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Research Article

Infinitely Many Solutions of Strongly Indefinite Semilinear Elliptic Systems

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Recommended by Wenming Zou

We proved a multiplicity result for strongly indefinite semilinear elliptic systems $-\Delta u + u = \pm 1/(1+|x|^a)|v|^{p-2}v$ in $\mathbb{R}^N$, $-\Delta v + v = \pm 1/(1+|x|^b)|u|^{q-2}u$ in $\mathbb{R}^N$ where a and b are positive numbers which are in the range we shall specify later.

1. Introduction

In this paper, we shall study the existence of multiple solutions of the semilinear elliptic systems

$$\begin{aligned} -\Delta u + u &= \pm \frac{1}{(1+|x|)^a}|v|^{p-2}v \quad \text{in } \mathbb{R}^N, \\ -\Delta v + v &= \pm \frac{1}{(1+|x|)^b}|u|^{q-2}u \quad \text{in } \mathbb{R}^N, \end{aligned} \tag{1.1}$$

where a and b are positive numbers which are in the range we shall specify later. Let us consider that the exponents $p, q > 2$ are below the critical hyperbola

$$1 > \frac{1}{p} + \frac{1}{q} > \frac{N-2}{N} \quad \text{for } N \geq 3, \tag{1.2}$$

so one of p and q could be larger than $2N/(N-2)$; for that matter, the quadratic part of the energy functional

$$I^{\pm}(u,v) = \pm\int(\nabla u\cdot\nabla v + uv)dx - \frac{1}{p}\int\frac{1}{(1+|x|)^{a}}|v|^{p}dx - \frac{1}{q}\int\frac{1}{(1+|x|)^{b}}|u|^{q}dx \tag{1.3}$$

has to be redefined, and we then need fractional Sobolev spaces.

Hence the energy functional $I^{\pm}$ is strongly indefinite, and we shall use the generalized critical point theorem of Benci [1] in a version due to Heinz [2] to find critical points of $I^{\pm}$. And there is a lack of compactness due to the fact that we are working in $\mathbb{R}^N$.

In [3], Yang shows that under some assumptions on the functions f and g there exist infinitely many solutions of the semilinear elliptic systems

$$\begin{aligned} -\Delta u + u &= \pm g(x,v) \quad \text{in } \mathbb{R}^N, \\ -\Delta v + v &= \pm f(x,u) \quad \text{in } \mathbb{R}^N. \end{aligned} \tag{1.4}$$

We shall propose herein a result similar to [3] for problem (1.1).

2. Abstract Framework and Fractional Sobolev Spaces

We recall some abstract results developed in [4] or [5].

We shall work with space E^s, which are obtained as the domains of fractional powers of the operator

$$-\Delta + id : H^2\left(\mathbb{R}^N\right)\cap H^1\left(\mathbb{R}^N\right)\subset L^2\left(\mathbb{R}^N\right)\longrightarrow L^2\left(\mathbb{R}^N\right). \tag{2.1}$$

Namely, $E^s = D((-\Delta + id)^{s/2})$ for $0\le s\le 2$, and the corresponding operator is denoted by $A^s : E^s \to L^2(\mathbb{R}^N)$. The spaces E^s, the usual fractional Sobolev space $H^s(\mathbb{R}^N)$, are Hilbert spaces with inner product

$$\langle u,v\rangle_{E^s} = \int A^s u A^s v dx \tag{2.2}$$

and associates norm

$$\|u\|_{E^s}^2 = \int |A^s u|^2 dx. \tag{2.3}$$

It is known that A^s is an isomorphism, and so we denote by A^{-s} the inverse of A^s.

Now let $s,t>0$ with $s+t=2$. We define the Hilbert space $E = E^s\times E^t$ and the bilinear form $B : E\times E\to\mathbb{R}$ by the formula

$$B\big((u,v),(\phi,\psi)\big) = \int A^s u A^t\psi + A^s\phi A^t v. \tag{2.4}$$

Using the Cauchy-Schwarz inequality, then it is easy to see that B is continuous and symmetric. Hence B induces a self-adjoint bounded linear operator $L : E \to E$ such that

$$B(z,\eta) = \langle Lz,\eta\rangle_E, \quad \text{for } z,\, \eta \in E. \tag{2.5}$$

Here and in what follows $\langle\cdot,\cdot\rangle_E$ denotes the inner product in E induced by $\langle\cdot,\cdot\rangle_{E^s}$ and $\langle\cdot,\cdot\rangle_{E^t}$ on the product space E in the usual way. It is easy to see that

$$Lz = L(u,v) = \left(A^{-s}A^t v, A^{-t}A^s u\right), \quad \text{for } z = (u,v) \in E. \tag{2.6}$$

We can then prove that L has two eigenvalues -1 and 1, whose corresponding eigenspaces are

$$\begin{aligned} E^- &= \left\{\left(u, -A^{-t}A^s u\right) : u \in E^s\right\}, \quad \text{for } \lambda = -1,\\ E^+ &= \left\{\left(u, A^{-t}A^s u\right) : u \in E^s\right\}, \quad \text{for } \lambda = 1, \end{aligned} \tag{2.7}$$

which give a natural splitting $E = E^+ \oplus E^-$. The spaces E^+ and E^- are orthogonal with respect to the bilinear form B, that is,

$$B\left(z^+, z^-\right) = 0, \quad \text{for } z^+ \in E^+,\, z^- \in E^-. \tag{2.8}$$

We can also define the quadratic form $Q : E \to \mathbb{R}$ associated to B and L as

$$Q(z) = \frac{1}{2}B(z,z) = \frac{1}{2}\langle Lz, z\rangle_E = \int A^s u A^t v \tag{2.9}$$

for all $z = (u,v) \in E$. It follows then that

$$\frac{1}{2}\|z\|_E^2 = Q\left(z^+\right) - Q\left(z^-\right), \tag{2.10}$$

where $z = z^+ + z^-$, $z^+ \in E^+$, $z^- \in E^-$. If $z = (u,v) \in E^+$, that is, $v = A^{-t}A^s u$, we have

$$Q(z) = \frac{1}{2}\|z\|_E^2 = \frac{1}{2}\left\|\left(u, A^{-t}A^s u\right)\right\|_E^2 = \|A^s u\|^2 = \|u\|_{E^s}^2. \tag{2.11}$$

Similarly

$$Q(z) = \left\| A^t v \right\|^2 = \|v\|_{E^t}^2 \tag{2.12}$$

for $z \in E^-$.

If $w(x) := 1/(1+|x|)^c$ where c is a number satisfying the condition

$$2c > 2N - \gamma(N-2s), \quad 2 < \gamma < \frac{2N}{N-2s} \tag{2.13}$$

and $m := (2N/(N-2s))/(2N/(N-2s)-\gamma)$, it follows by (2.13) that $w \in L^m(\mathbb{R}^N)$ and by Hölder inequalities that

$$\int w(x)|u(x)|^\gamma dx \leq |w|_m |u|_{2N/(N-2s)}^\gamma \leq c|w|_m \|u\|_{E^s}^\gamma. \tag{2.14}$$

In the sequel $|\cdot|_m$ denotes the norm in $L^m(\mathbb{R}^N)$, and we denote by $L^\gamma(w,\mathbb{R}^N)$ the weighted function spaces with the norm defined on E^s by $|u|_{w,\gamma} = (\int w(x)|u(x)|^\gamma)^{1/\gamma}$. According to the properties of interpolation space, we have the following embedding theorem.

Theorem 2.1. *Let $s > 0$. one defines the operator $\Theta : H^s(\mathbb{R}^N) \to H^{-s}(\mathbb{R}^N)$ as follows: for u, $\phi \in H^s(\mathbb{R}^N)$,*

$$\langle \Theta(u), \phi \rangle = \int w(x)|u|^{\gamma-2} u \phi dx. \tag{2.15}$$

Then the inclusion of $H^s(\mathbb{R}^N)$ into $L^\gamma(w,\mathbb{R}^N)$ is compact if $2 < \gamma < 2N/(N-2s)$.

Proof. Observe that, by Hölder's inequality and (2.14), we have

$$\left|\langle \Theta(u), \phi \rangle\right| \leq \int \left| w(x)^{1/\gamma'} |u|^{\gamma-1} w(x)^{1/\gamma} \phi \right| \leq \left(\int w(x)|u|^\gamma \right)^{1/\gamma'} \left(\int w(x)\left|\phi\right|^\gamma \right)^{1/\gamma} < \infty, \tag{2.16}$$

where $1/\gamma + 1/\gamma' = 1$; hence Θ is well defined. □

Then we will claim that Θ is compact. Since $w(x) \in L^m(\mathbb{R}^N)$, for any $\varepsilon > 0$, there exists $K > 0$, such that $(\int_{|x|>K} w(x)^m)^{1/m} < \epsilon$. Now, suppose $u_n \rightharpoonup u$ weakly in $H^s(\mathbb{R}^N)$. We estimate

$$\begin{aligned}
&\|\Theta(u_n) - \Theta(u)\|_{H^{-s}} \\
&\quad = \sup_{\|\phi\|_{E^s}\le 1} \left|\langle \Theta(u_n) - \Theta(u), \phi\rangle\right| \\
&\quad = \sup_{\|\phi\|_{E^s}\le 1} \left|\int w(x)\left(|u_n|^{\gamma-2}u_n - |u|^{\gamma-2}u\right)\phi\right| \\
&\quad = \sup_{\|\phi\|_{E^s}\le 1} \left|(\gamma-1)\int w(x)|\theta|^{\gamma-2}(u_n - u)\phi\right|, \quad \text{where } |\theta| \le |u_n| + |u| \\
&\quad \le C \sup_{\|\phi\|_{E^s}\le 1} \int |w(x)|\left(|u_n|^{\gamma-2} + |u|^{\gamma-2}\right)|u_n - u|\left|\phi\right| \\
&\quad \le C \sup_{\|\phi\|_{E^s}\le 1} \int \left(|w(x)||u_n|^{\gamma-2}|u_n - u|\left|\phi\right| + |w(x)||u|^{\gamma-2}|u_n - u|\left|\phi\right|\right) \\
&\quad \le C\Bigg(\sup_{\|\phi\|_{E^s}\le 1} \int_{|x|\le K} \left(|w(x)||u_n|^{\gamma-2}|u_n - u|\left|\phi\right| + |w(x)||u|^{\gamma-2}|u_n - u|\left|\phi\right|\right) \\
&\qquad + \sup_{\|\phi\|_{E^s}\le 1} \int_{|x|>K} \left(|w(x)||u_n|^{\gamma-2}|u_n - u|\left|\phi\right| + |w(x)||u|^{\gamma-2}|u_n - u|\left|\phi\right|\right)\Bigg),
\end{aligned} \tag{2.17}$$

letting

$$m_1 = \frac{2N/(N-2s)}{2N/(N-2s) - \gamma} = m, \quad m_2 = \frac{2N/(N-2s)}{\gamma - 2}, \quad m_3 = \frac{2N}{N-2s} = m_4, \tag{2.18}$$

we have

$$\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} + \frac{1}{m_4} = 1, \tag{2.19}$$

so that by Hölder's inequality, we observe that, for any $\varepsilon > 0$, we can choose a $K > 0$ so that the integral over $(|x| > K)$ is smaller than $\varepsilon/2$ for all n, while for this fixed K, by strong convergence of u_n to u in $L^{2N/(N-2s)}(\mathbb{R}^N)$ on any bounded region, the integral over $(|x| \le K)$ is smaller than $\varepsilon/2$ for n large enough. We thus have proved that $\Theta(u_n) \to \Theta(u)$ strongly in $H^{-s}(\mathbb{R}^N)$; that is, the inclusion of $H^s(\mathbb{R}^N)$ into $L^\gamma(w, \mathbb{R}^N)$ is compact if $2 < \gamma < 2N/(N-2s)$.

3. Main Theorem

We consider below the problem of finding multiple solutions of the semilinear elliptic systems

$$-\Delta u + u = \pm \frac{1}{(1+|x|)^a}|v|^{p-2}v \quad \text{in } \mathbb{R}^N,$$
$$-\Delta v + v = \pm \frac{1}{(1+|x|)^b}|u|^{q-2}u \quad \text{in } \mathbb{R}^N. \tag{3.1}$$

Now if we choose $s, t > 0$, $s + t = 2$, such that

$$\left(1-\frac{1}{q}\right)\max\{p,q\} < \frac{1}{2}+\frac{s}{N},$$
$$\left(1-\frac{1}{p}\right)\max\{p,q\} < \frac{1}{2}+\frac{t}{N}, \tag{3.2}$$

and we assume that

(H) $2 < p < 2N/(N-2t)$, $2 < q < 2N/(N-2s)$ and a and b are positive numbers such that

$$2a > 2N - p(N-2t), \quad 2b > 2N - q(N-2s). \tag{3.3}$$

We set

$$r(x) := \frac{1}{(1+|x|)^a}, \quad s(x) := \frac{1}{(1+|x|)^b} \tag{3.4}$$

and we let

$$\alpha := \frac{2N/(N-2t)}{2N/(N-2t)-p}, \quad \beta := \frac{2N/(N-2s)}{2N/(N-2s)-q} \tag{3.5}$$

so that, under assumption (H), Theorem 2.1 holds, respectively, with $w(x) := r(x)$ and $\gamma := p$, and $w(x) := s(x)$ and $\gamma := q$; that is, the inclusion of $H^s(\mathbb{R}^N)$ into $L^q(s,\mathbb{R}^N)$ and the inclusion of $H^t(\mathbb{R}^N)$ into $L^p(r,\mathbb{R}^N)$ are compact.

If $z = (u,v) \in E = E^s \times E^t$, we let

$$I^{\pm}(u,v) = \pm\int A^s u A^t v - \frac{1}{p}\int r(x)|v|^p dx - \frac{1}{q}\int s(x)|u|^q dx \tag{3.6}$$

denote the energy of z. It is well known that under assumption (H) the energy functional $I^{\pm}(u,v)$ is well defined and continuously differentiable on E, and for all $\eta = (\phi, \varphi) \in E^s \times E^t$ we have

$$\pm\int A^s u A^t \varphi - \int r(x)|v|^{p-2} v\varphi = 0, \tag{3.7}$$

$$\pm\int A^s \phi A^t v - \int s(x)|u|^{q-2} u\phi = 0, \tag{3.8}$$

and it is also well known that the critical points of $I^{\pm}$ are weak solutions of problem (3.1). The main theorem is the following.

Theorem 3.1. *Under assumption (H), problem* (3.1) *possesses infinitely many solutions* $\pm(u,v)$.

Since the functional $I^{\pm}$ are strongly indefinite, a modified multiplicity critical points theorem Heinz [2] which is the generalized critical point theorem of Benci [1] will be used. For completeness, we state the result from here.

Theorem 3.2. *(see [2]) Let E be a real Hilbert space, and let $I \in C^1(E,\mathbb{R})$ be a functional with the following properties:*

(i)*I has the form*

$$I(z) = \frac{1}{2}(Lz, z) + \varphi(z) \quad \forall z \in E, \tag{3.9}$$

where L is an invertible bounded self-adjoint linear operator in E and where $\varphi \in C^1(E,\mathbb{R})$ is such that $\varphi(0) = 0$ and the gradient $\nabla\varphi : E \to E$ is a compact operator;

(ii) *I is even, that is $I(-z) = I(z)$ for all $z \in E$;*

(iii) *I satisfies the Palais-Smale condition. Furthermore, let*

$$E = E^+ \oplus E^- \tag{3.10}$$

be an orthogonal splitting into L-invariant subspaces E^+, E^- such that $\pm(Lz,z) \geq 0$ for all $z \in E^{\pm}$. Then,

(a) *suppose that there is an m-dimensional linear subspace E_m of $E^+(m \in \mathbb{N})$ such that for the spaces $V := E^+$, $W = E^- \oplus E_m$ one has*

(iv) $\exists \rho_0 > 0$ *such that* $\inf\{I(z) : z \in V, \|z\| = \rho\} > 0$ *for all* $\rho \in (0, \rho_0]$;

(v) $\exists c_\infty \in \mathbb{R}$ *such that $I(z) \leq c_\infty$ for all $z \in W$.Then there exist at least m pairs $(z_j, -z_j)$ of critical points of I such that $0 < I(z_j) \leq c_\infty$ $(j = 1, \ldots, m)$;*

(b) *a similar result holds when $E_m \subset E^-$, and one takes $V := E^-$, $W = E^+ \oplus E_m$.*

It is known from Section 2 that the operator L induced by the bilinear form B is an invertible bounded self-adjoint linear operator satisfying $\pm\langle Lz, z\rangle_E \geq 0$ for all $z \in E^{\pm}$. We shall need some finite dimensional subspace of E. Let (e_j), $j = 1, 2, \ldots$, be a complete orthogonal system in $H^s(\mathbb{R}^N)$. Let H_n denote the finite dimensional subspaces of $H^s(\mathbb{R}^N)$ generated by (e_j), $j = 1, 2, \ldots, n$. Since $A^s : H^s(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ and $A^t : H^t(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ are isomorphisms, we know that $\widehat{e}_j = A^{-t}A^s e_j$, $j = 1, 2, \ldots$, is a complete orthogonal system in $H^t(\mathbb{R}^N)$. Let $\widehat{H}_n$ denote the finite dimensional subspaces of $H^t(\mathbb{R}^N)$ generated by $(\widehat{e}_j)$, $j = 1, 2, \ldots, n$. For each $n \in \mathbb{N}$, we introduce the following subspaces of E^+ and E^- :

$$\begin{aligned} E_n^+ &= \text{subspace of } E^+ \text{ generated by } \left(e_j, \widehat{e}_j\right), \quad j = 1, 2, \ldots, n, \\ E_n^- &= \text{subspace of } E^- \text{ generated by } \left(e_j, -\widehat{e}_j\right), \quad j = 1, 2, \ldots, n. \end{aligned} \tag{3.11}$$

Lemma 3.3. *The functional $I^\pm$ defined in* (3.6) *satisfies conditions (ii), (iv), and (v) of Theorem 3.2.*

Proof. Condition (ii) is an immediate consequence of the definition of $I^\pm$. For condition (iv), by (2.11) and Theorem 2.1, for $z \in V := E^\pm$,

$$\begin{aligned} I^\pm(z) &= \pm \int A^s u A^t v\, dx - \frac{1}{p}\int r(x)|v|^p dx - \frac{1}{q}\int s(x)|u|^q dx \\ &\geq \frac{1}{2}\|z\|_E^2 - C\|z\|_E^p - C\|z\|_E^q, \end{aligned} \tag{3.12}$$

and since $p, q > 2$, we conclude that $I^\pm(z) > 0$ for $z \in E^\pm$ with $\|z\|$ small.

Next, let us prove condition (v). Let $n \in \mathbb{N}$ be fixed, let $z \in W = E_n^\pm \oplus E^\mp$, and write $z = (u, v)$ and $z = z^+ + z^-$. We have

$$\begin{aligned} I^\pm(z) &= \pm\left[Q\left(z^+\right) + Q\left(z^-\right)\right] - \frac{1}{p}\int r(x)|v|^p dx - \frac{1}{q}\int s(x)|u|^q dx \\ &= -\frac{1}{2}\left\|z^\mp\right\|_E^2 + \frac{1}{2}\left\|z^\pm\right\|_E^2 - \frac{1}{p}\int r(x)|v|^p dx - \frac{1}{q}\int s(x)|u|^q dx. \end{aligned} \tag{3.13}$$

Let $z^+ = (u^+, v^+) \in E^+$ and $z^- = (u^-, v^-) \in E^-$. Then we have $v^+ = A^{-t}A^s u^+$ and $v^- = -A^{-t}A^s u^-$. Furthermore, we may write $u^\mp = \lambda u^\pm + \widehat{u}$, where $\widehat{u}$ is orthogonal to $u^\pm$ in $L^2(s, \mathbb{R}^N)$. We also have $v^\mp = \tau v^\pm + \widehat{v}$, where $\widehat{v}$ is orthogonal to $v^\pm$ in $L^2(r, \mathbb{R}^N)$. It is easy to see that either λ or τ is positive. Suppose $\lambda > 0$. Then we have

$$\begin{aligned} (1+\lambda)\int s(x)\left|u^\pm\right|^2 dx &= \int s(x)\left[(1+\lambda)u^\pm + \widehat{u}\right]u^\pm dx \\ &\leq |u|_{s,\alpha}\left|u^\pm\right|_{s,\alpha'}. \end{aligned} \tag{3.14}$$

Using the fact that the norms in $E_n^{\pm}$ are equivalent we obtain

$$\left|u^{\pm}\right|_{s,\alpha'} \leq C|u|_{s,\alpha} \tag{3.15}$$

with constant $C > 0$ independent of u. So from (3.13) and (2.11) we obtain

$$\begin{aligned} I^{\pm}(z) &\leq -\frac{1}{2}\left\|z^{\mp}\right\|_E^2 + \frac{1}{2}\left\|z^{\pm}\right\|_E^2 - C\left|u^{\pm}\right|_{s,\alpha}^{\alpha} \\ &= -\frac{1}{2}\left\|z^{\mp}\right\|_E^2 + \left\|u^{\pm}\right\|_{E^s}^2 - C\left|u^{\pm}\right|_{s,\alpha}^{\alpha}. \end{aligned} \tag{3.16}$$

The same arguments can be applied if $\tau > 0$.So the result follows from (3.16). □

A sequence $\{z_n\}$ is said to be the Palais-Smale sequence for $I^{\pm}$ ((PS)-sequence for short) if $|I^{\pm}(z_n)| \leq C$ uniformly in n and $\nabla I^{\pm}(z_n) \overset{n}{\rightarrow} 0$ in E^*. We say that $I^{\pm}$ satisfies the Palais-Smale condition ((PS)-condition for short) if every (PS)-sequence of $I^{\pm}$ is relatively compact in E.

Lemma 3.4. *Under assumption (H), the functional $I^{\pm}$ satisfies the (PS)-condition.*

Proof. We first prove the boundedness of (PS)-sequences of $I^{\pm}$. Let $z_n = (u_n, v_n) \in E$ be a (PS)-sequence of $I^{\pm}$ such that

$$\left|I^{\pm}(z_n)\right| = \left|\pm\int A^s u_n A^t v_n dx - \frac{1}{p}\int r(x)|v_n|^p dx - \frac{1}{q}\int s(x)|u_n|^q dx\right| \leq c, \tag{3.17}$$

$$\left|\left\langle \nabla I^{\pm}(z_n), \eta\right\rangle\right| \leq \epsilon_n \left\|\eta\right\|_E \text{where } \epsilon_n = o(1) \text{ as } n \rightarrow \infty \text{ an } \eta \in E. \tag{3.18}$$

Taking $\eta = z_n$ in (3.18), it follows from (3.17), (3.18), that

$$\begin{aligned} c + \epsilon_n\|z_n\|_E &\geq -\frac{1}{p}\int r(x)|v_n|^p dx - \frac{1}{q}\int s(x)|u_n|^q dx + \frac{1}{2}\int r(x)|v_n|^p dx + \frac{1}{2}\int s(x)|u_n|^q dx \\ &= \left(\frac{1}{2} - \frac{1}{p}\right)\int r(x)|v_n|^p dx + \left(\frac{1}{2} - \frac{1}{q}\right)\int s(x)|u_n|^q dx. \end{aligned} \tag{3.19}$$

Next, we estimate $\|u_n\|_{E^s}$ and $\|v_n\|_{E^t}$. From (3.18) with $\eta = (\phi, 0)$, we have

$$\left\langle \nabla I^{\pm}(z_n), \eta\right\rangle = \int A^s \phi A^t v_n dx - \int s(x)|u_n|^{q-2} u_n \phi dx \leq \epsilon_n \left\|\phi\right\|_{E^s} \tag{3.20}$$

for all $\phi \in E^s$. Using Hölder's inequality and by (3.20), we obtain

$$\begin{aligned}\left|\int A^s\phi A^t v_n dx\right| &\le \left|\int s(x)|u_n|^{q-2}u_n\phi dx\right| + \epsilon_n\|\phi\|_{E^s} \\ &\le \int\left|s(x)^{1/q'}|u_n|^{q-1}s(x)^{1/q}\phi\right|dx + \epsilon_n\|\phi\|_{E^s} \\ &\le \left(\int s(x)|u_n|^q\right)^{1/q'}\left(\int s(x)|\phi|^q\right)^{1/q} + \epsilon_n\|\phi\|_{E^s} \\ &\le \left(C|u_n|_{s,q}^{q-1} + C\right)\|\phi\|_{E^s}\end{aligned} \tag{3.21}$$

for all $\phi \in E^s$, which implies that

$$\|v_n\|_{E^t} \le C|u_n|_{s,q}^{q-1} + C. \tag{3.22}$$

Similarly, we prove that

$$\|u_n\|_{E^s} \le C|v_n|_{r,p}^{p-1} + C. \tag{3.23}$$

Adding (3.22) and (3.23) we conclude that

$$\|u_n\|_{E^s} + \|v_n\|_{E^t} \le C\left(|u_n|_{s,q}^{q-1} + |v_n|_{r,p}^{p-1} + 1\right). \tag{3.24}$$

Using this estimate in (3.19), we get

$$|u_n|_{s,q}^{q} + |v_n|_{r,p}^{p} \le C\left(|u_n|_{s,q}^{q-1} + |v_n|_{r,p}^{p-1}\right) + C. \tag{3.25}$$

Since $q > q-1$ and $p > p-1$, we conclude that both $|u_n|_{s,q}$ and $|v_n|_{r,p}$ are bounded, and consequently $\|u_n\|_{E^s}$ and $\|v_n\|_{E^t}$ are also bounded in terms of (3.24).

Finally, we show that $\{z_n\}$ contains a strongly convergent subsequence. It follows from $\|u_n\|_{E^s}$ and $\|v_n\|_{E^t}$ which are bounded and Theorem 2.1 that $\{z_n\}$ contains a subsequence, denoted again by $\{z_n\} = \{(u_n, v_n)\}$, such that

$$\begin{aligned}&u_n \rightharpoonup u \text{ in } E^s, \quad v_n \rightharpoonup v \text{ in } E^t, \\ &u_n \longrightarrow u \text{ in } L^q\left(s, \mathbb{R}^N\right), \quad 2 < q < \frac{2N}{N-2s}, \\ &v_n \longrightarrow v \text{ in } L^p\left(r, \mathbb{R}^N\right), \quad 2 < p < \frac{2N}{N-2t}.\end{aligned} \tag{3.26}$$

It follows from (3.18) that

$$\left|\pm\int A^s\phi A^t v_n - \int s(x)|u_n|^{q-2}u_n\phi\right| \le \epsilon_n\|\phi\|_{E^s}, \quad \phi\in E^s,$$
$$\left|\pm\int A^s u_n A^t\psi - \int r(x)|v_n|^{p-2}v_n\psi\right| \le \epsilon_n\|\psi\|_{E^t}, \quad \psi\in E^t. \tag{3.27}$$

Therefore,

$$\begin{aligned}\|v_n - v\|_{E^t} &= \sup\frac{\left|\int A^s\phi A^t(v_n - v)\right|}{\|\phi\|_{E^s}}\\ &\le \sup\frac{\left|\int s(x)\left(|u_n|^{q-2}u_n - |u|^{q-2}u\right)\phi\right|}{\|\phi\|_{E^s}},\end{aligned} \tag{3.28}$$

$$\begin{aligned}\|u_n - u\|_{E^s} &= \sup\frac{\left|\int A^s(u_n - u)A^t\psi\right|}{\|\psi\|_{E^t}}\\ &= \sup\frac{\left|\int r(x)\left(|v_n|^{p-2}v_n - |v|^{p-2}v\right)\psi\right|}{\|\psi\|_{E^t}},\end{aligned} \tag{3.29}$$

and by Theorem 2.1, we conclude that $v_n \to v$ strongly in E^t and $u_n \to u$ strongly in E^s. □

Proof of Theorem 3.1. Applying Lemmas 3.3 and 3.4 and Theorem 3.2, we can obtain the conclusion of Theorem 3.1. □

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