-1/2^{m^*+1}$. Let $N_{m^*}^* = \{k \in N_{m^*} : k \ge m^*\}$. Now $u_m, m \in N_{m^*}^*$ satisfies the integral equation

$$u_m(t) = u_m\left(\frac{1}{2}\right) + u_m'\left(\frac{1}{2}\right)\left(t - \frac{1}{2}\right) + \int_{1/2}^{t} (s-t)\left(\overline{g}_m(s,u_m) + \mu h(s,u_m)\right) ds \tag{2.117}$$

for $t \in [1/2^{m^*+1}, 1-1/2^{m^*+1}]$. Notice (take $t = 2/3$ say) that $\{u_m(1/2)\}$, $m \in N_{m^*}^*$ is a bounded sequence since $\underline{\underline{u}}(t) \le u_m(t) \le \chi(t)$ for $t \in [0,1]$. Thus $\{u_m(1/2)\}_{m\in N_{m^*}^*}$ has a convergent subsequence; for convenience we will let $\{u_m(1/2)\}_{m\in N_{m^*}^*}$ denote this subsequence also, and let $\tau \in R$ be its limit. Now for the above fixed t, and letting $m \to \infty$ through N_k^* to obtain

$$u(t) = u\left(\frac{1}{2}\right) + \tau\left(t - \frac{1}{2}\right) + \int_{1/2}^{t} (s-t)\left(g(s,u) + \mu h(s,u)\right) ds. \tag{2.118}$$

we can do this argument for each $t \in (0,1)$ and so

$$-u''(t) = g(t,u) + \mu h(t,u) \quad \text{for } t \in (0,1). \tag{2.119}$$

Also reasoning as before we have that u is continuous at 0 and 1.

Thus we have

$$-u'' = g(t,u) + \mu h(t,u),$$
$$u(0) = u(1) = 0. \tag{2.120}$$

Now let $\lambda_2^* = \sup \Lambda > 0$. Then

(i) if $0 < \lambda < \lambda_2^*$, (1.4) has at least one solution $u \in C[0,1] \cap C^1(0,1)$ and $u > 0$ for $t \in (0,1)$;

(ii) if $\lambda > \lambda_2^*$, (1.4) has no solutions. □

2.4. The Proof of Theorem 1.4

Claim 4. Let

$$\lambda^* = \frac{1}{\max_{t\in[0,1]}\int_0^1 N(t,s)h_2(s,a_3+\phi_1)ds} > 0; \tag{2.121}$$

here

$$a_3 = 1 + \frac{1}{4}\int_0^1 \left(g_2(s,1) + h_2\left(s, \frac{1}{2} + \phi_1(s)\right) + \frac{\phi_1(s)}{2|\phi_1|_\infty} \right) ds. \tag{2.122}$$

Then $(0, \lambda^*) \in \Lambda$.

Proof of Claim 4. Let $\lambda \in (0, \lambda^*)$ be fixed. From assumption (G6), it follows that there is $\tau \geq \tau_1$ and $c_3 \in (0,1)$, such that if $n > 2/c_3$, $0 < k < c_3/2 < 1$, we have

$$\begin{gathered} 0 < k|\phi_1|_\infty < \frac{c_3}{2}, \qquad 0 < \frac{1}{n} + k\phi_1(t) < c_3, \\ \frac{\tau(1/n + k\phi_1(t)) + g^-(t, 1/n + k\phi_1(t))}{h(t, 1/n + k\phi_1(t))} \leq \lambda. \end{gathered} \tag{2.123}$$

Thus,

$$\frac{\tau k\phi_1(t) + g^-(t, 1/n + k\phi_1(t))}{h(t, 1/n + k\phi_1(t))} \leq \lambda. \tag{2.124}$$

Then, for $n > 2/c_3$,

$$\tau k\phi_1(t) + g^-\left(t, \frac{1}{n} + k\phi_1(t)\right) \leq \lambda h\left(t, \frac{1}{n} + k\phi_1(t)\right), \tag{2.125}$$

and we have

$$\begin{aligned} \tau k\phi_1(t) &\leq \lambda h\left(t, \frac{1}{n} + k\phi_1(t)\right) - g^-\left(t, \frac{1}{n} + k\phi_1(t)\right) \\ &\leq g^+\left(t, \frac{1}{n} + k\phi_1(t)\right) - g^-\left(t, \frac{1}{n} + k\phi_1(t)\right) + \lambda h\left(t, \frac{1}{n} + k\phi_1(t)\right) \\ &= g\left(t, \frac{1}{n} + k\phi_1(t)\right) + \lambda h\left(t, \frac{1}{n} + k\phi_1(t)\right) \\ &\quad - \lambda h(t, k\phi_1(t)) + \lambda h(t, k\phi_1(t)) \\ &= g\left(t, \frac{1}{n} + k\phi_1(t)\right) + \lambda h(t, k\phi_1(t)) + \delta_n(t), \end{aligned} \tag{2.126}$$

where

$$\delta_n(t) = \lambda h\left(t, \frac{1}{n} + k\phi_1(t)\right) - \lambda h\big(t, k\phi_1(t)\big). \tag{2.127}$$

Let $\overline{u}(t) = k\phi_1(t)$. We have

$$-\overline{u}''(t) = \tau_1 k\phi_1(t) \le \tau k\phi_1(t) \le g\left(t, \frac{1}{n} + \overline{u}(t)\right) + \lambda h(t, \overline{u}(t)) + \delta_n(t) \quad \text{for } t \in (0,1). \tag{2.128}$$

Let

$$\psi(t,s) = g_2(t,s) + \lambda h_2\left(t, \frac{1}{2} + \phi_1(t)\right) + \frac{\phi_1(t)}{2|\phi_1|_\infty}. \tag{2.129}$$

From (G1) notice that ψ satisfies the assumptions of Lemma 2.3, so there exist $\omega, \omega_n \in C[0,1]$ such that

$$\begin{gathered}
-\omega_n''(t) = g_2\left(t, \frac{1}{n} + \omega_n\right) + \lambda h_2\left(t, \frac{1}{2} + \phi_1(t)\right) + \frac{\phi_1(t)}{2|\phi_1|_\infty} \quad \text{for } t \in (0,1), \\
\omega_n(0) = \omega_n(1) = 0, \\
\omega(t) = \lim_{n\to\infty} \omega_n(t) \quad \text{for } t \in [0,1], \\
\omega_n(t) \le 1 + \frac{1}{4}\int_0^1 \left(g_2(s,1) + h_2\left(s, \frac{1}{2} + \phi_1(s)\right) + \frac{\phi_1(s)}{2|\phi_1|_\infty}\right) ds = a_3 \quad \text{for } t \in [0,1],\ n \in N.
\end{gathered} \tag{2.130}$$

Consider the boundary value problem

$$\begin{gathered}
-\tilde{v}''(t) = \lambda h_2(t, \omega_n + \tilde{v}) \quad \text{for } t \in (0,1), \\
\tilde{v}(0) = \tilde{v}(1) = 0.
\end{gathered} \tag{2.131}$$

Let $\Phi : C[0,1] \to C[0,1]$ be the operator defined by

$$(\Phi v)(t) := \lambda \int_0^1 G(t,s) h_2(s, \omega_n + v) ds \quad \text{for } v \in C[0,1],\ t \in [0,1]. \tag{2.132}$$

It is easy to see that Φ is a continuous and completely continuous operator. Also if $0 \le v(t) \le \phi_1(t)$ for $t \in [0,1]$, then

$$\begin{aligned} 0 \le \Phi(v)(t) &= \lambda \int_0^1 G(t,s) h_2(s, \omega_n + v)\, ds \\ &\le \lambda^* \int_0^1 G(t,s) h_2(s, a_3 + \phi_1)\, ds \\ &= \frac{\phi_1(t) \int_0^1 N(t,s) h_2(s, a_3 + \phi_1)\, ds}{\max_{t\in[0,1]} \int_0^1 N(t,s) h_2(s, a_3 + \phi_1)\, ds} \\ &\le \phi_1(t) \quad \text{for } t \in [0,1]. \end{aligned} \tag{2.133}$$

Thus Schauder's fixed point theorem guarantees that there exists $\tilde{v}_n \in [0, \phi_1]$ such that $\Phi(\tilde{v}_n) = \tilde{v}_n$, that is,

$$\begin{gathered} -\tilde{v}_n''(t) = \lambda h_2(t, \omega_n + \tilde{v}_n), \\ \tilde{v}_n(0) = \tilde{v}_n(1) = 0. \end{gathered} \tag{2.134}$$

Let

$$\hat{u}_n(t) = \omega_n(t) + \tilde{v}_n(t), \quad \hat{u}(t) = \omega(t) + \phi_1(t) \quad \text{for } t \in [0,1]. \tag{2.135}$$

Then $\hat{u}_n, \hat{u} \in C[0,1]$, $\hat{u}_n(0) = \hat{u}_n(1) = 0$, $\hat{u}(0) = \hat{u}(1) = 0$, $0 \le \hat{u}_n(t) \le \hat{u}(t)$ for $t \in [0,1]$, and

$$\begin{aligned} -\hat{u}_n''(t) &= -\omega_n''(t) - \tilde{v}_n''(t) \\ &= g_2\left(t, \frac{1}{n} + \omega_n\right) + \lambda h_2\left(t, \frac{1}{2} + \phi_1(t)\right) + \frac{\phi_1(t)}{2|\phi_1|} + \lambda h_2(t, \omega_n + \tilde{v}_n) \\ &\ge g_2\left(t, \frac{1}{n} + \hat{u}_n\right) + \lambda h_2\left(t, \frac{1}{2} + \phi_1(t)\right) + \frac{\phi_1(t)}{2|\phi_1|} + \lambda h_2(t, \hat{u}_n) \quad \text{for } t \in (0,1). \end{aligned} \tag{2.136}$$

We next prove that

$$\overline{u}(t) \le \hat{u}_n(t) \quad \text{for } t \in [0,1]. \tag{2.137}$$

Suppose (2.137) is not true. Let $y(t) = \overline{u}(t) - \hat{u}_n(t)$ and $\sigma \in (0,1)$ be the point where $y(t)$ attains its maximum over $(0,1)$. We have

$$y(\sigma) > 0, \qquad y''(\sigma) \le 0. \tag{2.138}$$

On the other hand, since $\overline{u}(\sigma) > \widehat{u}_n(\sigma)$, we have

$$
\begin{aligned}
-\overline{u}''(\sigma) &\leq g\left(\sigma, \frac{1}{n} + \overline{u}(\sigma)\right) + \lambda h(\sigma, \overline{u}(\sigma)) + \delta_n(\sigma) \\
&= g\left(\sigma, \frac{1}{n} + \overline{u}(\sigma)\right) + \lambda h\left(\sigma, \frac{1}{n} + \overline{u}(\sigma)\right) \\
&\leq g_2\left(\sigma, \frac{1}{n} + \overline{u}(\sigma)\right) + \lambda h_2\left(\sigma, \frac{1}{2} + \phi_1(\sigma)\right) \\
&< g_2\left(\sigma, \frac{1}{n} + \widehat{u}_n(\sigma)\right) + \lambda h_2\left(\sigma, \frac{1}{2} + \phi_1(\sigma)\right) + \frac{\phi_1(\sigma)}{2|\phi_1|} + \lambda h_2(\sigma, \widehat{u}_n(\sigma)) \\
&\leq -\widehat{u}_n''(\sigma).
\end{aligned}
\tag{2.139}
$$

Thus $y''(\sigma) = \overline{u}''(\sigma) - \widehat{u}_n''(\sigma) > 0$, and this is a contradiction. As a result, (2.137) is true.

On the other hand, we have

$$
\begin{aligned}
|\delta_n(t)| &\leq \lambda \left| h\left(t, \frac{1}{n} + k\phi_1(t)\right) - h(t, k\phi_1(t)) \right| \\
&\leq 2\lambda h_2\left(t, \frac{1}{2} + |\phi_1|\right)
\end{aligned}
\tag{2.140}
$$

for $n > 2/c_3$. Consequently, for $t \in (0,1)$, $\delta_n \to 0$ and $n \to \infty$.

From assumptions $(G2)$ and $(H5)$, there exists a $\gamma, \tau \in M$, $n > 2/c_3$, so that $g(t, 1/n + r) + h(t,r) + a(t)r$ is increasing in $(0, |\widehat{u}|_\infty)$, where $a(t) = \gamma(t) + \tau(t)$. Let $\overline{u}_n = \overline{u}(t)$. For $v \in D_{\overline{u}_n}^{\widehat{u}_n}$, we have

$$
\begin{aligned}
&- \overline{u}_n''(t) + a(t)\overline{u}_n(t) \\
&\quad\leq g\left(t, \frac{1}{n} + \overline{u}_n(t)\right) + \lambda h(t, \overline{u}_n(t)) + \delta_n(t) + a(t)\overline{u}_n(t) \\
&\quad\leq g\left(t, \frac{1}{n} + \overline{u}_n(t)\right) + \lambda h(t, \overline{u}_n(t)) + a(t)\overline{u}_n(t) \\
&\qquad + \lambda h\left(t, \frac{1}{n} + \overline{u}_n(t)\right) - \lambda h(t, \overline{u}_n(t)) \\
&\quad\leq g\left(t, \frac{1}{n} + \widehat{u}_n(t)\right) + \lambda h(t, \widehat{u}_n(t)) + a(t)\widehat{u}_n(t) + \lambda h\left(t, \frac{1}{n} + \overline{u}_n(t)\right) \\
&\quad\leq g_2\left(t, \frac{1}{n} + \widehat{u}_n(t)\right) + \lambda h_2(t, \widehat{u}_n(t)) + \frac{\phi_1(t)}{2|\phi_1|_\infty} \\
&\qquad + \lambda h_2\left(t, \frac{1}{2} + \phi_1(t)\right) + a(t)\widehat{u}_n(t) \\
&\quad\leq -\widehat{u}_n''(t) + a_n(t)\widehat{u}_n(t).
\end{aligned}
\tag{2.141}
$$

Reasoning as in the proof of Theorem 1.1, Lemma 2.2 guarantees that (1.4) has a solution $u \in C[0,1] \cap C^1(0,1)$.

Thus (1.4) has a solution for $\lambda \in (0, \lambda^*)$ so Claim 4 holds. In particular, $\Lambda \neq \emptyset$ and $\sup \Lambda > 0$. □

Claim 5. If $\lambda \in \Lambda$, then $(0, \lambda] \in \Lambda$.

Proof of Claim 5. We may assume that $\lambda > 0$. Let χ be a positive solution of (1.4), that is,

$$-\chi'' = g(t,\chi) + \lambda h(t,\chi), \quad t \in (0,1),$$
$$\chi(0) = 0 = \chi(1). \tag{2.142}$$

We first prove that there exists $\rho > 0$ such that

$$\chi(t) \geq \rho l(t) \quad \text{for } t \in [0,1]. \tag{2.143}$$

By $(G6)$, there exists $\sigma > 0$ such that for all $r \in (0, \sigma)$, we have

$$\frac{\tau r + g^-(t,r)}{h(t,r)} \leq \lambda, \tag{2.144}$$

that is,

$$\tau r \leq \lambda h(t,r) - g^-(t,r) \quad \text{for } t \in (0,1),\ r \in (0,\sigma). \tag{2.145}$$

From the continuity of χ and $\chi(0) = 0 = \chi(1)$, it follows that there is $0 < \delta < 1/2$ such that

$$\chi(t) < \sigma \quad \text{for } t \in [0,\delta] \cup [1-\delta, 1]. \tag{2.146}$$

Then

$$\begin{aligned} -\chi'' &= g(t,\chi) + \lambda h(t,\chi) \\ &= g^+(t,\chi) + \lambda h(t,\chi) - g^-(t,\chi) \\ &\geq \lambda h(t,\chi) - g^-(t,\chi) \\ &\geq \tau \chi(t) \quad \text{for } t \in [0,\delta] \cup [1-\delta,1]. \end{aligned} \tag{2.147}$$

The next part is similar to the proof of (2.72), that is, there exists $\rho > 0$ such that

$$\chi(t) \geq \rho l(t) \quad \text{for } t \in [0,1]. \tag{2.148}$$

We consider the boundary value problem

$$-u'' = g(t, u \wedge \chi) + \mu h(t, u \wedge \chi),$$
$$u(0) = u(1) = 0, \tag{2.149}$$

where $\mu \in (0, \lambda)$. Let $\tilde{g}_1(t,u) = g_1(t, u \wedge \chi), \tilde{g}_2(t,u) = g_2(t, u \wedge \chi), \tilde{h}_1(t,u) = h_1(t, u \wedge \chi)$, and $\tilde{h}_2(t,u) = h_2(t, u \wedge \chi)$. We easily prove that the conditions of [6, Theorem 1.2] are satisfied so (2.149) has a positive solution $u \in C^1(0,1) \cap C[0,1]$. We next prove that

$$u(t) \le \chi(t) \quad \text{for } t \in [0,1]. \tag{2.150}$$

Suppose (2.150) is not true. Let $y(t) = u(t) - \chi(t)$ and $\sigma \in (0,1)$ be the point where $y(t)$ attains its maximum over $(0,1)$. We have

$$y(\sigma) > 0, \qquad y''(\sigma) \le 0. \tag{2.151}$$

On the other hand, since $u(\sigma) > \chi(\sigma)$, we have

$$\begin{aligned} y''(\sigma) &= u''(\sigma) - \chi''(\sigma) \\ &= -g(\sigma, u \wedge \chi) - \mu h(\sigma, u \wedge \chi) + g(\sigma, \chi) + \lambda h(\sigma, \chi) \\ &= -g(\sigma, \chi(\sigma)) - \mu h(\sigma, \chi(\sigma)) + g(\sigma, \chi(\sigma)) + \lambda h(\sigma, \chi(\sigma)) \\ &= (\lambda - \mu) h(\sigma, \chi(\sigma)) \\ &> 0. \end{aligned} \tag{2.152}$$

This is a contradiction, so

$$u(t) \le \chi(t) \quad \text{for } t \in [0,1]. \tag{2.153}$$

Thus we have

$$-u'' = g(t, u) + \mu h(t, u),$$
$$u(0) = u(1) = 0. \tag{2.154}$$

Let $\lambda_3^* = \sup \Lambda > 0$. Then

(i) if $0 < \lambda < \lambda_3^*$, (1.4) has at least one solution $u \in C[0,1] \cap C^1(0,1)$ and $u > 0$ for $t \in (0,1)$;

(ii) if $\lambda > \lambda_3^*$, (1.4) has no solutions. □

3. Example

Example 3.1. Consider the boundary value problem

$$-u'' = -\frac{1}{\sqrt{u}} + \lambda q(u) \quad \forall 0 < t < 1,$$
$$u(0) = u(1) = 0, \tag{3.1}$$

where $\lambda > 1$.

Define $\{x_n\}_{n=1}^{\infty}$ as $x_1 = 2, x_{2n} = x_{2n-1}^4, x_{2n+1} = x_{2n} + 1$, and

$$q(r) = \begin{cases} r^2, & \text{if } r \in [0,2], \\ x_{2n-1}^2, & \text{if } r \in [x_{2n-1}, x_{2n}], \\ \dfrac{x_{2n+1}^2 - \sqrt{x_{2n}}}{x_{2n+1} - x_{2n}}(r - x_{2n}) + \sqrt{x_{2n}}, & \text{if } r \in [x_{2n}, x_{2n+1}]. \end{cases} \tag{3.2}$$

Then, Theorem 1.1 implies that there exists $\lambda_1^* > 0$ such that for every $\lambda \geq \lambda_1^*$, (3.1) has at least one positive solution $u \in C[0,1] \cap C^1(0,1)$ and $u > 0$ for $t \in (0,1)$.

To see this, let

$$g_1(t,r) = g_2(t,r) = \frac{1}{\sqrt{r}} \quad \text{for } (t,r) \in (0,1) \times (0,\infty),$$
$$h_2(t,r) = q(r) \quad \text{for } (t,r) \in (0,1) \times (0,\infty), \tag{3.3}$$
$$h_1(t,r) = \begin{cases} \sqrt{r} & \text{for } (t,r) \in (0,1) \times (16,\infty), \\ q(r) & \text{for } (t,r) \in (0,1) \times (0,16). \end{cases}$$

It is easy to see that $(G1), (H1), (H2)$, and $(H3)$ are satisfied.

For all $r_2 > r_1 > 0$, let $\gamma(t) = 1/2r_1\sqrt{r_1}$. Then $g_2(t,r) + (1/2r_1\sqrt{r_1})r$ is increasing in (r_1, r_2).

On the other hand, $a_1 = 1 + \int_0^1 (1/\sqrt{s})ds = 3$ and let $R_j = x_{2j} - 3$, so we have

$$\begin{aligned} \lim_{j\to\infty} \frac{h_2(s, R_j + a_1)}{R_j} &= \lim_{j\to\infty} \frac{h_2(s, x_{2j})}{x_{2j}} \cdot \frac{x_{2j}}{x_{2j} - 3} \\ &= \lim_{j\to\infty} \frac{\sqrt{x_{2j}}}{x_{2j}} \cdot \frac{x_{2j}}{x_{2j} - 3} \\ &= 0. \end{aligned} \tag{3.4}$$

Thus $(G2)$ and $(H4)$ are satisfied. Then Theorem 1.1 implies that there exists $\lambda_1^* > 0$ such that for every $\lambda \geq \lambda_1^*$, (3.1) has at least one positive solution $u \in C[0,1] \cap C^1(0,1)$ and $u > 0$ for $t \in (0,1)$.

Example 3.2. Consider the boundary value problem

$$-u'' = g(t,u) + \lambda h(t,u), \quad t \in (0,1),$$
$$u(0) = 0 = u(1), \tag{3.5}$$

where

$$g(t,r) = \begin{cases} \dfrac{1}{r^\alpha}\left|\sin\dfrac{1}{r}\right|, & 0 < r \le \dfrac{1}{\pi}, \\ -\dfrac{1}{r^\alpha}\sin\dfrac{1}{r}, & \dfrac{1}{\pi} < r, \end{cases}$$
$$h(t,r) = r^2, \tag{3.6}$$

with $\alpha > 0$. Then Theorem 1.2 guarantees that there exists $\lambda_2^* > 0$ such that

(i) if $0 < \lambda < \lambda_2^*$, (3.5) has at least one solution $u \in C[0,1] \cap C^1(0,1)$ and $u > 0$ for $t \in (0,1)$;

(ii) if $\lambda > \lambda_2^*$, (3.5) has no solutions.

To see this, let $\beta = \min\{1/2, \alpha/2\}$, $g_1(t,r) = 1/r^\beta + \pi^\alpha$, and $g_2(t,r) = 1/r^\alpha$, for $(t,r) \in (0,1) \times (0,\infty)$, and $h_1(t,r) = h_2(t,r) = r^2$, for $(t,r) \in (0,1) \times [0,\infty)$. Notice that $(G1)$, $(H1)$, and $(H2)$ are satisfied.

For all $r_2 > r_1 > 0$, let

$$\gamma(t) \equiv \sup_{r\in\Lambda}\left|\frac{\partial g}{\partial r}\right| + 1 < \infty, \tag{3.7}$$

where $\Lambda = (r_1, r_2) \setminus \{n\pi \mid n \in N\}$, so we have $g(t,r) + \gamma(t)r$ is increasing in (r_1, r_2).

Let $c_1 = 1/\pi$ and we have

$$0 \le g(t,r), \quad t \in (0,1),\ 0 < r < c_1. \tag{3.8}$$

Let n_0 be fixed such that

$$2^{1/(\alpha-\beta)} < n_0\pi + \frac{\pi}{6}. \tag{3.9}$$

Let $c_2 \in (0, c_1)$ be such that

$$c_2^3 < \frac{1}{2(2-\beta)\pi^{1-\beta}}\sum_{n=n_0}^{\infty}\left[\left(\frac{6}{6n+1}\right)^{2-\beta} - \left(\frac{6}{6n+5}\right)^{2-\beta}\right], \tag{3.10}$$

and we have for $n \geq n_0, r \in (0, c_2)$,

$$\frac{1}{(rt)^{\alpha}}\left|\sin\frac{1}{rt}\right| \geq \frac{1}{(rt)^{\beta}} \quad \text{for } t \in \left[\frac{1}{r(n\pi + 5\pi/6)}, \frac{1}{r(n\pi + \pi/6)}\right]. \tag{3.11}$$

Also we have

$$\begin{aligned}
\int_0^1 t(1-t)\overline{g}_m(t, rl(t))dt &\geq \int_0^{1/2} t(1-t)\overline{g}_m(t, rl(t))dt \\
&\geq \frac{1}{2}\int_0^{1/2} t\overline{g}_m(t, rl(t))dt \\
&\geq \frac{1}{2}\sum_{n=n_0}^{\infty}\int_{1/r(n\pi+5\pi/6)}^{1/r(n\pi+\pi/6)} t\frac{1}{(rt)^{\beta}}\, dt \\
&\geq \frac{1}{2r^{\beta}}\sum_{n=n_0}^{\infty}\int_{1/r(n\pi+5\pi/6)}^{1/r(n\pi+\pi/6)} t^{1-\beta}dt \\
&= \frac{1}{2r^2(2-\beta)\pi^{2-\beta}}\sum_{n=n_0}^{\infty}\left[\left(\frac{6}{6n+1}\right)^{2-\beta} - \left(\frac{6}{6n+5}\right)^{2-\beta}\right] \\
&\geq r\pi.
\end{aligned} \tag{3.12}$$

Thus $(G5)$ is satisfied.

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