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Research Article

A Fourth-Order Boundary Value Problem with One-Sided Nagumo Condition

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The aim of this paper is to study a fourth-order separated boundary value problem with the right-hand side function satisfying one-sided Nagumo-type condition. By making a series of a priori estimates and applying lower and upper functions techniques and Leray-Schauder degree theory, the authors obtain the existence and location result of solutions to the problem.

1. Introduction

In this paper we apply the lower and upper functions method to study the fourth-order nonlinear equation

$$u^{(4)}(t) = f\left(t, u(t), u'(t), u''(t), u'''(t)\right), \quad 0 < t < 1, \tag{1.1}$$

with $f : [0,1] \times \mathbb{R}^4 \to \mathbb{R}$ being a continuous function.

This equation can be used to model the deformations of an elastic beam, and the type of boundary conditions considered depends on how the beam is supported at the two endpoints [1, 2]. We consider the separated boundary conditions

$$\begin{gathered} u(0) = u(1) = 0, \\ au''(0) - bu'''(0) = A, \\ cu''(1) + du'''(1) = B \end{gathered} \tag{1.2}$$

with $a, b, c, d \in \mathbb{R}^+ = (0, +\infty)$, $A, B \in \mathbb{R}$.

For the fourth-order differential equation

$$u^{(4)}(t) = f\left(t, u(t), u'(t), u''(t), u'''(t)\right), \quad 0 < t < 1, \\ u(0) = u(1) = u''(0) = u''(1), \tag{1.3}$$

the authors in [3] obtained the existence of solutions with the assumption that f satisfies the two-sided Nagumo-type conditions. For more related works, interested readers may refer to [1–14]. The one-sided Nagumo-type condition brings some difficulties in studying this kind of problem, as it can be seen in [15–18].

Motivated by the above works, we consider the existence of solutions when f satisfies one-sided Nagumo-type conditions. This is a generalization of the above cases. We apply lower and upper functions technique and topological degree method to prove the existence of solutions by making a priori estimates for the third derivative of all solutions of problems (1.1) and (1.2). The estimates are essential for proving the existence of solutions.

The outline of this paper is as follows. In Section 2, we give the definition of lower and upper functions to problems (1.1) and (1.2) and obtain some a priori estimates. Section 3 will be devoted to the study of the existence of solutions. In Section 4, we give an example to illustrate the conclusions.

2. Definitions and A Priori Estimates

Upper and lower functions will be an important tool to obtain a priori bounds on u, u', and u''. For this problem we define them as follows.

Definition 2.1. The functions $\alpha, \beta \in C^4(0,1) \cap C^3[0,1]$ verifying

$$\alpha''(t) \leq \beta''(t), \quad \forall t \in [0,1], \tag{2.1}$$

define a pair of lower and upper functions of problems (1.1) and (1.2) if the following conditions are satisfied:

(i) $\alpha^{(4)}(t) \geq f(t, \alpha(t), \alpha'(t), \alpha''(t), \alpha'''(t))$, $\beta^{(4)}(t) \leq f(t, \beta(t), \beta'(t), \beta''(t), \beta'''(t))$,

(ii) $\alpha(0) \leq 0$, $\alpha(1) \leq 0$, $a\alpha''(0) - b\alpha'''(0) \leq A$, $c\alpha''(1) + d\alpha'''(1) \leq B$, $\beta(0) \geq 0$, $\beta(1) \geq 0$, $a\beta''(0) - b\beta'''(0) \geq A$, $c\beta''(1) + d\beta'''(1) \geq B$,

(iii) $\alpha'(0) - \beta'(0) \leq \min\{\beta(0) - \beta(1), \alpha(1) - \alpha(0), 0\}$.

Remark 2.2. By integration, from (iii) and (2.1), we obtain

$$\alpha(t) \leq \beta(t), \quad \alpha'(t) \leq \beta'(t), \quad \forall t \in [0,1], \tag{2.2}$$

that is, lower and upper functions, and their first derivatives are also well ordered.

To have an a priori estimate on u''', we need a one-sided Nagumo-type growth condition, which is defined as follows.

Definition 2.3. Given a set $E \subset [0,1] \times \mathbb{R}^4$, a continuous $f : E \to \mathbb{R}$ is said to satisfy the one-sided Nagumo-type condition in E if there exists a real continuous function $h_E : \mathbb{R}_0^+ \to [k, +\infty)$, for some $k > 0$, such that

$$f(t, x_0, x_1, x_2, x_3) \leq h_E(|x_3|), \quad \forall (t, x_0, x_1, x_2, x_3) \in E, \tag{2.3}$$

with

$$\int_0^{+\infty} \frac{s}{h_E(s)} ds = +\infty. \tag{2.4}$$

Lemma 2.4. *Let $\Gamma_i(t), \gamma_i(t) \in C([0,1], \mathbb{R})$ satisfy*

$$\Gamma_i(t) \geq \gamma_i(t), \quad \forall t \in [0,1], \ i = 0, 1, 2, \tag{2.5}$$

and consider the set

$$E = \left\{ (t, x_0, x_1, x_2, x_3) \in [0,1] \times \mathbb{R}^4 : \gamma_i(t) \leq x_i \leq \Gamma_i(t), \ i = 0, 1, 2 \right\}. \tag{2.6}$$

Let $f : [0,1] \times \mathbb{R}^4 \to \mathbb{R}$ be a continuous function satisfying one-sided Nagumo-type condition in E.

Then, for every $\rho > 0$, there exists an $R > 0$ such that for every solution $u(t)$ of problems (1.1) *and* (1.2) *with*

$$u'''(0) \leq \rho, \qquad u'''(1) \geq -\rho, \tag{2.7}$$

$$\gamma_i(t) \leq u^{(i)}(t) \leq \Gamma_i(t), \tag{2.8}$$

for $i = 0, 1, 2$ and every $t \in [0,1]$, one has $\|u'''\|_\infty < R$.

Proof. Let u be a solution of problems (1.1) and (1.2) such that (2.7) and (2.8) hold. Define

$$\eta := \max\{\Gamma_2(1) - \gamma_2(0), \Gamma_2(0) - \gamma_2(1)\}. \tag{2.9}$$

Assume that $\rho \geq \eta$, and suppose, for contradiction, that $|u'''(t)| > \rho$ for every $t \in (0,1)$. If $u'''(t) > \rho$ for every $t \in (0,1)$, then we obtain the following contradiction:

$$\Gamma_2(1) - \gamma_2(0) \geq u''(1) - u''(0) = \int_0^1 u'''(t) dt > \int_0^1 \rho \, dt \geq \Gamma_2(1) - \gamma_2(0). \tag{2.10}$$

If $u'''(t) < -\rho$ for every $t \in (0,1)$, a similar contradiction can be derived. So there is a $\tilde{t} \in (0,1)$ such that $|u'''(\tilde{t})| \leq \rho$. By (2.4) we can take $R_1 > \rho$ such that

$$\int_\rho^{R_1} \frac{s}{h_E(s)} ds > \max_{t \in [0,1]} \Gamma_2(t) - \min_{t \in [0,1]} \gamma_2(t). \tag{2.11}$$

If $|u'''(t)| \leq \rho$ for every $t \in [0,1]$, then we have trivially $|u'''(t)| < R_1$. If not, then we can take $t_1 \in [0,1)$ such that $u'''(t_1) < -\rho$ or $t_1 \in (0,1]$ such that $u'''(t_1) > \rho$. Suppose that the first case holds. By (2.7) we can consider $t_1 < t_0 \leq 1$ such that

$$u'''(t_0) = -\rho, \quad u'''(t) < -\rho, \quad \forall t \in [t_1, t_0). \tag{2.12}$$

Applying a convenient change of variable, we have, by (2.3) and (2.11),

$$\begin{aligned} \int_{-u'''(t_0)}^{-u'''(t_1)} \frac{s}{h_E(s)} ds &= \int_{t_0}^{t_1} \frac{-u'''(t)}{h_E(-u'''(t))} \left(-u^{(4)}(t)\right) dt \\ &= \int_{t_1}^{t_0} \frac{-u'''(t)}{h_E(-u'''(t))} f(t, u(t), u'(t), u''(t), u'''(t)) dt \\ &\leq \int_{t_1}^{t_0} -u'''(t) dt = u''(t_1) - u''(t_0) \\ &\leq \max_{t\in[0,1]} \Gamma_2(t) - \min_{t\in[0,1]} \gamma_2(t) < \int_{\rho}^{R_1} \frac{s}{h_E(s)} ds. \end{aligned} \tag{2.13}$$

Hence, $u'''(t_1) > -R_1$. Since t_1 can be taken arbitrarily as long as $u'''(t_1) < -\rho$, we conclude that $u'''(t) > -R_1$ for every $t \in [0,1)$ provided that $u'''(t) < -\rho$.

In a similar way, it can be proved that $u'''(t) < R_1$, for every $t \in (0,1]$ if $u'''(t) > \rho$. Therefore,

$$\left|u'''(t)\right| < R_1, \quad \forall t \in [0,1]. \tag{2.14}$$

Consider now the case $\eta > \rho$, and take $R_2 > \eta$ such that

$$\int_{\eta}^{R_2} \frac{s}{h_E(s)} ds > \max_{t\in[0,1]} \Gamma_2(t) - \min_{t\in[0,1]} \gamma_2(\mathrm{t}). \tag{2.15}$$

In a similar way, we may show that

$$\left|u'''(t)\right| < R_2, \quad \forall t \in [0,1]. \tag{2.16}$$

Taking $R = \max\{R_1, R_2\}$, we have $\|u'''\|_\infty < R$. □

Remark 2.5. Observe that the estimation R depends only on the functions h_E, γ_2, Γ_2, and ρ and it does not depend on the boundary conditions.

3. Existence and Location Result

In the presence of an ordered pair of lower and upper functions, the existence and location results for problems (1.1) and (1.2) can be obtained.

Theorem 3.1. *Suppose that there exist lower and upper functions $\alpha(t)$ and $\beta(t)$ of problems* (1.1) *and* (1.2)*, respectively. Let $f : [0,1] \times \mathbb{R}^4 \to \mathbb{R}$ be a continuous function satisfying the one-sided Nagumo-type conditions* (2.3) *and* (2.4) *in*

$$E_* = \left\{(t, x_0, x_1, x_2, x_3) \in [0,1] \times \mathbb{R}^4 : \alpha(t) \leq x_0 \leq \beta(t),\ \alpha'(t) \leq x_1 \leq \beta'(t),\ \alpha''(t) \leq x_2 \leq \beta''(t)\right\} \tag{3.1}$$

If f verifies

$$f\left(t, \alpha(t), \alpha'(t), x_2, x_3\right) \geq f(t, x_0, x_1, x_2, x_3) \geq f\left(t, \beta(t), \beta'(t), x_2, x_3\right) \tag{3.2}$$

for $(t, x_2, x_3) \in [0,1] \times \mathbb{R}^2$ and

$$\left(\alpha(t), \alpha'(t)\right) \leq (x_0, x_1) \leq \left(\beta(t), \beta'(t)\right), \tag{3.3}$$

where $(x_0, x_1) \leq (y_0, y_1)$ means $x_0 \leq y_0$ and $x_1 \leq y_1$, then problems (1.1) *and* (1.2) *has at least one solution $u(t) \in C^4[0,1]$ satisfying*

$$\alpha(t) \leq u(t) \leq \beta(t), \qquad \alpha'(t) \leq u'(t) \leq \beta'(t), \qquad \alpha''(t) \leq u''(t) \leq \beta''(t) \tag{3.4}$$

for $t \in [0,1]$.

Proof. Define the auxiliary functions

$$\delta_i(t, x_i) = \begin{cases} \alpha^{(i)}(t), & x_i < \alpha^{(i)}(t), \\ x_i, & \alpha^{(i)}(t) \leq x_i \leq \beta^{(i)}(t), \quad i = 0, 1, 2, \\ \beta^{(i)}(t), & x_i > \beta^{(i)}(t). \end{cases} \tag{3.5}$$

For $\lambda \in [0,1]$, consider the homotopic equation

$$u^{(4)}(t) = \lambda f\left(t, \delta_0(t, u(t)), \delta_1\left(t, u'(t)\right), \delta_2\left(t, u''(t)\right), u'''(t)\right) + u''(t) - \lambda\delta_2\left(t, u''(t)\right), \tag{3.6}$$

with the boundary conditions

$$\begin{gathered} u(0) = u(1) = 0, \\ u'''(0) = \frac{\lambda}{b}\left[a u''(0) - A\right], \\ u'''(1) = \frac{\lambda}{d}\left[B - c u''(1)\right]. \end{gathered} \tag{3.7}$$

Take $r_1 > 0$ large enough such that, for every $t \in [0,1]$,

$$-r_1 < \alpha''(t) \le \beta''(t) < r_1, \tag{3.8}$$

$$f(t, \alpha(t), \alpha'(t), \alpha''(t), 0) - r_1 - \alpha''(t) < 0, \tag{3.9}$$

$$f(t, \beta(t), \beta'(t), \beta''(t), 0) + r_1 - \beta''(t) > 0, \tag{3.10}$$

$$\frac{|A|}{a} < r_1, \qquad \frac{|B|}{c} < r_1. \tag{3.11}$$

Step 1. Every solution $u(t)$ of problems (3.6) and (3.7) satisfies

$$\left|u^{(i)}(t)\right| < r_1, \quad \forall t \in [0,1] \tag{3.12}$$

for $i = 0, 1, 2$, for some r_1 independent of $\lambda \in [0,1]$.

Assume, for contradiction, that the above estimate does not hold for $i = 2$. So there exist $\lambda \in [0,1]$, $t \in [0,1]$, and a solution u of (3.6) and (3.7) such that $|u''(t)| \ge r_1$. In the case $u''(t) \ge r_1$ define

$$\max_{t\in[0,1]} u''(t) := u''(t_0) \ge r_1. \tag{3.13}$$

If $t_0 \in (0,1)$, then $u'''(t_0) = 0$ and $u^{(4)}(t_0) \le 0$. Then, by (3.2) and (3.10), for $\lambda \in (0,1]$, the following contradiction is obtained:

$$\begin{aligned} 0 \ge u^{(4)}(t_0) &= \lambda f(t_0, \delta_0(t_0, u(t_0)), \delta_1(t_0, u'(t_0)), \delta_2(t_0, u''(t_0)), u'''(t_0)) + u''(t_0) - \lambda\delta_2(t_0, u''(t_0)) \\ &= \lambda f(t_0, \delta_0(t_0, u(t_0)), \delta_1(t_0, u'(t_0)), \beta''(t_0), 0) + u''(t_0) - \lambda\beta''(t_0) \\ &\ge \lambda f(t_0, \beta(t_0), \beta'(t_0), \beta''(t_0), 0) + u''(t_0) - \lambda\beta''(t_0) \\ &= \lambda\left[f(t_0, \beta(t_0), \beta'(t_0), \beta''(t_0), 0) + r_1 - \beta''(t_0)\right] + u''(t_0) - \lambda r_1 > 0. \end{aligned} \tag{3.14}$$

For $\lambda = 0$,

$$0 \ge u^{(4)}(t_0) = u''(t_0) \ge r_1 > 0. \tag{3.15}$$

If $t_0 = 0$, then

$$\max_{t\in[0,1]} u''(t) := u''(0) \ge r_1 > 0 \tag{3.16}$$

and $u'''(0^+) = u'''(0) \le 0$. If $\lambda = 0$, then $u'''(0) = 0$ and so $u^{(4)}(0) \le 0$. Therefore, the above computations with t_0 replaced by 0 yield a contradiction. For $\lambda \in (0,1]$, by (3.11), we get the following contradiction:

$$0 \geq u'''(0) = \frac{\lambda}{b}\left[au''(0) - A\right] \geq \frac{\lambda}{b}\left[ar_1 - A\right] > 0. \tag{3.17}$$

The case $t_0 = 1$ is analogous. Thus, $u''(t) < r_1$ for every $t \in [0,1]$. In a similar way, we may prove that $u''(t) > -r_1$ for every $t \in [0,1]$.

By the boundary condition (3.7) there exists a $\xi \in (0,1)$, such that $u'(\xi) = 0$. Then by integration we obtain

$$\begin{aligned} \left|u'(t)\right| &= \left|\int_{\xi}^{t} u''(s)ds\right| < r_1|t-\xi| \leq r_1, \\ |u(t)| &= \left|\int_{0}^{t} u'(s)ds\right| < r_1 t \leq r_1. \end{aligned} \tag{3.18}$$

Step 2. There is an $R > 0$ such that for every solution $u(t)$ of problems (3.6) and (3.7)

$$\left|u'''(t)\right| < R, \quad \forall t \in [0,1], \tag{3.19}$$

with R independent of $\lambda \in [0,1]$.

Consider the set

$$E_{r_1} = \left\{(t, x_0, x_1, x_2, x_3) \in [0,1] \times \mathbb{R}^4 : -r_1 \leq x_i \leq r_1,\ i = 0,1,2\right\} \tag{3.20}$$

and for $\lambda \in [0,1]$ the function $F_\lambda : E_{r_1} \longrightarrow \mathbb{R}$ given by

$$F_\lambda(t, x_0, x_1, x_2, x_3) = \lambda f(t, \delta_0(t, x_0), \delta_1(t, x_1), \delta_2(t, x_2), x_3) + x_2 - \lambda\delta_2(t, x_2). \tag{3.21}$$

In the following we will prove that the function F_λ satisfies the one-sided Nagumo-type conditions (2.3) and (2.4) in E_{r_1} independently of $\lambda \in [0,1]$. Indeed, as f verifies (2.3) in E_*, then

$$\begin{aligned} F_\lambda(t, x_0, x_1, x_2, x_3) &= \lambda f(t, \delta_0(t, x_0), \delta_1(t, x_1), \delta_2(t, x_2), x_3) + x_2 - \lambda\delta_2(t, x_2) \\ &\leq h_{E_*}(|x_3|) + r_1 - \lambda\alpha''(t) \leq h_{E_*}(|x_3|) + 2r_1. \end{aligned} \tag{3.22}$$

So, defining $h_{E_{r_1}}(t) = h_{E_*}(|x_3|) + 2r_1$ in $\mathbb{R}_0^+$, we see that F_λ verifies (2.3) with E and h_E replaced by E_{r_1} and $h_{E_{r_1}}$, respectively. The condition (2.4) is also verified since

$$\int_0^{+\infty} \frac{s}{h_{E_{r_1}}(s)} ds = \int_0^{+\infty} \frac{s}{h_{E_*(s)+2r_1}} ds \geq \frac{1}{1 + 2r_1/k} \int_0^{+\infty} \frac{s}{h_{E_*(s)}} ds = +\infty. \tag{3.23}$$

Therefore, F_λ satisfies the one-sided Nagumo-type condition in E_{r_1} with h_E replaced by $h_{E_{r_1}}$, with r_1 independent of $\lambda \in [0,1]$.

Moreover, for

$$\rho := \max\left\{ \frac{ar_1 + |A|}{b}, \frac{|B| + cr_1}{d} \right\}, \tag{3.24}$$

every solution u of (3.6) and (3.7) satisfies

$$\begin{aligned} u'''(0) &= \frac{\lambda}{b}\left[au''(0) - A\right] \leq \frac{\lambda}{b}\left[ar_1 + |A|\right] \leq \rho, \\ u'''(1) &= \frac{\lambda}{d}\left[B - cu''(1)\right] \geq -\frac{\lambda}{d}\left[|B| + cr_1\right] \geq -\rho. \end{aligned} \tag{3.25}$$

Define

$$\gamma_i(t) := -r_1, \quad \Gamma_i(t) := r_1, \quad \text{for } i = 0, 1, 2. \tag{3.26}$$

The hypotheses of Lemma 2.4 are satisfied with E replaced by E_{r_1}. So there exists an $R > 0$, depending on r_1 and $h_{E_{r_1}}$, such that $|u'''(t)| < R$ for every $t \in [0,1]$. As r_1 and $h_{E_{r_1}}$ do not depend on λ, we see that R is maybe independent of λ.

Step 3. For $\lambda = 1$, the problems (3.6) and (3.7) has at least one solution $u_1(t)$.

Define the operators

$$L : C^4([0,1]) \subset C^3([0,1]) \longrightarrow C([0,1]) \times \mathbb{R}^4 \tag{3.27}$$

by

$$Lu = \left(u^{(4)} - u'', u(0), u(1), u'''(0), u'''(1)\right) \tag{3.28}$$

and for $\lambda \in [0,1]$, $\mathcal{N}_\lambda : C^3([0,1]) \to C([0,1]) \times \mathbb{R}^4$ by

$$\mathcal{N}_\lambda u = \left(\lambda f\left(t, \delta_0(t, u(t)), \delta_1\left(t, u'(t)\right), \delta_2\left(t, u''(t)\right), u'''(t)\right) - \lambda \delta_2\left(t, u''(t)\right), 0, 0, A_\lambda, B_\lambda\right), \tag{3.29}$$

with

$$\begin{aligned} A_\lambda &:= \frac{\lambda}{b}\left[au''(0) - A\right], \\ B_\lambda &:= \frac{\lambda}{d}\left[B - cu''(1)\right]. \end{aligned} \tag{3.30}$$

Observe that L has a compact inverse. Therefore, we can consider the completely continuous operator

$$T_\lambda : \left(C^3([0,1]), \mathbb{R}\right) \longrightarrow \left(C^3([0,1]), \mathbb{R}\right) \tag{3.31}$$

given by

$$T_\lambda(u) = L^{-1}\mathcal{N}_\lambda(u). \tag{3.32}$$

For R given by Step 2, take the set

$$\Omega = \left\{x \in C^3([0,1]) : \left\|x^{(i)}\right\|_\infty < r_1,\ i = 0,1,2,\ \|x'''\|_\infty < R\right\}. \tag{3.33}$$

By Steps 1 and 2, degree $d(I - T_\lambda, \Omega, 0)$ is well defined for every $\lambda \in [0,1]$ and by the invariance with respect to a homotopy

$$d(I - T_0, \Omega, 0) = d(I - T_1, \Omega, 0). \tag{3.34}$$

The equation $x = T_0(x)$ is equivalent to the problem

$$\begin{gathered} u^{(4)}(t) = u''(t), \\ u(0) = u(1) = u'''(0) = u'''(1) = 0 \end{gathered} \tag{3.35}$$

and has only the trivial solution. Then, by the degree theory,

$$d(I - T_0, \Omega, 0) = \pm 1. \tag{3.36}$$

So the equation $T_1(x) = x$ has at least one solution, and therefore the equivalent problem

$$\begin{gathered} u^{(4)}(t) = f\left(t, \delta_0(t, u(t)), \delta_1\left(t, u'(t)\right), \delta_2\left(t, u''(t)\right), u'''(t)\right) + u''(t) - \delta_2\left(t, u''(t)\right), \\ u(0) = u(1) = 0, \\ au''(0) - bu'''(0) = A, \\ cu''(1) + du'''(1) = B \end{gathered} \tag{3.37}$$

has at least one solution $u_1(t)$ in Ω.

Step 4. The function $u_1(\mathrm{t})$ is a solution of the problems (1.1) and (1.2).

The proof will be finished if the above function $u_1(t)$ satisfies the inequalities

$$\alpha(t) \leq u_1(t) \leq \beta(t), \qquad \alpha'(t) \leq u_1'(t) \leq \beta'(t), \qquad \alpha''(t) \leq u_1''(t) \leq \beta''(t). \tag{3.38}$$

Assume, for contradiction, that there is a $\bar{t} \in [0,1]$ such that $u_1''(\bar{t}) > \beta''(\bar{t})$, and define

$$\max_{t\in[0,1]} \left[u_1''(t) - \beta''(t)\right] := u_1''(t_2) - \beta''(t_2) > 0. \tag{3.39}$$

If $t_2 \in (0,1)$, then $u_1'''(t_2) = \beta'''(t_2)$ and $u_1^{(4)}(t_2) \leq \beta^{(4)}(t_2)$. Therefore, by (3.2) and Definition 2.1, we obtain the contradiction

$$\begin{aligned} u_1^{(4)}(t_2) &= f\left(t_2, \delta_0(t_2, u_1(t_2)), \delta_1\left(t_2, u_1'(t_2)\right), \delta_2\left(t_2, u_1''(t_2)\right), u_1'''(t_2)\right) \\ &\quad + u_1''(t_2) - \delta_2\left(t_2, u_1''(t_2)\right) \\ &= f\left(t_2, \delta_0(t_2, u_1(t_2)), \delta_1\left(t_2, u_1'(t_2)\right), \beta''(t_2), \beta'''(t_2)\right) + u_1''(t_2) - \beta''(t_2) \\ &\geq f\left(t_2, \beta(t_2), \beta'(t_2), \beta''(t_2), \beta'''(t_2)\right) \geq \beta^{(4)}(t_2). \end{aligned} \tag{3.40}$$

If $t_2 = 0$, then we have

$$\begin{gathered} \max_{t\in[0,1]} \left[u_1''(t) - \beta''(t)\right] := u_1''(0) - \beta''(0) > 0, \\ u_1'''(0) - \beta'''(0) = u_1'''(0^+) - \beta'''(0^+) \leq 0. \end{gathered} \tag{3.41}$$

By Definition 2.1 this yields a contradiction

$$u_1'''(0) = \frac{1}{b}\left[a u_1''(0) - A\right] > \frac{1}{b}\left[a\beta''(0) - A\right] \geq \beta'''(0). \tag{3.42}$$

Then $t_2 \neq 0$ and, by similar arguments, we prove that $t_2 \neq 1$. Thus,

$$u_1''(t) \leq \beta''(t), \quad \forall t \in [0,1]. \tag{3.43}$$

Using an analogous technique, it can be deduced that $\alpha''(t) \leq u_1''(t)$ for every $t \in [0,1]$. So we have

$$\alpha''(t) \leq u_1''(t) \leq \beta''(t). \tag{3.44}$$

On the other hand, by (1.2),

$$0 = u_1(1) - u_1(0) = \int_0^1 u_1'(t)dt = \int_0^1 \left(u_1'(0) + \int_0^t u_1''(s)ds\right)dt = u_1'(0) + \int_0^1 \int_0^t u_1''(s)ds\, dt, \tag{3.45}$$

that is,

$$u_1'(0) = -\int_0^1 \int_0^t u_1''(s)ds\,dt. \tag{3.46}$$

Applying the same technique, we have

$$-\int_0^1 \int_0^t \beta''(s)ds\,dt = -\int_0^1 \beta'(t)dt + \beta'(0) = \beta(0) - \beta(1) + \beta'(0), \tag{3.47}$$

and then by Definition 2.1 (iii), (3.44) and (3.46), we obtain

$$\begin{aligned}
\alpha'(0) &\le \beta'(0) - \beta(1) + \beta(0) \\
&= -\int_0^1 \int_0^t \beta''(s)ds\,dt \le -\int_0^1 \int_0^t u_1''(s)ds\,dt = u_1'(0), \\
\beta'(0) &\ge \alpha'(0) - \alpha(1) + \alpha(0) \\
&= -\int_0^1 \int_0^t \alpha''(s)ds\,dt \ge -\int_0^1 \int_0^t u_1''(s)ds\,dt = u_1'(0),
\end{aligned} \tag{3.48}$$

that is,

$$\alpha'(0) \le u_1'(0) \le \beta'(0). \tag{3.49}$$

Since, by (3.44), $\beta'(t) - u_1'(t)$ is nondecreasing, we have by (3.49)

$$\beta'(t) - u_1'(t) \ge \beta'(0) - u_1'(0) \ge 0, \tag{3.50}$$

and, therefore, $\beta'(t) \ge u'_1(t)$ for every $t \in [0,1]$. By the monotonicity of $\beta(t) - u_1(t)$,

$$\beta(t) - u_1(t) \ge \beta(0) - u_1(0) = \beta(0) \ge 0, \tag{3.51}$$

and so $\beta(t) \ge u_1(t)$ for every $t \in [0,1]$.

The inequalities $u_1'(t) \ge \alpha'(t)$ and $u_1(t) \ge \alpha(t)$ for every $t \in [0,1]$ can be proved in the same way. Then $u_1(t)$ is a solution of problems (1.1) and (1.2). □

4. An Example

The following example shows the applicability of Theorem 3.1 when f satisfies only the one-sided Nagumo-type condition.

Example 4.1. Consider now the problem

$$u^{(4)}(t) = -[3+u(t)]\left[e^{u'(t)}\right]\left[u''(t)-2\right]^2 - \left[u'''(t)\right]^4, \tag{4.1}$$

$$\begin{gathered} u(0) = u(1) = 0, \\ u''(0) - u'''(0) = A, \\ u''(1) + u'''(1) = B \end{gathered} \tag{4.2}$$

with $A, B \in \mathbb{R}$. The nonlinear function

$$f(t, x_0, x_1, x_2, x_3) = -(3+x_0)e^{x_1}(x_2-2)^2 - (x_3)^4 \tag{4.3}$$

is continuous in $[0,1] \times \mathbb{R}^4$. If $A, B \in [-2,2]$, then the functions $\alpha, \beta : [0,1] \to \mathbb{R}$ defined by

$$\alpha(t) = -t^2 - t, \qquad \beta(t) = t^2 + t \tag{4.4}$$

are, respectively, lower and upper functions of (4.1) and (4.2). Moreover, define

$$E = \left\{(t, x_0, x_1, x_2, x_3) \in [0,1] \times \mathbb{R}^4 : -t^2 - t \le x_0 \le t^2 + t,\ -2t-1 \le x_1 \le 2t+1,\ -2 \le x_2 \le 2\right\}. \tag{4.5}$$

Then f satisfies condition (3.2) and the one-sided Nagumo-type condition with $h_E(|x_3|) = 1$, in E.

Therefore, by Theorem 3.1, there is at least one solution $u(t)$ of Problem (4.1) and (4.2) such that, for every $t \in [0,1]$,

$$-t^2 - t \le u(t) \le t^2 + t, \quad -2t-1 \le u'(t) \le 2t+1, \quad -2 \le u''(t) \le 2. \tag{4.6}$$

Notice that the function

$$f(t, x_0, x_1, x_2, x_3) = -(3+x_0)e^{x_1}(x_2-2)^2 - (x_3)^4 \tag{4.7}$$

does not satisfy the two-sided Nagumo condition.

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