Hindawi Publishing Corporation
Boundary Value Problems
Volume 2009, Article ID 834158, 12 pages
doi:10.1155/2009/834158

Research Article

Existence of Positive Solutions for Multipoint Boundary Value Problem on the Half-Line with Impulses

Jianli Li[1] and Juan J. Nieto[2]

[1] *Department of Mathematics, Hunan Normal University, Changsha, Hunan 410081, China*
[2] *Departamento de Análisis Matemático, Facultad de Matemáticas, Universidad de Santiago de Compostela, 15782 Santiago de Compostela, Spain*

Correspondence should be addressed to Jianli Li, ljianli@sina.com

Received 7 March 2009; Revised 18 April 2009; Accepted 25 April 2009

Recommended by Donal O'Regan

We consider a multi-point boundary value problem on the half-line with impulses. By using a fixed-point theorem due to Avery and Peterson, the existence of at least three positive solutions is obtained.

1. Introduction

Impulsive differential equations are a basic tool to study evolution processes that are subjected to abrupt changes in their state. For instance, many biological, physical, and engineering applications exhibit impulsive effects (see [1–3]). It should be noted that recent progress in the development of the qualitative theory of impulsive differential equations has been stimulated primarily by a number of interesting applied problems [4–24].

In this paper, we consider the existence of multiple positive solutions of the following impulsive boundary value problem (for short BVP) on a half-line:

$$\begin{gathered} u''(t) + q(t)f(t,u) = 0, \quad 0 < t < \infty,\ t \neq t_k, \\ \Delta u(t_k) = I_k(u(t_k)), \quad k = 1, \ldots, p, \\ u(0) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i), \quad u'(\infty) = 0, \end{gathered} \tag{1.1}$$

where $u'(\infty) = \lim_{t\to+\infty} u'(t), 0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < +\infty$, $0 < t_1 < t_2 < \cdots < t_p < +\infty$, $\Delta u(t_k) = u(t_k^+) - u(t_k^-)$, and α_i, f, q, and I_k satisfy

(H_1) $0 < \sum_{i=1}^{m-2} \alpha_i < 1$;

(H_2) $f(t,u) \in C([0,\infty)\times[0,+\infty),[0,+\infty))$, $I_k(u) \in C([0,+\infty),[0,+\infty))$, and when $u/(1+t)$ is bounded, $f(t,u)$ and $I_k(u)$ are bounded on $[0,+\infty)$;

(H_3) $q(t) \in C([0,\infty),[0,+\infty))$ and $q(t)$ is not identically zero on any compact subinterval of $(0,+\infty)$. Furthermore $q(t)$ satisfies

$$\sup_{t\in[0,+\infty)} \int_0^{+\infty} G(t,s)q(s)ds < +\infty, \tag{1.2}$$

where

$$G(t,s) = \begin{cases} s, & 0 \le s \le t < +\infty, \\ t, & 0 \le t \le s < +\infty. \end{cases} \tag{1.3}$$

Boundary value problems on the half-line arise quite naturally in the study of radially symmetric solutions of nonlinear elliptic equations and there are many results in this area, see [8, 13, 14, 20, 25–27], for example.

Lian et al. [25] studied the following boundary value problem of second-order differential equation with a p-Laplacian operator on a half-line:

$$\begin{gathered} \left(\varphi_p(u'(t))\right)' + \phi(t)f\left(t,u,u'\right) = 0, \quad 0 < t < +\infty, \\ \alpha u(0) - \beta u'(0) = 0, \qquad u'(\infty) = 0. \end{gathered} \tag{1.4}$$

They showed the existence at least three positive solutions for (1.4) by using a fixed point theorem in a cone due to Avery-Peterson [28].

Yan [20], by using Leray-Schauder theorem and fixed point index theory presents some results on the existence for the boundary value problems on the half-line with impulses and infinite delay.

However to the best knowledge of the authors, there is no paper concerned with the existence of three positive solutions to multipoint boundary value problems of impulsive differential equation on infinite interval so far. Motivated by [20, 25], in this paper, we aim to investigate the existence of triple positive solutions for BVP (1.1). The method chosen in this paper is a fixed point technique due to Avery and Peterson [28].

2. Preliminaries

In this section, we give some definitions and results that we will use in the rest of the paper.

Definition 2.1. Suppose P is a cone in a Banach. The map α is a nonnegative continuous concave functional on P provided $\alpha : P \to [0,\infty)$ is continuous and

$$\alpha\left(tx + (1-t)y\right) \ge t\alpha(x) + (1-t)\alpha\left(y\right) \tag{2.1}$$

for all $x, y \in P$, and $t \in [0,1]$. Similarly, the map β is a nonnegative continuous convex functional on P provided $\beta : P \to [0,\infty)$ is continuous and

$$\beta\big(tx + (1-t)y\big) \le t\beta(x) + (1-t)\beta(y) \tag{2.2}$$

for all $x, y \in P$, and $t \in [0,1]$.

Let γ, θ be nonnegative, continuous, convex functionals on P and α be a nonnegative, continuous, concave functionals on P, and ψ be a nonnegative continuous functionals on P. Then, for positive real numbers a, b, c, and d, we define the convex sets

$$\begin{gathered} P(\gamma,d) = \{x \in P : \gamma(x) < d\}, \\ P(\gamma,\alpha,b,d) = \{x \in P : b \le \alpha(x),\ \gamma(x) \le d\}, \\ P(\gamma,\theta,\alpha,b,c,d) = \{x \in P : b \le \alpha(x),\ \theta(x) \le c,\ \gamma(x) \le d\}, \end{gathered} \tag{2.3}$$

and the closed set

$$R(\gamma,\psi,a,d) = \{x \in P : a \le \psi(x),\ \gamma(x) \le d\}. \tag{2.4}$$

To prove our main results, we need the following fixed point theorem due to Avery and Peterson in [28].

Theorem 2.2. *Let P be a cone in a real Banach space E. Let γ and θ be nonnegative continuous convex functionals on a cone P, α be a nonnegative continuous concave functional on P, and ψ be a nonnegative continuous functional on P satisfying $\psi(\lambda x) \le \lambda\psi(x)$ for $0 \le \lambda \le 1$, such that for some positive numbers M and d*

$$\alpha(x) \le \psi(x), \qquad \|x\| \le M\gamma(x) \tag{2.5}$$

for all $x \in \overline{P(\gamma,d)}$. Suppose

$$\Phi : \overline{P(\gamma,d)} \longrightarrow \overline{P(\gamma,d)} \tag{2.6}$$

is completely continuous and there exist positive numbers a, d, and c with $a < b$ such that

(i) *$\{x \in P(\gamma,\theta,\alpha,b,c,d) : \alpha(x) > b\} \ne \emptyset$ and $\alpha(\Phi x) > b$ for $x \in P(\gamma,\theta,\alpha,b,c,d)$;*

(ii) *$\alpha(\Phi x) > b$ for $x \in P(\gamma,\alpha,b,d)$ with $\theta(\Phi x) > c$;*

(iii) *$0 \notin R(\gamma,\psi,a,d)$ and $\psi(Tx) < a$ for $x \in R(\gamma,\psi,a,d)$, with $\psi(\Phi x) = a$.*

Then Φ has at least three fixed points $x_1, x_2, x_3 \in \overline{P(\gamma,d)}$ such that

$$\begin{gathered} \gamma(x_i) \le d, \quad \text{for } i = 1,2,3, \quad \psi(x_1) < a, \\ a < \psi(x_2) \quad \text{with } \alpha(x_2) < b, \quad \alpha(x_3) > b. \end{gathered} \tag{2.7}$$

3. Some Lemmas

Define $PC[0,+\infty) = \{u : [0,+\infty) \to R \mid u(t)$ is continuous at each $t \neq t_k$, left continuous at $t = t_k$, $u(t_k^+)$ exists, $k = 1,\ldots,p\}$.

By a solution of (1.1) we mean a function u in $PC[0,\infty)$ satisfying the relations in (1.1).

Lemma 3.1. *$u(t)$ is a solution of* (1.1) *if and only if $u(t)$ is a solution of the following equation:*

$$\begin{aligned} u(t) &= \int_0^{+\infty} G(t,s)q(s)f(s,u(s))ds + \sum_{0<t_k<t} I_k(u) \\ &\quad + \frac{\sum_{i=1}^{m-2}\alpha_i}{1-\sum_{i=1}^{m-2}\alpha_i}\left[\int_0^{+\infty} G(\xi_i,s)q(s)f(s,u(s))ds + \sum_{0<t_k<\xi_i} I_k(u)\right] \\ &:= Tu(t), \end{aligned} \tag{3.1}$$

where $G(t,s)$ is defined as (1.3).

The proof is similar to Lemma 3 in [9], and here we omit it.

For $t_p < a^* < b^* < +\infty$, let $c^* = \min\{a^*/(1+a^*), 1/(1+b^*)\}$. Then

$$\frac{G(t,s)}{1+t} \ge c^* \frac{G(r,s)}{1+r}, \quad \frac{1}{1+t} \ge \frac{c^*}{1+r}, \quad \text{for } t \in [a^*,b^*],\ r \in [0,+\infty),\ s \in [0,+\infty). \tag{3.2}$$

It is clear that $0 < c^* < 1$. Consider the space E defined by

$$E = \left\{ u \in PC[0,+\infty) : \sup_{t\in[0,\infty)} \frac{|u(t)|}{1+t} < +\infty \right\}. \tag{3.3}$$

E is a Banach space, equipped with the norm $\|u\| = \sup_{0\le t<+\infty}(|u(t)|/(1+t)) < +\infty$. Define the cone $P \subset E$ by

$$P = \left\{ u \in E : u(t) \ge 0,\ t \in [0,+\infty),\ \min_{t\in[a^*,b^*]} \frac{u(t)}{1+t} \ge c^*\|u\| \right\}. \tag{3.4}$$

Lemma 3.2 (see [20, Theorem 2.2]). *Let $M \subset PC[0,+\infty)$. Then M is compact in $PC[0,+\infty)$, if the following conditions hold:*

(a) *M is bounded in $PC[0,+\infty)$;*

(b) *the functions belonging to M are piecewise equicontinuous on any interval of $[0,+\infty)$;*

(c) *the functions from M are equiconvergent, that is, given $\varepsilon > 0$, there corresponds $\tau(\varepsilon) > 0$ such that $|f(t) - f(+\infty)| < \varepsilon$ for any $t \ge \tau(\varepsilon)$ and $f \in M$.*

Lemma 3.3. *$T : P \to P$ is completely continuous.*

Proof. Firstly, for $u \in P$, from (H_1)–(H_3), it is easy to check that Tu is well defined, and $Tu(t) \geq 0$ for all $t \in [0, +\infty)$. For $t \in [a^*, b^*]$

$$\begin{aligned}
\frac{1}{1+t}(Tu)(t) &= \frac{1}{1+t}\int_0^{+\infty} G(t,s)q(s)f(s,u(s))ds + \frac{1}{1+t}\sum_{k=1}^{p} I_k(u) \\
&\quad + \frac{1}{1+t}\frac{\sum_{i=1}^{m-2}\alpha_i}{1-\sum_{i=1}^{m-2}\alpha_i}\left[\int_0^{+\infty} G(\xi_i,s)q(s)f(s,u(s))ds + \sum_{0<t_k<\xi_i} I_k(u)\right] \\
&\geq c^*\left[\int_0^{+\infty} \frac{G(r,s)}{1+r}q(s)f(s,u(s))ds + \frac{1}{1+r}\sum_{k=1}^{p} I_k(u)\right] \\
&\quad + c^*\frac{\sum_{i=1}^{m-2}\alpha_i}{1-\sum_{i=1}^{m-2}\alpha_i}\left[\int_0^{+\infty} \frac{G(\xi_i,s)}{1+r}q(s)f(s,u(s))ds + \frac{1}{1+r}\sum_{0<t_k<\xi_i} I_k(u)\right] \\
&\geq c^*\frac{Tu(r)}{1+r}, \quad \text{for } r \in [0,+\infty)
\end{aligned} \tag{3.5}$$

so

$$\min_{t\in[a^*,b^*]} \frac{Tu(t)}{1+t} \geq c^*\|Tu\|, \tag{3.6}$$

which shows $TP \subseteq P$.

Now we prove that T is continuous and compact, respectively. Let $u_n \to u$ as $n \to +\infty$ in P. Then there exists r_0 such that $\sup_{n\in N\setminus\{0\}}\|u_n\| < r_0$. By (H_2) we have $f(t,u)$ is bounded on $[0,+\infty) \times [0,r_0]$. Set $B_0 = \sup\{f(t,u) : (t, u/(1+t)) \in [0,+\infty) \times [0,r_0]\}$, and we have

$$\int_0^{+\infty} \frac{G(t,s)}{1+t}q(s)\left|f(s,u_n) - f(s,u)\right|ds \leq 2B_0\int_0^{+\infty} \frac{G(t,s)}{1+t}q(s)ds. \tag{3.7}$$

Therefore by the Lebesgue dominated convergence theorem and continuity of f and I_k, one arrives at

$$\begin{aligned}
&\|Tu_n - Tu\| \\
&\quad\leq \sup_{t\in[0,+\infty)}\frac{1}{1+t}\left\{\int_0^{+\infty} G(t,s)q(s)\left|f(s,u_n) - f(s,u)\right|ds + \sum_{0<t_k<t}|I_k(u_n) - I_k(u)| + \frac{\sum_{i=1}^{m-2}\alpha_i}{1-\sum_{i=1}^{m-2}\alpha_i}\right. \\
&\qquad\left.\times\left[\int_0^{+\infty} G(\xi_i,s)q(s)\left|f(s,u_n) - f(s,u)\right|ds + \sum_{0<t_k<\xi_i}|I_k(u_n) - I_k(u)|\right]\right\} \\
&\quad\longrightarrow 0 \quad \text{as } n \longrightarrow +\infty.
\end{aligned} \tag{3.8}$$

Therefore T is continuous.

Let Ω be any bounded subset of P. Then there exists $r > 0$ such that $\|u\| \le r$ for all $u \in \Omega$. Set $B_1 = \sup\{f(t,u) : (t, u/(1+t)) \in [0,+\infty)\times[0,r]$, $B_{2k} = \sup\{I_k(u) : u/(1+t) \in [0,r]\}$, then

$$\begin{aligned}\|Tu\| &= \sup_{t\in[0,+\infty)} \frac{1}{1+t}\left\{\int_0^{+\infty} G(t,s)q(s)\left|f(s,u)\right|ds + \sum_{0<t_k<t}|I_k(u)|\right.\\ &\qquad \left. + \frac{\sum_{i=1}^{m-2}\alpha_i}{1-\sum_{i=1}^{m-2}\alpha_i}\left[\int_0^{+\infty} G(\xi_i,s)q(s)\left|f(s,u)\right|ds + \sum_{0<t_k<\xi_i}|I_k(u)|\right]\right\}\\ &\le B_1\left[\int_0^{+\infty} G(t,s)q(s)ds + \frac{\sum_{i=1}^{m-2}\alpha_i}{1-\sum_{i=1}^{m-2}\alpha_i}\int_0^{+\infty} G(\xi_i,s)q(s)ds\right]\\ &\quad + \left(1 + \frac{\sum_{i=1}^{m-2}\alpha_i}{1-\sum_{i=1}^{m-2}\alpha_i}\right)\sum_{k=1}^{p} B_{2k}.\end{aligned} \tag{3.9}$$

So $T\Omega$ is bounded.

Moreover, for any $\nu \in (0,+\infty)$ and $t', t'' \in (t_k, t_{k+1}] \subset [0,\nu] (t' < t'')$, and $u \in \Omega$, then

$$\begin{aligned}\left|\frac{Tu(t'')}{1+t''} - \frac{Tu(t')}{1+t'}\right| &\le \frac{\sum_{i=1}^{m-2}\alpha_i}{1-\sum_{i=1}^{m-2}\alpha_i}\left[B_1\int_0^{+\infty} G(\xi_i,s)q(s)ds + \sum_{0<t_k<\xi_i} B_{2k}\right]\left|\frac{1}{1+t''} - \frac{1}{1+t'}\right|\\ &\quad + B_1\int_0^{+\infty}\left|\frac{G(t'',s)}{1+t''} - \frac{G(t',s)}{1+t'}\right|q(s)ds + \sum_{0<t_k<t'} B_{2k}\left|\frac{1}{1+t''} - \frac{1}{1+t'}\right|\\ &\longrightarrow 0, \quad \text{uniformly as } t' \longrightarrow t''.\end{aligned} \tag{3.10}$$

So $T\Omega$ is quasi-equicontinuous on any compact interval of $[0,+\infty)$.

Finally, we prove for any ε, there exists sufficiently large $N_1 > 0$ such that

$$\left|\frac{Tu(t'')}{1+t''} - \frac{Tu(t')}{1+t'}\right| < \varepsilon, \quad \forall t', t'' \ge N_1,\ u \in \Omega. \tag{3.11}$$

Since $\int_0^{+\infty} G(t,s)q(s)ds < +\infty$, we can choose $N_1 > 0$ such that

$$\begin{gathered}\frac{\sum_{i=1}^{m-2}\alpha_i}{N_1\left(1-\sum_{i=1}^{m-2}\alpha_i\right)}\left[B_1\int_0^{+\infty} G(\xi_i,s)q(s)ds + \sum_{0<t_k<\xi_i} B_{2k}\right] < \frac{\varepsilon}{6},\\ \frac{B_1\int_0^{+\infty} G(t,s)q(s)ds}{N_1} < \frac{\varepsilon}{6}, \qquad \sum_{k=1}^{p}\frac{B_{2k}}{N_1} \le \frac{\varepsilon}{6}.\end{gathered} \tag{3.12}$$

For $t', t'' \geq N_1$, it follows that

$$\left|\frac{(Tu)(t')}{1+t'} - \frac{(Tu)(t'')}{1+t''}\right| \leq \frac{\sum_{i=1}^{m-2}\alpha_i}{1-\sum_{i=1}^{m-2}\alpha_i}\left[B_1\int_0^{+\infty} G(\xi_i, s)q(s)ds + \sum_{0<t_k<\xi_i} B_{2k}\right]\left(\frac{1}{1+t''} + \frac{1}{1+t'}\right)$$
$$+ B_1\int_0^{+\infty}\frac{G(t', s)}{1+t'}q(s)ds + B_1\int_0^{+\infty}\frac{G(t'', s)}{1+t''}q(s)ds$$
$$+ \sum_{k=1}^{p} B_{2k}\left(\frac{1}{1+t''} + \frac{1}{1+t'}\right)$$
$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{6} + \frac{\varepsilon}{6} + \frac{\varepsilon}{3} = \varepsilon. \tag{3.13}$$

That is (3.11) holds. By Lemma 3.2, $T\Omega$ is relatively compact. In sum, $T : P \to P$ is completely continuous. □

4. Existence of Three Positive Solutions

Let the nonnegative continuous concave functional α, the nonnegative continuous convex functionals γ and θ, and the nonnegative continuous functionals ψ be defined on the cone P by

$$\gamma(u) = \psi(u) = \theta(u) = \sup_{t\in[0,+\infty)}\frac{u(t)}{1+t}, \qquad \alpha(u) = \min_{t\in[a^*,b^*]}\frac{u(t)}{1+t}. \tag{4.1}$$

For notational convenience, we denote by

$$M = \min_{t\in[a^*,b^*]}\int_0^{+\infty}\frac{G(t,s)}{1+t}q(s)ds,$$
$$M_1 = \frac{\sum_{i=1}^{m-2}\alpha_i}{1-\sum_{i=1}^{m-2}\alpha_i}\int_0^{+\infty} G(\xi_i, s)q(s)ds. \tag{4.2}$$

The main result of this paper is the following.

Theorem 4.1. *Assume (H_1)–(H_3) hold. Let $a_k \geq 0$, $0 < a < b/c^* < c = d$, $b/M < c^*d/2(M+M_1)$ and suppose that f, I_k satisfy the following conditions:*

(A_1) $f(t,u) \leq c^*d/2(M + M_1), I_k(u) \leq dc^*/2M_2$ for $(t, u/(1+t)) \in [0,+\infty) \times [0, d]$,

(A_2) $f(t,u) > b/M$ *for* $(t, u/(1+t)) \in [a^*, b^*] \times [b, c]$,

(A_3) $f(t,u) < c^*a/2(M+M_1)$, $I_k(u) \leq ac^*a_k/2M_2$ for $t \in (t, u/(1+t)) \in [0,+\infty) \times [0, a]$, *where* $M_2 = \sum_{k=1}^{p} a_k/(1-\sum_{i=1}^{m-2}\alpha_i)$. *Then* (1.1) *has at least three positive solutions* u_1, u_2 *and* u_3 *such that*

$$\gamma(u_i) \le d, \quad \text{for } i = 1,2,3, \qquad \psi(u_1) < a, \quad a < \psi(u_2) \quad \text{with } \alpha(u_2) < b,\ \alpha(u_3) > b. \tag{4.3}$$

Proof.
Step 1. From the definition α, ψ, and γ, we easily show that

$$\alpha(u) \le \psi(u), \quad \|u\| \le \gamma(u) \quad \text{for } u \in \overline{P(\gamma,d)}. \tag{4.4}$$

Next we will show that

$$T : \overline{P(\gamma,d)} \longrightarrow \overline{P(\gamma,d)}. \tag{4.5}$$

In fact, for $u \in \overline{P(\gamma,d)}$, then

$$\sup_{t\in[0,+\infty)} \frac{u(t)}{1+t} \le d. \tag{4.6}$$

From condition (A_1), we obtain

$$f(t,u) \le \frac{dc^*}{2(M+M_1)}, \qquad I_k(u) \le \frac{dc^*}{2M_2}. \tag{4.7}$$

It follows that

$$\begin{aligned}
\gamma(Tu) &= \sup_{t\in[0,+\infty)} \frac{Tu(t)}{1+t} \le \frac{1}{c^*} \min_{t\in[a^*,b^*]} \frac{Tu(t)}{1+t} \\
&\le \frac{1}{c^*} \min_{t\in[a^*,b^*]} \Bigg[\frac{1}{1+t} \int_0^{+\infty} G(t,s)q(s)f(s,u)\,ds + \frac{1}{1+t}\sum_{k=1}^{p} I_k(u) \\
&\qquad\qquad + \frac{1}{1+t} \cdot \frac{\sum_{i=1}^{m-2}\alpha_i}{1-\sum_{i=1}^{m-2}\alpha_i} \left(\int_0^{+\infty} G(\xi_i,s)q(s)f(s,u)\,ds + \sum_{0<t_k<\xi_i} I_k(u) \right) \Bigg] \\
&\le \frac{1}{c^*} \cdot \frac{c^* d}{2(M+M_1)} \left[\min_{t\in[a^*,b^*]} \int_0^{+\infty} \frac{G(t,s)}{1+t} q(s)\,ds + \frac{\sum_{i=1}^{m-2}\alpha_i}{1-\sum_{i=1}^{m-2}\alpha_i} \int_0^{+\infty} G(\xi_i,s)q(s)\,ds \right] \\
&\quad + \frac{1}{c^*} \cdot \frac{c^* d \sum_{k=1}^{p} a_k}{2M_2} \left[1 + \frac{\sum_{i=1}^{m-2}\alpha_i}{1-\sum_{i=1}^{m-2}\alpha_i} \right] \\
&\le \frac{d}{2} + \frac{d}{2} = d.
\end{aligned} \tag{4.8}$$

Thus (4.5) holds.

Step 2. We show that condition (i) in Theorem 2.2 holds. Taking $u(t) = (1+t)((b+d)/2)$, then $u \in P(\gamma, \theta, \alpha, b, c, d)$ and $\alpha(u) > b$, which shows $\{u \in P(\gamma, \theta, \alpha, b, c, d) \mid \alpha(u) > b\} \neq \emptyset$. Thus for $u \in P(\gamma, \theta, \alpha, b, c, d)$, there is

$$b \leq \frac{u(t)}{1+t} \leq \quad \text{for } t \in [a^*, b^*]. \tag{4.9}$$

Hence by (A_2), we have

$$\begin{aligned}
\alpha(Tu) &= \min_{t\in[a^*,b^*]} \frac{Tu(t)}{1+t} \\
&\geq \min_{t\in[a^*,b^*]} \int_0^{+\infty} \frac{G(t,s)}{1+t} q(s) f(s,u) ds \\
&> \frac{b}{M} \cdot \min_{t\in[a^*,b^*]} \int_0^{+\infty} \frac{G(t,s)}{1+t} q(s) ds = b.
\end{aligned} \tag{4.10}$$

Therefore we have

$$\alpha(Tu) > b, \quad \forall u \in P(\gamma, \theta, \alpha, b, c, d). \tag{4.11}$$

This shows the condition (i) in Theorem 2.2 is satisfied.

Step 3. We now prove (ii) in Theorem 2.2 holds. For $u \in P(\gamma, \alpha, b, d)$ with $\theta(Tu) > c$, we have

$$\alpha(Tu) = \min_{t\in[a^*,b^*]} \frac{Tu(t)}{1+t} \geq c^* \|Tu\| = c^* \theta(Tu) > c^* c > b. \tag{4.12}$$

Hence, condition (ii) in Theorem 2.2 is satisfied.

Step 4. Finally, we prove (iii) in Theorem 2.2 is satisfied. Since $\psi(0) = 0 < a$, so $0 \notin R(\gamma, \psi, a, d)$. Suppose that $u \in R(\gamma, \theta, a, d)$ with $\psi(u) = a$, then

$$0 \leq \frac{u(t)}{1+t} \leq a, \tag{4.13}$$

by the condition (A_3) of this theorem,

$$\begin{aligned}
\psi(Tu) &= \sup_{t\in[0,+\infty)} \frac{Tu(t)}{1+t} \leq \frac{1}{c^*} \min_{t\in[a^*,b^*]} \frac{Tu(t)}{1+t} \\
&\leq \frac{1}{c^*} \cdot \frac{c^* a}{2(M+M_1)} \left[\min_{t\in[a^*,b^*]} \int_0^{+\infty} \frac{G(t,s)}{1+t} q(s) ds + \frac{\sum_{i=1}^{m-2} \alpha_i}{1-\sum_{i=1}^{m-2} \alpha_i} \int_0^{+\infty} G(\xi_i, s) q(s) ds \right] \\
&\quad + \frac{1}{c^*} \cdot \frac{c^* a \sum_{k=1}^{p} a_k}{2M_2} \left[1 + \frac{\sum_{i=1}^{m-2} \alpha_i}{1 - \sum_{i=1}^{m-2} \alpha_i} \right] \\
&\leq \frac{a}{2} + \frac{a}{2} = a.
\end{aligned} \tag{4.14}$$

Thus condition (iii) in Theorem 2.2 holds. Therefore an application of Theorem 2.2 implies the boundary value problem (1.1) has at least three positive solutions such that

$$\sup_{t\in[0,+\infty)}\frac{u_i(t)}{1+t}\le d,\quad i=1,2,3,$$

$$\sup_{t\in[0,+\infty)}\frac{u_1(t)}{1+t}<a,\quad a<\sup_{t\in[0,+\infty)}\frac{u_2(t)}{1+t}\quad\text{with}\ \min_{t\in[a^*,b^*]}\frac{u_2(t)}{1+t}<b,\tag{4.15}$$

$$\min_{t\in[a^*,b^*]}\frac{u_3(t)}{1+t}>b.$$

□

5. An Example

Now we consider the following boundary value problem

$$\begin{gathered}
u''(t)+q(t)f(t,u)=0,\quad 0<t<+\infty, t\neq t_1,\\
\Delta u(t_1)=I_1(u(t_1)),\quad t_1=1,\\
u(0)=\frac{1}{4}u\left(\frac{1}{4}\right)+\frac{1}{4}u(4),\qquad u'(\infty)=0\\
f(t,u)=\begin{cases}\dfrac{1}{100}e^{-t}+4\left(\dfrac{u}{1+t}\right)^7, & \dfrac{u}{1+t}\le 1,\\ \dfrac{1}{100}e^{-t}+4, & \dfrac{u}{1+t}\ge 1,\end{cases}\\
I_1(u)=\begin{cases}\dfrac{1}{2}\left(\dfrac{u}{1+t}\right)^4, & \dfrac{u}{1+t}\le 1,\\ \dfrac{1}{2}, & \dfrac{u}{1+t}\ge 1,\end{cases}
\end{gathered}\tag{5.1}$$

$q(t)=e^{-t}$. Choose $a_1=1/2$, $a=1/2$, $b=3/4$, $c=d=48$. If taking $a^*=2$, $b^*=3$, then $c^*=1/4$, and $M=(1-e^{-3})/4$, $M_1=2-e^{-1/4}-e^{-4}$, $M_2=1$. Consequently, $f(t,u)$, $I_k(u)$ satisfies the following:

(1) $f(t,u)\le 1/100+4<c^*d/2(M+M_1)$, $I_1(u)\le 1/2<3=c^*a_1d/2M_2$, for $(t,u/(1+t))\in[0,+\infty)\times[0,48]$;

(2) $f(t,u)>4>b/M$, for $(t,u/(1+t))\in[2,3]\times[3/4,48]$;

(3) $f(t,u)<1/100+(1/2)^5\le c^*a/2(M+M_1)$, $I_1(u)\le 1/32=c^*a_1a/2M_2$, for $(t,u/(1+t))\in[0,+\infty)\times[0,1/2]$.

Then all conditions of Theorem 4.1 hold, so by Theorem 4.1, boundary value problem (5.1) has at least three positive solutions.

Acknowledgments

This work is supported by the NNSF of China (no. 60671066), A project supported by Scientific Research Fund of Hunan Provincial Education Department (07B041) and Program for Young Excellent Talents in Hunan Normal University, The research of J. J. Nieto has been partially supported by Ministerio de Educacion y Ciencia and FEDER, Project MTM2007-61724, and by Xunta de Galicia and FEDER, Project PGIDIT06PXIB207023PR.

References

[1] M. Benchohra, J. Henderson, and S. Ntouyas, *Impulsive Differential Equations and Inclusions*, vol. 2 of *Contemporary Mathematics and Its Applications*, Hindawi Publishing Corporation, New York, NY, USA, 2006.

[2] V. Lakshmikantham, D. D. Baĭnov, and P. S. Simeonov, *Theory of Impulsive Differential Equations*, vol. 6 of *Series in Modern Applied Mathematics*, World Scientific, Teaneck, NJ, USA, 1989.

[3] S. T. Zavalishchin and A. N. Sesekin, *Dynamic Impulse Systems: Theory and Application*, vol. 394 of *Mathematics and Its Applications*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1997.

[4] J.-M. Belley and M. Virgilio, "Periodic Liénard-type delay equations with state-dependent impulses," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 64, no. 3, pp. 568–589, 2006.

[5] J. Chu and J. J. Nieto, "Impulsive periodic solutions of first-order singular differential equations," *Bulletin of the London Mathematical Society*, vol. 40, no. 1, pp. 143–150, 2008.

[6] T. Cardinali and P. Rubbioni, "Impulsive semilinear differential inclusions: topological structure of the solution set and solutions on non-compact domains," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 69, no. 1, pp. 73–84, 2008.

[7] L. Di Piazza and B. Satco, "A new result on impulsive differential equations involving non-absolutely convergent integrals," *Journal of Mathematical Analysis and Applications*, vol. 352, no. 2, pp. 954–963, 2009.

[8] D. Guo, "Existence of positive solutions for nth-order nonlinear impulsive singular integro-differential equations in Banach spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 68, no. 9, pp. 2727–2740, 2008.

[9] S. Gao, L. Chen, J. J. Nieto, and A. Torres, "Analysis of a delayed epidemic model with pulse vaccination and saturation incidence," *Vaccine*, vol. 24, no. 35-36, pp. 6037–6045, 2006.

[10] J. Jiao, L. Chen, J. J. Nieto, and A. Torres, "Permanence and global attractivity of stage-structured predator-prey model with continuous harvesting on predator and impulsive stocking on prey," *Applied Mathematics and Mechanics*, vol. 29, no. 5, pp. 653–663, 2008.

[11] J. Li, J. J. Nieto, and J. Shen, "Impulsive periodic boundary value problems of first-order differential equations," *Journal of Mathematical Analysis and Applications*, vol. 325, no. 1, pp. 226–236, 2007.

[12] Z. Luo and J. Shen, "Stability of impulsive functional differential equations via the Liapunov functional," *Applied Mathematics Letters*, vol. 22, no. 2, pp. 163–169, 2009.

[13] J. Li and J. Shen, "Existence of positive solution for second-order impulsive boundary value problems on infinity intervals," *Boundary Value Problems*, vol. 2006, Article ID 14594, 11 pages, 2006.

[14] S. Liang and J. Zhang, "The existence of three positive solutions for some nonlinear boundary value problems on the half-line," *Positivity*, vol. 13, no. 2, pp. 443–457, 2009.

[15] J. J. Nieto and D. O'Regan, "Variational approach to impulsive differential equations," *Nonlinear Analysis: Real World Applications*, vol. 10, no. 2, pp. 680–690, 2009.

[16] J. J. Nieto, "Impulsive resonance periodic problems of first order," *Applied Mathematics Letters*, vol. 15, no. 4, pp. 489–493, 2002.

[17] G. Tr. Stamov, "On the existence of almost periodic solutions for the impulsive Lasota-Wazewska model," *Applied Mathematics Letters*, vol. 22, no. 4, pp. 516–520, 2009.

[18] J. R. Wang, X. Xiang, W. Wei, and Q. Chen, "Bounded and periodic solutions of semilinear impulsive periodic system on Banach spaces," *Fixed Point Theory and Applications*, vol. 2008, Article ID 401947, 15 pages, 2008.

[19] X. Xian, D. O'Regan, and R. P. Agarwal, "Multiplicity results via topological degree for impulsive boundary value problems under non-well-ordered upper and lower solution conditions," *Boundary Value Problems*, vol. 2008, Article ID 197205, 21 pages, 2008.

[20] B. Yan, "Boundary value problems on the half-line with impulses and infinite delay," *Journal of Mathematical Analysis and Applications*, vol. 259, no. 1, pp. 94–114, 2001.

[21] J. Yan, A. Zhao, and J. J. Nieto, "Existence and global attractivity of positive periodic solution of periodic single-species impulsive Lotka-Volterra systems," *Mathematical and Computer Modelling*, vol. 40, no. 5-6, pp. 509–518, 2004.

[22] H. Zhang, L. Chen, and J. J. Nieto, "A delayed epidemic model with stage-structure and pulses for pest management strategy," *Nonlinear Analysis: Real World Applications*, vol. 9, no. 4, pp. 1714–1726, 2008.

[23] X. Zhang, Z. Shuai, and K. Wang, "Optimal impulsive harvesting policy for single population," *Nonlinear Analysis: Real World Applications*, vol. 4, no. 4, pp. 639–651, 2003.

[24] G. Zeng, F. Wang, and J. J. Nieto, "Complexity of a delayed predator-prey model with impulsive harvest and Holling type II functional response," *Advances in Complex Systems*, vol. 11, no. 1, pp. 77–97, 2008.

[25] H. Lian, H. Pang, and W. Ge, "Triple positive solutions for boundary value problems on infinite intervals," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 67, no. 7, pp. 2199–2207, 2007.

[26] Y. Liu, "Existence and unboundedness of positive solutions for singular boundary value problems on half-line," *Applied Mathematics and Computation*, vol. 144, no. 2-3, pp. 543–556, 2003.

[27] D. O'Regan, *Theory of Singular Boundary Value Problems*, World Scientific, River Edge, NJ, USA, 1994.

[28] R. I. Avery and A. C. Peterson, "Three positive fixed points of nonlinear operators on ordered Banach spaces," *Computers & Mathematics with Applications*, vol. 42, no. 3–5, pp. 313–322, 2001.