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Research Article

Infinitely Many Solutions for a Semilinear Elliptic Equation with Sign-Changing Potential

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We consider a similinear elliptic equation with sign-changing potential $-\Delta u - V(x)u = f(x,u)$, $u \in H^1(\mathbb{R}^N)$, where $V(x)$ is a function possibly changing sign in $\mathbb{R}^N$. Under certain assumptions on f, we prove that the equation has infinitely many solutions.

1. Introduction

In this paper, the existence of solutions of the following elliptic equation:

$$-\Delta u - V(x)u = f(x,u), \quad u \in H^1\left(\mathbb{R}^N\right) \tag{P}$$

is studied, where $V(x)$ is a function possibly changing sign, f is a continuous function on $\mathbb{R}^N \times \mathbb{R}$.

Problem (P) arises in various branches of applied mathematics and has been studied extensively in recent years. For example, Rabinowitz [1] has studied the existence of a nontrivial solution of this kind of equation on a bounded domain. Lien et al. [2] studied the existence of positive solutions of problem (P) with $V(x) \equiv \lambda$ (λ is a positive constant) and $f(x,u) = |u|^{p-2}u$. And Grossi et al. [3] established some existence results for $-\Delta u = \lambda u + a(x)g(u)$, where $a(x)$ is a function possibly changing sign, $g(u)$ has superlinear growth and λ is a positive real parameter; he discussed both the cases of subcritical and critical growth for $g(u)$ and proved the existence of linking type solutions.

Cerami et al. [4] prove that the problem (P) has infinitely many solutions, where $a(x)$ is a regular function such that $\liminf_{|x|\to\infty} a(x) = a_\infty > 0$ and some suitable decay assumptions, $f(x,u) = |u|^{p-2}u$. Kryszewski and Szulkin [5] considered the existence of a nontrivial solution of (P) in a situation where $f(x,u)$ and $V(x)$ are periodic in the x-variable, $f(x,u)$ is superlinear at $u = 0$ and $\pm\infty$, and 0 lies in a spectral gap of $-\Delta u + V$. If in addition $f(x,u)$ is odd in u, (P) has infinitely many solutions.

In [6], Zeng and Li proved existence of $m-n$ pairs of nontrivial solutions ($m > n$, m and n are integers) of (P), under the assumption that $V(x)$ is a function possibly changing sign in $\mathbb{R}^N$ and $f(x,u)$ satisfies some growth conditions.

In this paper, we prove the existence of infinitely many solutions of (P), under the assumption that $V(x)$ is a function possibly changing sign in $\mathbb{R}^N$ and $f(x,u)$ also satisfies some growth conditions. One difficulty in considering problem (P) is the loss of compactness because of $\mathbb{R}^N$; the other is that $V(x)$ may change sign, which leads to difficulty in verifying the Palais-Smale condition and applying the well-known theorem.

Notation. We use the following notations. A strip region is a domain like this: for $d > 0$, $\widetilde{\Omega} = \{x \in \mathbb{R}^N; -d < x_i < d$ at least for some fixed $i\}$. $V(x) = V^+(x) - V^-(x)$, where $V^\pm = \max\{\pm V(x), 0\}$. $\Omega_1 = \{x \in \mathbb{R}^N; V^-(x) \neq 0\}$, $\Omega_2 = \{x \in \mathbb{R}^N; V^-(x) = 0\}$.

X is defined as the completion of $D(\mathbb{R}^N)$ with respect to the inner product

$$\langle u, v\rangle_1 := \int_{\mathbb{R}^N} (\nabla u \cdot \nabla v + V^-(x)uv)\,dx. \tag{1.1}$$

The functional associated with (P) is

$$I(u) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + V^-(x)u^2dx - \frac{1}{2}\int_{\mathbb{R}^N} V^+(x)u^2dx - \int_{\mathbb{R}^N} F(x,u)dx, \tag{1.2}$$

for $u \in X$, where $F(x,u) = \int_0^u f(x,t)dt$.

Our fundamental assumptions are as follows:

($\mathbb{A}_1$) $V^+(x) \in L^{N/2}(\mathbb{R}^N)$, $\text{meas}\{x \in \mathbb{R}^N; V^+(x) \neq 0\} > 0$. $V^-(x) \in L^\infty(\mathbb{R}^N)$, Ω_2 is a strip region, $\lim_{|x|\to\infty} V^-(x) = a > 0$ in Ω_1.

($\mathbb{A}_2$) $f \in C(\mathbb{R}^N \times \mathbb{R})$ and there are constants $C_1 > 0$ and $2 < p \leq q < 2^*$ such that $|f(x,t)| \leq C_1(|t|^{p-1} + |t|^{q-1})$.

($\mathbb{A}_3$) There exists $\alpha > 2$ such that $0 < \alpha F(x,t) \leq t f(x,t)$ for every $x \in \mathbb{R}^N$ and $t \neq 0$.

($\mathbb{A}_4$) $\lim_{|x|\to\infty} \sup_{|t|\leq r}(|f(x,t)|/|t|) = 0$ for every $r > 0$.

($\mathbb{A}_5$) For any $t \in \mathbb{R}$, $f(x,t) = -f(x,-t)$.

Here 2^* denotes the critical Sobolev exponent, that is, $2^* = 2N/(N-2)$ for $N \geq 3$ and $2^* = \infty$ for $N = 1, 2$.

Theorem 1.1. *Under the assumptions* ($\mathbb{A}_1$)–($\mathbb{A}_5$), (P) *possesses infinitely many solutions on* X.

Remark 1.2. It is easily seen that ($\mathbb{A}_2$)–($\mathbb{A}_5$) hold for nonlinearities of the form $f(x,t) = \sum_{i=1}^k a_i(x)|t|^{p_i-2}t$ with $2 < p_i < 2^*$ and for $i = 1, \ldots, k$, the nonnegative function $a_i(x) \in L^\infty(\mathbb{R}^N)$, $\lim_{|x|\to\infty} a_i(x) = 0$.

2. Preliminaries

We define the Palais-Smale (denoted by (PS)) sequences, (PS)-values, and (PS)-conditions in X for I as follows.

Definition 2.1 (cf. [7]). (i) For $c \in \mathbb{R}$, a sequence $\{u_n\}$ is a $(PS)_c$-sequence in X for I if $I(u_n) = c + \circ(1)$ and $I'(u_n) = \circ(1)$ strongly in X' as $n \to \infty$;

(ii) $c \in \mathbb{R}$ is a (PS)-value in X for I if there is a $(PS)_c$-sequence in X for I;

(iii) I satisfies the $(PS)_c$-condition in X if every $(PS)_c$-sequence in X for I contains a convergent subsequence;

(iv) I satisfies the (PS)-condition in X if for every $c \in \mathbb{R}$, I satisfies the $(PS)_c$-condition in X.

Lemma 2.2 (cf. [6, Lemma 2.1]). *Under the assumption* $(\mathbb{A}_1)$, *the inner product*

$$\langle u, v\rangle_1 := \int_{\mathbb{R}^N} \left(\nabla u \cdot \nabla v + V^-(x)uv\right)dx \tag{2.1}$$

is well defined; therefore the corresponding norm $\|u\|_1 := \sqrt{\langle u, u\rangle_1}$ *is well defined too, which is equivalent to the norm* $\|u\| = \left(\int_{\mathbb{R}^N}(|\nabla u|^2 + u^2 dx\right)^{1/2}$.

Lemma 2.3 (cf. [8]). *Under the assumption that* $V^+(x) \in L^{N/2}(\mathbb{R}^N)$ *for the eigenvalue problem*

$$-\Delta u + V^-(x)u = \mu V^+(x)u, \quad u \in E \tag{2.2}$$

there exists a sequence of eigenvalues $\mu_n \to \infty$ *such that the eigenfunction sequence* φ_n *is an orthonormal basis of* E.

When $(PS)_c$-condition is satisfied for all $c \in \mathbb{R}$, there are known methods of obtaining an unbounded sequence of critical values of φ (see, e.g., [9]).

Theorem 2.4 (cf. [10, Theorem 6.5]). *Suppose that* E *is an infinite-dimensional Banach space and suppose* $\varphi \in C^1(E, \mathbb{R})$ *satisfies* (PS)*-condition,* $\varphi(u) = \varphi(-u)$ *for all* u, *and* $\varphi(0) = 0$. *Suppose* $E = E^- \oplus E^+$, *where* E^- *is finite dimensional, and assume the following conditions:*

(i) *there exist* $\zeta > 0$ *and* $\varrho > 0$ *such that if* $\|u\| = \varrho$ *and* $u \in E^+$, *then* $\varphi(u) \geq \zeta$;

(ii) *for any finite-dimensional subspace* $W \subset E$ *there exists* $R = R(W)$ *such that* $\varphi(u) \leq 0$ *for* $u \in W$, $\|u\| \geq R$.

Then φ possesses an unbounded sequence of critical values.

3. The $(\mathrm{PS})_c$-Condition

Lemma 3.1. *Under the assumptions* $(\mathbb{A}_1)$, $(\mathbb{A}_2)$, and $(\mathbb{A}_3)$, *for every* $c \in \mathbb{R}$, *any* $(PS)_c$*-sequence is bounded.*

Proof. By the eigenvalue problem in Lemma 2.3, there exist $k \in N$ such that eigenvalues are $\mu_1 < \mu_2 \leq \mu_3 \leq \cdots \leq \mu_k \leq \lambda < \mu_{k+1} \leq \cdots$ for some $\lambda \geq 1$; the corresponding eigenfunction is $\varphi_1, \varphi_2, \varphi_3, \ldots, \varphi_k, \varphi_{k+1}, \ldots$, then we denote $X = X_1 \bigoplus X_2$, with $X_1 = \bigoplus_{i=1}^k \mathrm{span}\{\varphi_i\}$, $X_2 = X_1^{\perp}$, and denote $u_n \in X$ as $u_n = v_n + w_n$, where $v_n \in X_1$, $w_n \in X_2$. It's obvious that

$$\int_{\mathbb{R}^N} \left(|\nabla u|^2 + V^-(x)u^2 - \lambda V^+(x)u^2 \right) dx \le 0, \quad \forall u \in X_1, \tag{3.1}$$

and there exist $\delta > 0$ such that

$$\int_{\mathbb{R}^N} \left(|\nabla u|^2 + V^-(x)u^2 - V^+(x)u^2 \right) dx \ge \delta \|u\|_1^2, \quad \forall u \in X_2 \tag{3.2}$$

by Lemma 2.3. For any $\epsilon > 0$, there exists $C_\epsilon > 0$ such that $|F(x,u)| \ge C_\epsilon |u|^\alpha - \epsilon |u|^2$ from $(\mathbb{A}_2)$ and $(\mathbb{A}_3)$. Choose $2 < \alpha' < \alpha$, then

$$\begin{aligned} &\int_{\mathbb{R}^N} F(x,u_n)dx - \frac{1}{\alpha'} \int_{\mathbb{R}^N} u_n f(x,u_n)dx \\ &\quad \le \int_{\mathbb{R}^N} \left(1 - \frac{\alpha}{\alpha'}\right) F(x,u_n)dx \\ &\quad \le \left(1 - \frac{\alpha}{\alpha'}\right) \int_{\mathbb{R}^N} \left(C_\epsilon |u_n|^\alpha - \epsilon |u_n|^2 \right) dx. \end{aligned} \tag{3.3}$$

Let $\{u_n\}$ be the sequence such that $I(u_n) \to c$, $I'(u_n) \to 0$. By inequality (3.2) and $u_n = v_n + w_n$, $v_n \in X_1$, $w_n \in X_2$, and then

$$\begin{aligned} c + 1 + \|u\|_1 &\ge I(u_n) - \frac{1}{\alpha'} \langle I'(u_n), u_n \rangle \\ &= \frac{1}{2} \int_{\mathbb{R}^N} \left(|\nabla u_n|^2 - V(x)u_n^2 \right) dx - \int_{\mathbb{R}^N} F(x,u_n)dx \\ &\quad - \frac{1}{\alpha'} \int_{\mathbb{R}^N} \left(|\nabla u_n|^2 - V(x)u_n^2 \right) dx + \frac{1}{\alpha'} \int_{\mathbb{R}^N} u_n f(x,u_n)dx \\ &= \left(\frac{1}{2} - \frac{1}{\alpha'} \right) \int_{\mathbb{R}^N} \left(|\nabla w_n|^2 - V(x)w_n^2 + |\nabla v_n|^2 - V(x)v_n^2 \right) dx \\ &\quad - \int_{\mathbb{R}^N} F(x,u_n)dx + \frac{1}{\alpha'} \int_{\mathbb{R}^N} u_n f(x,u_n)dx \\ &\ge \left(\frac{1}{2} - \frac{1}{\alpha'} \right) \delta \|w_n\|_1^2 + \left(\frac{1}{2} - \frac{1}{\alpha'} \right) \|v_n\|_1^2 - \left(\frac{1}{2} - \frac{1}{\alpha'} \right) \int_{\mathbb{R}^N} \left(V^+(x)|v_n|^2 \right) dx \\ &\quad + \left(\frac{\alpha}{\alpha'} - 1 \right) \int_{\mathbb{R}^N} \left(C_\epsilon |u_n|^\alpha - \epsilon |u_n|^2 \right) dx. \end{aligned} \tag{3.4}$$

Choose $\epsilon > 0$ small, then for suitable C_2, C_3, the above inequality becomes

$$c + 1 + \|u\|_1 \geq C_2\|u_n\|_1^2 + C_3|u_n|_\alpha^\alpha - \left(\frac{1}{2} - \frac{1}{\alpha'}\right)|V^+|_{N/2}|v_n|_{2^*}^2. \tag{3.5}$$

Due to $\alpha > 2$, it follows that $\{u_n\}$ is bounded. □

The following lemma is the same as [6, Lemma 3.2]. For the completeness, we prove it.

Lemma 3.2. *Under the assumptions* $(\mathbb{A}_1)$, $(\mathbb{A}_2)$, $(\mathbb{A}_3)$, *and* $(\mathbb{A}_4)$, *I satisfies the* (PS)*-condition in* X.

Proof. By Lemma 3.1, we know that any $(PS)_c$ sequence u_n is bounded in X. Up to a subsequence, we may assume that $u_n \rightharpoonup u$ in X. In order to establish strong convergence it suffices to show

$$\|u_n\|_1 \longrightarrow \|u\|_1. \tag{3.6}$$

Since $\langle I'(u_n), u_n - u\rangle \to 0$, we infer that

$$\begin{aligned} 0 &\leq \limsup_{n\to\infty}\left(\|u_n\|_1^2 - \|u\|_1^2\right) \\ &= \limsup_{n\to\infty}(u_n, u_n - u) \\ &= \limsup_{n\to\infty}\int_{\mathbb{R}^N} f(x,u_n)(u_n - u)dx. \end{aligned} \tag{3.7}$$

We restrict our attention to the case $N \geq 3$, but the cases $N = 1, 2$ can be treated similarly. Let $\epsilon > 0$, for $r \geq 1$, then

$$\begin{aligned} \int_{|u_n|\geq r} f(x,u_n)(u_n - u)dx &\leq C_4\int_{|u_n|\geq r}|u_n|^{p-1}|u_n - u|dx \\ &\leq C_4 r^{p-2^*}\int_{|u_n|\geq r}|u_n|^{2^*-1}|u_n - u|dx \\ &\leq C_4 r^{p-2^*}|u_n|_{2^*}^{2^*-1}|u_n - u|_{2^*}. \end{aligned} \tag{3.8}$$

Since $p < 2^*$, we may fix r large enough such that

$$\int_{|u_n|\geq r} f(x,u_n)(u_n - u)dx \leq \frac{\epsilon}{3} \tag{3.9}$$

for all n. Moreover, by $(\mathbb{A}_4)$ there exists $R_1 > 0$ such that

$$\int_{(|u_n|\leq r_{\cap|x|\geq R_1})} f(x,u_n)(u_n - u)dx \leq |u_n|_2|u_n - u|_2 \sup_{|t|\leq r,|x|\geq R_1}\frac{|f(x,t)|}{|t|} \leq \frac{\epsilon}{3} \tag{3.10}$$

for all n. Finally, since $u_n \to u$ in $L^s(B_{R_1}(0))$ for $s \in [2, 2^*)$, we can use $(\mathbb{A}_2)$ again to derive

$$\int_{(|u_n| \le r \cap |x| \le R_1)} f(x, u_n)(u_n - u)dx \le \frac{\epsilon}{3} \tag{3.11}$$

for n large enough. Combining (3.9)–(3.11) we conclude that

$$\int_{\mathbb{R}^N} f(x, u_n)(u_n - u)dx \le \epsilon \tag{3.12}$$

for n large enough. From this and (3.7), we deduce (3.6) and complete the proof. □

4. Infinitely Many Solutions

We can obtain an infinite sequence of critical values from Theorem 2.4.

Proof of Theorem 1.1. We apply Theorem 2.4 with $E = X$, $\varphi = I$. It is clear that $I \in C^1(X, \mathbb{R})$ is even because of $(\mathbb{A}_1)$, $(\mathbb{A}_2)$, and $(\mathbb{A}_5)$. $I(0) = 0$. By lemma 3.2, the (PS)-condition is satisfied. From the proof of Lemma 3.1, we have $X = X_1 \bigoplus X_2$, where $X_1 = \bigoplus_{i=1}^{k} \operatorname{span}\{\varphi_i\}$, $X_2 = X_1^{\perp}$. That is $E^- = X_1$, $E^+ = X_2$. We only need to check conditions (i) and (ii).

Integrating $(\mathbb{A}_2)$, there is a constant $C_5 > 0$ such that for all $x \in \mathbb{R}^N$ and $t \in \mathbb{R}$,

$$|F(x,t)| \le C_5\left(|t|^p + |t|^q\right). \tag{4.1}$$

By the Sobolev embeding theorem and (3.2), we have the estimate

$$\begin{aligned} I(u) &\ge \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla u|^2 + V^-(x)u^2\right)dx - \frac{1}{2}\int_{\mathbb{R}^N} V^+(x)u^2 dx - C_5\int_{\mathbb{R}^N} \left(|u|^p + |u|^q\right)dx \\ &\ge \frac{\delta}{2}\|u\|_1^2 - C_6\|u\|_1^p - C_7\|u\|_1^q \end{aligned} \tag{4.2}$$

for $u \in X_2$. Let $\|u\|_1 = \varrho$ and $u \in X_2$,

$$I(u) \ge \frac{\delta}{2}\varrho^2 - C_6\varrho^p - C_7\varrho^q > 0 \tag{4.3}$$

for small ϱ. Thus condition (i) is fulfilled with $\zeta = (\delta/2)\varrho^2 - C_6\varrho^p - C_7\varrho^q$.

By $(\mathbb{A}_3)$, there is a constant C_8 such that $|F(x,t)| \ge C_8|t|^\alpha$ for every $x \in \mathbb{R}^N$ and $|t| > \epsilon$. Indeed, let $\epsilon > 0$ small be given. By integration of $(\mathbb{A}_3)$, we have for $x \in \mathbb{R}^N$ and $|t| > \epsilon$,

$$F(x,t) \ge \frac{F(x,\epsilon)}{\epsilon^\alpha}|t|^\alpha \ge C_8|t|^\alpha. \tag{4.4}$$

Let W be a finite-dimensional subspace of X. Since all norms are equivalent of W and since

$$I(u) \leq \frac{1}{2}\|u\|_1^2 - \frac{1}{2}\int_{\mathbb{R}^N} V^+ u^2 dx - C_9\|u\|_\alpha^\alpha. \tag{4.5}$$

Also since $\alpha > 2$, condition (ii) follows. Thus we complete the proof. □

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