

## Research Article

# On Some Generalizations Bellman-Bihari Result for Integro-Functional Inequalities for Discontinuous Functions and Their Applications

Angela Gallo<sup>1</sup> and Anna Maria Piccirillo<sup>2</sup>

<sup>1</sup> Department of Mathematics and Applications, "R.Caccioppoli" University of Naples "Federico II", Claudio street 21, 80125 Naples, Italy

<sup>2</sup> Department of Civil Engineering, Second University of Naples, Roma, street 21, 81100 Caserta, Italy

Correspondence should be addressed to Angela Gallo, [angallo@unina.it](mailto:angallo@unina.it)

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We present some new nonlinear integral inequalities Bellman-Bihari type with delay for discontinuous functions (integro-sum inequalities; impulse integral inequalities). Some applications of the results are included: conditions of boundedness (uniformly), stability by Lyapunov (uniformly), practical stability by Chetaev (uniformly) for the solutions of impulsive differential and integro-differential systems of ordinary differential equations.

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## 1. Introduction

The first generalizations of the Bihari result for discontinuous functions which satisfy nonlinear impulse inequality (integro-sum inequality) are connected with such types of inequalities:

(a)

$$v(t) \leq c + \int_{t_0}^t p(\tau) v^m(\tau) d\tau + \sum_{t_0 < t_i < t} \beta_i v(t_i - 0), \quad m > 0, m \neq 1, \quad (1.1)$$

(b)

$$v(t) \leq c + \int_{t_0}^t p(\tau) \varphi(v(\tau)) d\tau + \sum_{t_0 < t_i < t} \beta_i v(t_i - 0), \quad (1.2)$$

Which are studied in the publications by Bainov, Borysenko, Iovane, Lakshmikantham, Leela, Martynyuk, Mitropolskiy, Samoilenko ([1–13]), and in many others. In these investigations the method of integral inequalities for continuous functions is generalized to the case of piecewise continuous (one-dimensional inequalities) and discontinuous (multidimensional inequalities) functions.

For the generalization of the integral inequalities method for discontinuous functions and for their applications to qualitative analysis of impulsive systems: existence, uniqueness, boundedness, comparison, stability, and so forth. We refer to the results [2–5, 12, 14] and for periodic boundary value problems we cite [15–17]. More recently, a novel variational approach appeared in [18]. This approach to impulsive differential equations also used the critical point theory for the existence of solutions of a nonlinear Dirichlet impulsive problem and in [19] some new comparison principles and the monotone iterative technique to establish a more general existence theorem for a periodic boundary value problem. Reference [20] is very interesting in that it gives a complete overview of the state-of-the-art of the impulsive differential, inclusions.

In this paper, in Section 2, we investigate new analogies Bihari results for piece-wise continuous functions and, in Section 3, the conditions of boundedness, stability, practical stability of the solutions of nonlinear impulsive differential and integro-differential systems.

## 2. General Bihari Theorems for Integro-Functional Inequalities for Discontinuous Functions

Let us consider the class  $\wp$  of continuous functions  $p : R \rightarrow R$ ,  $p(t) \leq t$ ,  $\lim_{|t| \rightarrow \infty} p(t) = \infty$  ( $p = p(t)$  is the delaying argument). The following holds.

**Theorem 2.1.** (a) Let one suppose that for  $x \geq x_0$  the following integro-sum functional inequality holds:

$$u(x) \leq \varphi(x) + q(x) \int_{x_i}^x f(\tau) W(u(p(\tau))) d\tau + \sum_{x_0 < x_i < x} \beta_i u^m(x_i - 0), \quad (2.1)$$

where  $q(x) \geq 1$ ,  $\varphi(x)$  is a positive nondecreasing function,  $\beta_i = \text{const} \geq 0$ ,  $f : R_+ \rightarrow R_+$ ,  $m = \text{const} > 0$ ; function  $u(x)$  is a nonnegative piecewise-continuous, with 1-st kind of discontinuities in the points  $x_i : x_0 < x_1 < \dots \lim_{n \rightarrow \infty} x_n = \infty$ ,  $p(t)$  belongs to the class  $\wp$ .

(b) Function  $W(x)$  satisfies such conditions:

- (i)  $W(\gamma\beta) \leq W(\gamma)W(\beta)$ ;
- (ii)  $W : R_+ \rightarrow R_+$ ,  $W(0) = 0$ ;
- (iii)  $W$  is nondecreasing.

Then for arbitrary  $x \in ]x_0, \infty[$  the next estimate holds:

$$u(x) \leq \varphi(x)q(x)G_i^{-1} \left[ \int_{x_i}^x \frac{f(\tau)}{\varphi(\tau)} W[\varphi(p(\tau))q(p(\tau))] d\tau \right] \quad \text{for } x \in ]x_i, x_{i+1}[ \quad (2.2)$$

$$\int_{x_i}^x \frac{f(\tau)}{\varphi(\tau)} W[\varphi(p(\tau))q(p(\tau))] d\tau \in \text{Dom}(G_i^{-1}),$$

$$G_0(u) = \int_1^u \frac{d\sigma}{W(\sigma)}, \quad (2.3)$$

$$G_i(u) = \int_{c_i}^u \frac{d\sigma}{W(\sigma)}, \quad i = 1, 2, \dots, \quad (2.4)$$

$$c_i = \left(1 + \beta_i \varphi^{m-1}(x_i) q^m(x_i - 0)\right) G_{i-1}^{-1} \left( \int_{x_{i-1}}^{x_i} \frac{f(\tau)}{\varphi(\tau)} W[\varphi(p(\tau))q(p(\tau))] d\tau \right),$$

$$i = 1, 2, \dots \text{ if } m \in ]0, 1], \quad \forall x \geq x_0, \quad (2.5)$$

$$c_i = \left(1 + \beta_i \varphi^{m-1}(x_i) q^m(x_i - 0)\right) \left[ G_{i-1}^{-1} \left( \int_{x_{i-1}}^{x_i} \frac{f(\tau)}{\varphi(\tau)} W[\varphi(p(\tau))q(p(\tau))] d\tau \right) \right]^m,$$

$$i = 1, 2, \dots \text{ if } m \geq 1, \quad \forall x \geq x_0.$$

*Proof.* It follows from inequality (2.1)

$$\frac{u(x)}{\varphi(x)} \leq 1 + q(x) \int_{x_0}^x \frac{f(\tau) W(u(p(\tau)))}{\varphi(\tau)} d\tau + \sum_{x_0 < x_i < x} \beta_i \frac{u^m(x_i - 0)}{\varphi(x)}$$

$$\leq q(x) \left\{ 1 + \int_{x_0}^x \frac{f(\tau)}{\varphi(\tau)} W(u(p(\tau))) d\tau + \sum_{x_0 < x_i < x} \beta_i \varphi^{m-1}(x_i - 0) \left[ \frac{u(x_i - 0)}{\varphi(x_i - 0)} \right]^m \right\}. \quad (2.6)$$

Denoting by

$$u^*(x) = 1 + \int_{x_0}^x \frac{f(\tau)}{\varphi(\tau)} W(u(p(\tau))) d\tau + \sum_{x_0 < x_i < x} \beta_i \varphi^{m-1}(x_i - 0) \left[ \frac{u(x_i - 0)}{\varphi(x_i - 0)} \right]^m, \quad (2.7)$$

$$u^*(x) = 1 \quad \text{for } x = x_0,$$

then

$$u(x_i - 0) \leq \varphi(x_i - 0) q(x_i - 0) u^*(x_i - 0),$$

$$u(x) \leq \varphi(x) q(x) u^*(x),$$

$$u(p(\tau)) \leq \varphi(p(\tau)) q(p(\tau)) u^*(p(\tau)) \leq \varphi(p(\tau)) q(p(\tau)) u^*(\tau), \quad (2.8)$$

$$\implies u^*(x) \leq 1 + \int_{x_0}^x \frac{f(\tau)}{\varphi(\tau)} W[\varphi(p(\tau))q(p(\tau))] W(u^*(\tau)) d\tau$$

$$+ \sum_{x_0 < x_i < x} \beta_i \varphi^{m-1}(x_i - 0) q^m(x_i - 0) u^{*m}(x_i - 0).$$

Let us consider the interval  $I_1 = [x_0, x_1[$ . Then

$$u^*(x) \leq G_0^{-1} \left( \int_{x_0}^x \frac{f(\tau)}{\varphi(\tau)} W[\varphi(p(\tau))q(p(\tau))] d\tau \right), \quad (2.9)$$

$$\text{if only } \int_{x_0}^x \frac{f(\tau)}{\varphi(\tau)} W[\varphi(p(\tau))q(p(\tau))] d\tau \in \text{Dom}(G_0^{-1}),$$

where  $G_0(\xi) = \int_1^\xi (d\tau/W(\tau))$ . So it results in

$$u(x) \leq \varphi(x)q(x)G_0^{-1} \left[ \int_{x_0}^x \frac{f(\tau)}{\varphi(\tau)} W[\varphi(p(\tau))q(p(\tau))] d\tau \right], \quad (2.10)$$

and estimate (2.2) is valid in  $I_1$ .

Let us suppose that for  $x \in I_k = [x_{k-1}, x_k [$ ,  $k = 2, 3, \dots$  estimate (2.2) is fulfilled. Then for every  $x \in I_{k+1}$  we have

$$u^*(x) \leq G_k^{-1} \left( \int_{x_k}^x \frac{f(\tau)}{\varphi(\tau)} W[\varphi(p(\tau))q(p(\tau))] d\tau \right) \quad (2.11)$$

$$\text{with } \int_{x_k}^x \frac{f(\tau)}{\varphi(\tau)} W[\varphi(p(\tau))q(p(\tau))] d\tau \in \text{Dom}(G_k^{-1}),$$

where  $G_k(\xi)$  is determined from (2.3)–(2.5).

Taking into account such inequality

$$u(x) \leq \varphi(x)q(x)u^*(x), \quad (2.12)$$

we obtain estimate (2.2) for every  $x \in [x_0, \infty[$ .

Let us consider the class  $\mathfrak{J}$  of functions  $f$  such that

- (i)  $f(x)$ -positive, continuous, nondecreasing for  $x > 0$ ;
- (ii)  $\forall u \geq 1, v > 0 \Rightarrow u^{-1} f(v) < f(u^{-1} v)$ ;
- (iii)  $f(0) = 0$ .

The following result is proved. □

**Theorem 2.2.** *Suppose that the part (a) of Theorem 2.1 is valid and function  $W : [0, \infty[ \rightarrow [0, \infty[$  belongs to the class  $\mathfrak{J}$ . Then for arbitrary  $x_0 \leq x \leq x^*$  such estimate holds:*

$$u(x) \leq \varphi(x)q(x)G_i^{*-1} \left[ \int_{x_i}^x f(\tau)q(p(\tau))d\tau \right] \quad \text{for } I_i = [x_i, x_{i+1} [, \quad i = 0, 1, \dots, \quad (2.13)$$

where

$$\begin{aligned}
 G_0^*(\eta) &= \int_1^\eta \frac{d\sigma}{W(\sigma)}, & G_i^*(\eta) &= \int_{c_i^*}^\eta \frac{d\sigma}{W(\sigma)} \quad i = 1, 2, \dots, \\
 c_i^* &= \left(1 + \beta_i \varphi^{m-1}(x_i) q^m(x_i)\right) G_{i-1}^{*-1} \left( \int_{x_{i-1}}^{x_i} f(\tau) q(p(\tau)) d\tau \right) \quad \text{if } m \in ]0, 1], \\
 c_i^* &= \left(1 + \beta_i \varphi^{m-1}(x_i) q^m(x_i)\right) \left[ G_{i-1}^{*-1} \left( \int_{x_{i-1}}^{x_i} f(\tau) q(p(\tau)) d\tau \right) \right]^m \quad \text{if } m \geq 1,
 \end{aligned} \tag{2.14}$$

and  $x^* = \sup_x \{ \int_{x_{i-1}}^x f(\tau) q(p(\tau)) d\tau \in \text{Dom}(G_{i-1}^{*-1}) \}$ ,  $i = 1, 2, \dots$ .

*Proof.* By using the previous theorem we have  $u(x) \leq \varphi(x)g(x)u^*(x)$ ,  $u^*(x) = 1$   $x = x_0$ . On the interval  $I_1$

$$\frac{du^*(x)}{dx} = \frac{f(x)}{\varphi(x)} W(u(p(x))). \tag{2.15}$$

Then

$$\begin{aligned}
 u(p(x)) &\leq \varphi(p(x))q(p(x))u^*(p(x)) \leq \varphi(x)q(p(x))u^*(x), \\
 \frac{du^*(x)}{dx} &\leq \frac{f(x)}{\varphi(x)} W(q(p(x))\varphi(x)u^*(x)) \\
 &\leq \frac{f(x)q(p(x))}{\varphi(x)q(p(x))} W(q(p(x))\varphi(x)u^*(x)) \\
 &\leq f(x)q(p(x))W(u^*(x)).
 \end{aligned} \tag{2.16}$$

Taking into account estimate (2.16), we obtain

$$\begin{aligned}
 \int_{x_0}^x \frac{u^{*\prime}(\sigma)}{W(u^*(\sigma))} d\sigma &\leq \int_{x_0}^x f(\tau) q(p(\tau)) d\tau, \\
 \int_{x_0}^x \frac{u^{*\prime}(\sigma)}{W(u^*(\sigma))} d\sigma &= \int_{u^*(x_0)}^{u^*(x)} \frac{du}{W(u)} = G_0^*(u^*(x)) - G_0^*(u^*(x_0)), \\
 u^*(x_0) &= 1, \quad u^*(x) \geq 1, \quad G_0^*(u^*(x_0)) = G_0^*(1) = 0, \\
 G_0^*(u^*(x)) &\leq \int_{x_0}^x f(\tau) q(p(\tau)) d\tau.
 \end{aligned} \tag{2.17}$$

Then in  $I_1$  we have

$$u(x) \leq \varphi(x)q(x) G_0^{*-1} \left[ \int_{x_0}^x f(\tau)q(p(\tau))d\tau \right] \quad \text{if only } \int_{x_0}^x f(\tau)q(p(\tau))d\tau \in \text{Dom}(G_0^{*-1}). \quad (2.18)$$

As in the previously theorem, the proof is completed by using the inductive method.  $\square$

The following result is easily to obtain

**Theorem 2.3.** *Suppose that for  $x \geq x_0$  the next inequality holds:*

$$u(x) \leq u_0 + q(x) \left[ \int_{x_0}^x f(s)u(p(s))ds + \int_{x_0}^x f(s) \left( \int_{x_0}^x g(\tau)u(p(\tau))d\tau \right) ds \right] \\ + \int_{x_0}^x h(s)W(u(\sigma(s)))ds + \sum_{x_0 < x_i < x} \beta_i u^m(x_i - 0), \quad (2.19)$$

where functions  $u(x), f(x), q(x), g(x), h(x), p(x), \sigma(x)$  are real nonnegative for  $x \geq x_0 > 0, p(x), \sigma(x) \in \mathfrak{J}, q(x) \geq 1, \beta_i \geq 0$ , function  $W$  satisfies conditions (i), ..., (iii) of Theorem 2.1.

Then for  $x \geq x_0$  it results in

$$u(x) \leq \prod_{x_0 < x_i < x} \left( 1 + \beta_i q^m(x_i) u_0^{m-1} \right) \exp \left( \int_{x_0}^x q(p(\tau)) [f(\tau) + g(\tau)] d\tau \right) \\ \cdot \varphi_0^{-1} \left( \int_{x_0}^x h(\tau) W \left[ \prod_{x_0 < x_i < \sigma(\tau)} \left( 1 + \beta_i q^m(x_i) u_0^{m-1} \right) \right] W \right. \\ \left. \times \left[ q(\sigma(\tau)) \exp \left( \int_{x_0}^{\sigma(\tau)} q(p(s)) [f(s) + g(s)] ds \right) \right] d\tau \right), \quad \text{if } m \in ]0, 1[ \\ \int_{x_0}^x h(\tau) W \left[ \prod_{x_0 < x_i < \sigma(\tau)} \left( 1 + \beta_i q^m(x_i) u_0^{m-1} \right) \right] W \\ \times \left[ q(\sigma(\tau)) \exp \left( \int_{x_0}^{\sigma(\tau)} q(p(s)) [f(s) + g(s)] ds \right) \right] d\tau \in \text{Dom}(\varphi_0^{-1}), \quad (2.20)$$

where  $\psi_0(u) = \int_{u_0}^u (dv/W(v))$ ;

$$\begin{aligned}
 u(x) &\leq \prod_{x_0 < x_i < x} \left(1 + \beta_i q^m(x_i) u_0^{m-1}\right) \exp\left(m \int_{x_0}^x q(p(\tau)) [f(\tau) + g(\tau)]\right) \\
 &\quad \cdot \psi_0^{-1}\left(\int_{x_0}^x h(\tau) \left[\prod_{x_0 < x_i < \sigma(\tau)} \left(1 + \beta_i q^m(x_i) u_0^{m-1}\right)\right]\right. \\
 &\quad \left. \cdot W\left[q(\sigma(\tau)) \exp\left(m \int_{x_0}^{\sigma(\tau)} q(p(s)) [f(s) + g(s)] ds\right)\right] d\tau\right), \quad \text{if } m \geq 1, \quad (2.21) \\
 &\int_{x_0}^x h(\tau) W\left[\prod_{x_0 < x_i < \sigma(\tau)} \left(1 + \beta_i q^m(x_i) u_0^{m-1}\right)\right] W \\
 &\quad \times \left[q(\sigma(\tau)) \exp\left(m \int_{x_0}^{\sigma(\tau)} q(p(s)) [f(s) + g(s)] ds\right)\right] d\tau \in \text{Dom}(\psi_0^{-1}).
 \end{aligned}$$

The proof the same procedure as that of (Iovane [21, Theorems 2.1 and 3.1]).

**Corollary 2.4.** *Suppose that*

- (a)  $m = 1$ , then the result of Theorem 2.1 coincides with the result [22, Theorem 3.7.1, page 232];
- (b)  $m = 1$ ,  $\varphi(x) = c$ ,  $q(x) = 1$ ,  $p(t) = t$ , then the result of Theorem 2.1 coincides with result [12, Proposition 2.3, page 2143];
- (c)  $q(x) = 1$ ,  $W(u) = u$ ,  $p(t) = t$ , then one obtains the analogy of Gronwall- Bellman result for discontinuous functions [23, Lemma 1] and estimate (2.2) reduces in the following form:

$$u(x) \leq \varphi(x) \prod_{x_0 < x_i < x} \left(1 + \beta_i \varphi^{m-1}(x_i)\right) \exp\left(\int_{x_0}^x f(\tau) d\tau\right) \quad \text{if } m \in ]0, 1], \forall x \geq x_0, \quad (2.22)$$

$$u(x) \leq \varphi(x) \prod_{x_0 < x_i < x} \left(1 + \beta_i \varphi^{m-1}(x_i)\right) \exp\left(m \int_{x_0}^x f(\tau) d\tau\right) \quad \text{if } m \geq 1, \forall x \geq x_0.$$

- (d)  $q(x) = 1$ ,  $W(u) = u$ , then one obtains the result [21, Theorem 2.1] and estimate (2.2) are as follows:

$$u(x) \leq \varphi(x) \prod_{x_0 < x_i < x} \left(1 + \beta_i \varphi^{m-1}(x_i)\right) \exp\left(\int_{x_0}^x f(\tau) \frac{\varphi(p(\tau))}{\varphi(\tau)} d\tau\right), \quad \text{if } m \in ]0, 1], \forall x \geq x_0;$$

$$u(x) \leq \varphi(x) \prod_{x_0 < x_i < x} \left(1 + \beta_i \varphi^{m-1}(x_i)\right) \exp\left(m \int_{x_0}^x f(\tau) \frac{\varphi(p(\tau))}{\varphi(\tau)} d\tau\right) \quad \text{if } m \geq 1, \forall x \geq x_0. \quad (2.23)$$

(e)  $q(x) = 1, W(u) = u^m, m > 0, p(t) = t$ , then one obtains the analogy of Bihari result for discontinuous functions [23, Lemma 2] and estimate (2.2) reduces as follows are reduced:

$$u(x) \leq \varphi(x) \prod_{x_0 < x_i < x} \left(1 + \beta_i \varphi^{m-1}(x_i)\right) \left[1 + (1-m) \int_{x_0}^x \varphi^{m-1}(\tau) f(\tau) d\tau\right]^{1/(1-m)},$$

if  $0 < m < 1, \forall x \geq x_0$ ,

$$u(x) \leq \varphi(x) \prod_{x_0 < x_i < x} \left(1 + \beta_i m \varphi^{m-1}(x_i)\right) \left[1 - (m-1) \left[ \prod_{x_0 < x_i < x} \left(1 + \beta_i m \varphi^{m-1}(x_i)\right) \right]^{m-1} \times \int_{x_0}^x \varphi^{m-1}(\tau) f(\tau) d\tau\right]^{-1/(m-1)} \quad \forall x \geq x_0,$$

(2.24)

such that

$$\int_{x_0}^x \varphi^{m-1}(\tau) f(\tau) d\tau \leq \frac{1}{m}, \quad m > 1, \quad \prod_{x_0 < x_i < x} \left(1 + \beta_i \varphi^{m-1}(x_i)\right) < \left(1 + \frac{1}{m-1}\right)^{1/(m-1)}.$$

(2.25)

(f)  $W(u) = u^m, m > 0$ , then estimate (2.2) reduces as follows (see [21, Theorem 2.2]):

$$u(x) \leq \varphi(x) q(x) \prod_{x_0 < x_i < x} \left(1 + \beta_i \varphi^{m-1}(x_i) q^m(x_i)\right)$$

$$\times \left[1 + (1-m) \int_{x_0}^x \varphi^{m-1}(\tau) f(\tau) q^m(p(\tau)) \left[\frac{\varphi(p(\tau))}{\varphi(\tau)}\right]^m d\tau\right]^{1/(1-m)}$$

if  $0 < m < 1, \forall x \geq x_0$ ,

$$u(x) \leq \varphi(x) q(x) \prod_{x_0 < x_i < x} \left(1 + \beta_i m \varphi^{m-1}(x_i) q^m(x_i)\right)$$

$$\times \left\{1 - (m-1) \left[ \prod_{x_0 < x_i < x} \left(1 + \beta_i m \varphi^{m-1}(x_i) q^m(x_i)\right) \right]^{m-1} \times \int_{x_0}^x \varphi^{m-1}(\tau) f(\tau) q^m(p(\tau)) \left[\frac{\varphi(p(\tau))}{\varphi(\tau)}\right]^m d\tau\right\}^{-1/(m-1)} \quad \forall x \geq x_0$$



such that

$$\int_{x_0}^x \varphi^{m-1}(\tau) f(\tau) q^m(p(\tau)) \left[ \frac{\varphi(p(\tau))}{\varphi(\tau)} \right]^m d\tau \leq \frac{1}{m}, \quad m > 1,$$

$$\prod_{x_0 < x_i < x} \left( 1 + \beta_i m \varphi^{m-1}(x_i) q^m(x_i) \right) < \left( 1 + \frac{1}{m-1} \right)^{-1/(m-1)}.$$
(2.26)

(g) Suppose that in Theorem 2.3  $q(x) = 1, W(u) = u, \sigma(s) = p(s) = s$ , then estimates (2.20), (2.21) reduce as shown:

$$u(x) \leq u_0 \prod_{x_0 < x_i < x} \left( 1 + \beta_i u_0^{m-1}(x_i) \right) \exp \left[ \int_{x_0}^x [f(\xi) + g(\xi) + h(\xi)] d\xi \right] \quad \text{if } m \in ]0, 1], \forall x \geq x_0;$$

$$u(x) \leq u_0 \prod_{x_0 < x_i < x} \left( 1 + \beta_i u_0^{m-1}(x_i) \right) \exp \left[ m \int_{x_0}^x [f(\xi) + g(\xi) + h(\xi)] d\xi \right] \quad \text{if } m \geq 1, \forall x \geq x_0,$$
(2.27)

which coincide with result of [21, Theorem 3.1] for  $h(t) = u_0$ .

### 3. Applications

Let us consider the following system of differential equations

$$\frac{dx}{dt} = F(t, x), \quad t \neq t_i,$$

$$\Delta x|_{t=t_i} = I_i(x)$$
(3.1)

where  $x \in \mathfrak{R}^n, F \in \mathfrak{R}^n, I_i(x) \in \mathfrak{R}^n$  ( $i = 1, 2, \dots$ ),  $t \geq t_0 \geq 0, \lim_{i \rightarrow \infty} t_i = \infty, t_{i-1} < t_i$  for all  $i = 1, 2, \dots$

Let us assume that  $F(t, x)$  and  $I_i(x)$  are defined in the domain  $D = \{(t, x) : t \in \mathcal{J} = [t_0, T], T \leq \infty, \|x\| \leq h\}$  and satisfy such conditions:

(a)  $\|F(t, x)\| \leq f(t)W(\|x\|), f : \mathfrak{R}_+ \rightarrow \mathfrak{R}_+$ ,

$W$  satisfies conditions (i)–(iii) of Theorem 2.1;

(b)  $\|I_i(x)\| \leq \beta_i \|x\|^m, \beta_i = \text{const} > 0, m > 0$ .

Consider  $x(t) = x(t, t_0, x_0)$  the solution of Cauchy problem for system (3.1). Then

$$x(t, t_0, x_0) = x_0 + \int_{t_0}^t F(\tau, x(\tau, t_0, x_0)) d\tau + \sum_{t_0 < t_i < t} I_i(x(t_i - 0, t_0, x_0)),$$
(3.2)

from which it follows

$$\|x(t, t_0, x_0)\| \leq \|x_0\| + \int_{t_0}^t f(\tau) W(\|x(\tau, t_0, x_0)\|) d\tau + \sum_{t_0 < t_i < t} \beta_i \|x(t_i - 0, t_0, x_0)\|^m. \quad (3.3)$$

By using the result of Theorem 2.1 and estimate (2.2) we obtain

$$\|x(t, t_0, x_0)\| \leq \|x_0\| G_i^{-1} \left[ \int_{x_i}^x f(\tau) \frac{W(\|x_0\|)}{\|x_0\|} d\tau \right] \quad \text{for } x \in ]x_i, x_{i+1}[ , \quad (3.4)$$

$$\int_{x_i}^x f(\tau) \frac{W(\|x_0\|)}{\|x_0\|} d\tau \in \text{Dom}(G_i^{-1}),$$

where

$$G_0(u) = \int_1^u \frac{d\sigma}{W(\sigma)}, \quad G_i(u) = \int_{c_i}^u \frac{d\sigma}{W(\sigma)}, \quad i = 1, 2, \dots,$$

$$c_i = (1 + \beta_i \|x_0\|^{m-1}) G_{i-1}^{-1} \left( \int_{x_{i-1}}^{x_i} f(\tau) \frac{W(\|x_0\|)}{\|x_0\|} d\tau \right),$$

$$i = 1, 2, \dots \quad \text{if } m \in ]0, 1], \quad \forall x \geq x_0, \quad (3.5)$$

$$c_i = (1 + \beta_i \|x_0\|^{m-1}) \left[ G_{i-1}^{-1} \left( \int_{x_{i-1}}^{x_i} f(\tau) \frac{W(\|x_0\|)}{\|x_0\|} d\tau \right) \right]^m,$$

$$i = 1, 2, \dots \quad \text{if } m \geq 1, \quad \forall x \geq x_0.$$

Let us consider some particular cases of  $W$ .

If  $W(u) = u$ ,  $m = 1$ , estimate (3.4) is reduced in such form

$$\|x(t, t_0, x_0)\| \leq \|x_0\| \prod_{t_0 < t_i < t} (1 + \beta_i) \exp \left[ \int_{t_0}^t f(\tau) d\tau \right]. \quad (3.6)$$

Then such result holds.

**Proposition 3.1.** *Let the following conditions be fulfilled for system (3.1) :*

- (i)  $\|F(t, x)\| \leq f(t)\|x\|$ ;
- (ii)  $\|I_i(x)\| \leq \beta_i \|x\|$ ;
- (iii)  $\exists m_1(t_0) = \text{const.} > 0 : \prod_{t_0 < t_i < t} (1 + \beta_i) \leq m_1(t_0) < \infty$ ;
- (iv)  $\exists m_2(t_0) = \text{const.} > 0 : \int_{t_0}^t f(\tau) d\tau \leq m_2(t_0) < \infty, \forall t \geq t_0$ .

Then one has:

- (a) All solutions of system (3.1) are bounded (uniformly, if  $m_i(t_0)$  are independent of  $t_0$ ) and such estimate is valid:

$$\|x(t, t_0, x_0)\| \leq m_1(t_0) \exp[m_2(t_0)] \|x_0\|. \quad (3.7)$$

- (b) The trivial solution of system (3.1) is stable by Lyapunov (uniformly stable relative  $t_0$ , if  $m_i(t_0) = m_i$ ,  $i = 1, 2$ ).

*Remark 3.2.* If conditions I–IV of Proposition 3.1 are valid and  $\lambda/\Lambda < (m_1(t_0) \exp[m_2(t_0)])^{-1}$ , then the trivial solution is  $(\lambda, \Lambda, \mathcal{J})$ -stable by Chetaev (uniformly  $(\lambda, \Lambda, \mathcal{J})$ -stable, if  $m_i(t_0)$ ,  $i = 1, 2$  is independent of  $t_0$ ).

If  $W(u) = u^l$ ,  $l \neq 1$ ,  $m = 1$  the estimate (3.4) is reduced in such form

$$\|x(t, t_0, x_0)\| \leq \prod_{t_0 < t_i < t} (1 + \beta_i) \left[ \|x_0\|^{1-l} + (1-l) \int_{t_0}^t f(\tau) d\tau \right]^{1/(1-l)} \quad \forall t \geq t_0, \text{ if } 0 < l < 1, \quad (3.8)$$

$$\begin{aligned} \|x(t, t_0, x_0)\| &\leq \|x_0\| \prod_{t_0 < t_i < t} (1 + \beta_i) \\ &\times \left[ 1 - (l-1) \|x_0\|^{l-1} \cdot \left[ \prod_{t_0 < t_i < t} (1 + \beta_i) \right]^{l-1} \int_{t_0}^t f(\tau) d\tau \right]^{-1/(l-1)} \quad \forall t \geq t_0, \end{aligned} \quad (3.9)$$

$$\int_{t_0}^t f(\tau) d\tau < \left( (l-1) \left[ \|x_0\| \prod_{t_0 < t_i < t} (1 + \beta_i) \right]^{l-1} \right)^{-1}, \quad \text{if } l > 1. \quad (3.10)$$

From estimate (3.8) the next propositions follow.

**Proposition 3.3.** Suppose that such conditions occur:

- (a)  $\|F(t, x) - F(t, y)\| \leq f(t) \|x - y\|^l$ ,  $0 < l < 1$  for all  $x, y \in D$   
 (b) estimates ii–iv of Proposition 3.1 be fulfilled.

Then all the solutions of system (3.1) are bounded (uniformly if  $m_i(t_0) = m_i$ ,  $i = 1, 2$ ).

*Remark 3.4.* Suppose that conditions (a), (b) of Proposition 3.3 are valid and

$$\lambda^{1-l} + (1-l) m_2(t_0) < \left[ \frac{\Lambda}{m_1(t_0)} \right]^{1-l}. \quad (3.11)$$

Then trivial solution of system (3.1) is  $(\lambda, \Lambda, \mathcal{T})$ -stable by Chetaev (uniformly if  $m_i(t_0)$  is independent of  $t_0$ ).

**Proposition 3.5.** *Let conditions ii–iv of Proposition 3.1 be fulfilled for system (3.1), inequality (3.10) holds and*

$$\|F(t, x)\| \leq f(t)\|x\|^l, \quad l > 1. \quad (3.12)$$

Then trivial solution of system (3.1) is stable by Lyapunov (uniformly if  $m_i(t_0) = m_i$ ,  $i = 1, 2$ ).

*Remark 3.6.* If  $W(u) = u^l$ ,  $l > 0$ , and  $m \neq 1$  the conditions of boundedness, stability,  $(\lambda, \Lambda, \mathcal{T})$ -stability is investigated in [14, see Theorems 3.4–3.6]; the estimates of the solutions of system (3.1) with non-Lipschitz type of discontinuities are investigated in [23, see Proposition 1, Proposition 2].

Let us consider the following impulsive system of integro-differential equations:

$$\begin{aligned} \frac{dx}{dt} &= F(t, x, K[x(t)]), \quad t \neq t_i, \\ \Delta x|_{t=t_i} &= I_i(x), \end{aligned} \quad (3.13)$$

where  $x \in \mathfrak{R}^n$ ,  $F \in \mathfrak{R}^n$ ,  $I_i(x) \in \mathfrak{R}^n$  ( $i = 1, 2, \dots$ ) and defined in the domain  $D$ ,  $K[x(t)] = \int_{t_0}^t k(t, \tau, x(\tau)) d\tau$ .

We suppose that such conditions are valid:

- (i)  $\|F(t, x, y)\| \leq f(t)(\|x\| + \|y\|)$  for all  $x, y \in D$ ,  $f : \mathfrak{R}_+ \rightarrow \mathfrak{R}_+$ ;
- (ii)  $\|k(t, s, x)\| \leq g(t)\|x\|$  for all  $s \in [t_0, t]$ ,  $g : \mathfrak{R}_+ \rightarrow \mathfrak{R}_+$ ;
- (iii)  $\|I_i(x)\| \leq \beta_i\|x\|^m$  for all  $x, y \in D$ ,  $\beta_i = \text{const} > 0$ ,  $m > 0$ ,  $m \neq 1$ .

It is easy to see that

$$\begin{aligned} \|x(t, t_0, x_0)\| &\leq \|x_0\| + \int_{t_0}^t f(\tau)\|x(\tau, t_0, x_0)\|d\tau \\ &\quad + \int_{t_0}^t f(\tau) \left( \int_{t_0}^{\tau} g(\xi)\|x(\xi, t_0, x_0)\|d\xi \right) d\tau + \sum_{t_0 < t_i < t} \beta_i \|x(t_i - 0, t_0, x_0)\|^m \\ \implies \|x(t, t_0, x_0)\| &\leq \|x_0\| \prod_{t_0 < t_i < t} (1 + \beta_i \|x_0\|^{m-1}) \exp \int_{t_0}^t [f(\xi) + g(\xi)] d\xi, \\ &\quad \text{if } 0 < m \leq 1, t \geq t_0 \\ \|x(t)\| &\leq \|x_0\| \prod_{t_0 < t_i < t} (1 + \beta_i \|x_0\|^{m-1}) \exp \left( m \int_{t_0}^t [f(\xi) + g(\xi)] d\xi \right), \\ &\quad \text{if } 0 < m \geq 1, t \geq t_0. \end{aligned} \quad (3.14)$$

From estimate (3.15) such result follows.

**Proposition 3.7.** *Let one suppose that for system (3.13) conditions (i)–(iii) take place for  $m > 1$  and the following estimates are fulfilled:*

- (a)  $\exists m_3(t_0) = \text{const.} > 0 : \prod_{t_0 < t_i < t} (1 + \beta_i \|x_0\|^{m-1}) \leq m_3(t_0) < \infty;$   
 (b)  $\exists m_4(t_0) = \text{const.} > 0 : \int_{t_0}^t [f(\xi) + g(\xi)] d\xi \leq m_4(t_0) < \infty$  for all  $t \geq t_0$ .

Then we have:

- (i) *All solutions of system (3.13) are bounded and satisfy the estimate:*

$$\|x(t)\| \leq m_3(t_0) \exp[m_4(t_0)] \|x_0\|. \quad (3.16)$$

- (ii) *The trivial solution of system (3.13) is stable by Lyapunov (uniformly, if  $m_i(t_0) = m_i$ ,  $i = 3, 4$ ).*  
 (iii) *The trivial solution of system (3.13) is  $(\lambda, \Lambda, \mathfrak{J})$ -stable by Chetaev (uniformly if  $m_i(t_0)$  is independent of  $t_0$ ) and  $m_3(t_0) \exp[m_4(t_0)] < \Lambda/\lambda$ .*

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