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Research Article

Global Structure of Nodal Solutions for Second-Order *m*-Point Boundary Value Problems with Superlinear Nonlinearities

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We consider the nonlinear eigenvalue problems $u'' + \lambda f(u) = 0$, $0 < t < 1$, $u(0) = 0$, $u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i)$, where $m \geq 3$, $\eta_i \in (0,1)$, and $\alpha_i > 0$ for $i = 1, \ldots, m-2$, with $\sum_{i=1}^{m-2} \alpha_i < 1$, and $f \in C^1(\mathbb{R}\backslash\{0\}, \mathbb{R}) \cap C(\mathbb{R}, \mathbb{R})$ satisfies $f(s)s > 0$ for $s \neq 0$, and $f_0 = \infty$, where $f_0 = \lim_{|s| \to 0} f(s)/s$. We investigate the global structure of nodal solutions by using the Rabinowitz's global bifurcation theorem.

1. Introduction

We study the global structure of nodal solutions of the problem

$$u'' + \lambda f(u) = 0, \qquad t \in (0,1), \tag{1.1}$$

$$u(0) = 0, \qquad u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i). \tag{1.2}$$

Here $m \geq 3$, $\eta_i \in (0,1)$, and $\alpha_i > 0$ for $i = 1, \ldots, m-2$ with $\sum_{i=1}^{m-2} \alpha_i < 1$; λ is a positive parameter, and $f \in C^1(\mathbb{R} \setminus \{0\}, \mathbb{R}) \cap C(\mathbb{R}, \mathbb{R})$.

In the case that $f_0 \in (0, \infty)$, the global structure of nodal solutions of nonlinear second-order m-point eigenvalue problems (1.1), (1.2) have been extensively studied; see [1–5] and the references therein. However, relatively little is known about the global structure of solutions in the case that $f_0 = \infty$, and few global results were found in the available literature when $f_0 = \infty = f_\infty$. The likely reason is that the global bifurcation techniques cannot be used directly in the case. On the other hand, when m-point boundary value condition (1.2) is concerned, the discussion is more difficult since the problem is nonsymmetric and the corresponding operator is disconjugate. In [6], we discussed the global structure of positive solutions of (1.1), (1.2) with $f_0 = \infty$. However, to the best of our knowledge, there is no paper to discuss the global structure of nodal solutions of (1.1), (1.2) with $f_0 = \infty$.

In this paper, we obtain a complete description of the global structure of nodal solutions of (1.1), (1.2) under the following assumptions:

(A1) $\alpha_i > 0$ for $i = 1, \ldots, m-2$, with $0 < \sum_{i=1}^{m-2} \alpha_i < 1$;

(A2) $f \in C^1(\mathbb{R} \setminus \{0\}, \mathbb{R}) \cap C(\mathbb{R}, \mathbb{R})$ satisfies $f(s)s > 0$ for $s \neq 0$;

(A3) $f_0 := \lim_{|s| \to 0} f(s)/s = \infty$;

(A4) $f_\infty := \lim_{|s| \to \infty} f(s)/s \in [0, \infty]$.

Let $Y = C[0,1]$ with the norm

$$\|u\|_\infty = \max_{t \in [0,1]} |u(t)|. \tag{1.3}$$

Let

$$X = \left\{ u \in C^1[0,1] \mid u(0) = 0, u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i) \right\},$$
$$E = \left\{ u \in C^2[0,1] \mid u(0) = 0, u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i) \right\} \tag{1.4}$$

with the norm

$$\|u\|_X = \max\{\|u\|_\infty, \|u'\|_\infty\}, \qquad \|u\| = \max\{\|u\|_\infty, \|u'\|_\infty, \|u''\|_\infty\}, \tag{1.5}$$

respectively. Define $L : E \to Y$ by setting

$$Lu := -u'', \quad u \in E. \tag{1.6}$$

Then L has a bounded inverse $L^{-1} : Y \to E$ and the restriction of L^{-1} to X, that is, $L^{-1} : X \to X$ is a compact and continuous operator; see [1, 2, 6].

For any C^1 function u, if $u(x_0) = 0$, then x_0 is a simple zero of u if $u'(x_0) \neq 0$. For any integer $k \geq 1$ and any $\nu \in \{+, -\}$, define sets S_k^ν, $T_k^\nu \subset C^2[0,1]$ consisting of functions $u \in C^2[0,1]$ satisfying the following conditions:

S_k^ν : (i) $u(0) = 0, \nu u'(0) > 0$,

(ii) u has only simple zeros in $[0,1]$ and has exactly $k-1$ zeros in $(0,1)$;

T_k^ν : (i) $u(0) = 0$, $\nu u'(0) > 0$ and $u'(1) \neq 0$,

(ii) u' has only simple zeros in $(0,1)$ and has exactly k zeros in $(0,1)$,

(iii) u has a zero strictly between each two consecutive zeros of u'.

Remark 1.1. Obviously, if $u \in T_k^\nu$, then $u \in S_k^\nu$ or $u \in S_{k+1}^\nu$. The sets T_k^ν are open in E and disjoint.

Remark 1.2. The nodal properties of solutions of nonlinear Sturm-Liouville problems with separated boundary conditions are usually described in terms of sets similar to S_k^ν; see [7]. However, Rynne [1] stated that T_k^ν are more appropriate than S_k^ν when the multipoint boundary condition (1.2) is considered.

Next, we consider the eigenvalues of the linear problem

$$Lu = \lambda u, \quad u \in E. \tag{1.7}$$

We call the set of eigenvalues of (1.7) the spectrum of L and denote it by $\sigma(L)$. The following lemmas or similar results can be found in [1–3].

Lemma 1.3. *Let (A1) hold. The spectrum $\sigma(L)$ consists of a strictly increasing positive sequence of eigenvalues λ_k, $k = 1, 2, \ldots$, with corresponding eigenfunctions $\varphi_k(x) = \sin(\sqrt{\lambda_k}\, x)$. In addition,*

(i) $\lim_{k\to\infty} \lambda_k = \infty$;

(ii) $\varphi_k \in T_k^+$, *for each* $k \geq 1$, *and* φ_1 *is strictly positive on* $(0, 1)$.

We can regard the inverse operator $L^{-1} : Y \to E$ as an operator $L^{-1} : Y \to Y$. In this setting, each λ_k, $k = 1, 2, \ldots$, is a characteristic value of L^{-1}, with algebraic multiplicity defined to be $\dim \bigcup_{j=1}^{\infty} N((I - \lambda_k L^{-1})^j)$, where N denotes null-space and I is the identity on Y.

Lemma 1.4. *Let (A1) hold. For each $k \geq 1$, the algebraic multiplicity of the characteristic value λ_k, $k = 1, 2, \ldots$, of $L^{-1} : Y \to Y$ is equal to 1.*

Let $\mathbb{E} = \mathbb{R} \times E$ under the product topology. As in [7], we add the points $\{(\lambda, \infty) \mid \lambda \in \mathbb{R}\}$ to our space $\mathbb{E}$. Let $\Phi_k^\nu = \mathbb{R} \times T_k^\nu$. Let Σ_k^ν denote the closure of set of those solutions of (1.1), (1.2) which belong to Φ_k^ν. The main results of this paper are the following.

Theorem 1.5. *Let (A1)–(A4) hold.*

(a) *If $f_\infty = 0$, then there exists a subcontinuum $\mathcal{C}_k^\nu$ of Σ_k^ν with $(0, 0) \in \mathcal{C}_k^\nu$ and*

$$\mathrm{Proj}_{\mathbb{R}} \mathcal{C}_k^\nu = (0, \infty). \tag{1.8}$$

(b) *If $f_\infty \in (0, \infty)$, then there exists a subcontinuum $\mathcal{C}_k^\nu$ of Σ_k^ν with*

$$(0, 0) \in \mathcal{C}_k^\nu, \quad \mathrm{Proj}_{\mathbb{R}} \mathcal{C}_k^\nu \subseteq \left(0, \frac{\lambda_1}{f_\infty}\right). \tag{1.9}$$

(c) *If $f_\infty = \infty$, then there exists a subcontinuum $\mathcal{C}_k^\nu$ of Σ_k^ν with $(0, 0) \in \mathcal{C}_k^\nu$, $\mathrm{Proj}_{\mathbb{R}}\, \mathcal{C}_k^\nu$ is a bounded closed interval, and $\mathcal{C}_k^\nu$ approaches $(0, \infty)$ as $\|u\| \to \infty$.*

Theorem 1.6. *Let (A1)–(A4) hold.*

(a) *If* $f_\infty = 0$, *then* (1.1), (1.2) *has at least one solution in* T_k^ν *for any* $\lambda \in (0, \infty)$.

(b) *If* $f_\infty \in (0, \infty)$, *then* (1.1), (1.2) *has at least one solution in* T_k^ν *for any* $\lambda \in (0, \lambda_1 / f_\infty)$.

(c) *If* $f_\infty = \infty$, *then there exists* $\lambda_* > 0$ *such that* (1.1), (1.2) *has at least two solutions in* T_k^ν *for any* $\lambda \in (0, \lambda_*)$.

We will develop a bifurcation approach to treat the case $f_0 = \infty$. Crucial to this approach is to construct a sequence of functions $\{f^{[n]}\}$ which is asymptotic linear at 0 and satisfies

$$f^{[n]} \longrightarrow f, \quad \left(f^{[n]}\right)_0 \longrightarrow \infty. \tag{1.10}$$

By means of the corresponding auxiliary equations, we obtain a sequence of unbounded components $\{C_k^{\nu[n]}\}$ via Rabinowitz's global bifurcation theorem [8], and this enables us to find unbounded components $\mathcal{C}_k^\nu$ satisfying

$$(0,0) \in \mathcal{C}_k^\nu \subset \limsup C_k^{\nu[n]}. \tag{1.11}$$

The rest of the paper is organized as follows. Section 2 contains some preliminary propositions. In Section 3, we use the global bifurcation theorems to analyse the global behavior of the components of nodal solutions of (1.1), (1.2).

2. Preliminaries

Definition 2.1 (see [9]). Let W be a Banach space and $\{C_n \mid n = 1, 2, \ldots\}$ a family of subsets of W. Then the *superior limit* $\mathfrak{D}$ of $\{C_n\}$ is defined by

$$\mathfrak{D} := \limsup_{n \to \infty} C_n = \{x \in W \mid \exists \{n_i\} \subset \mathbb{N} \text{ and } x_{n_i} \in C_{n_i}, \text{ such that } x_{n_i} \longrightarrow x\}. \tag{2.1}$$

Lemma 2.2 (see [9]). *Each connected subset of metric space W is contained in a component, and each connected component of W is closed.*

Lemma 2.3 (see [6]). *Assume that*

(i) *there exist* $z_n \in C_n$ $n = 1, 2, \ldots$ *and* $z^* \in W$, *such that* $z_n \to z^*$;

(ii) $r_n = \infty$, *where* $r_n = \sup\{\|x\| \mid x \in C_n\}$;

(iii) *for all* $R > 0$, $(\bigcup_{n=1}^\infty C_n) \cap B_R$ *is a relative compact set of* W, *where*

$$B_R = \{x \in W \mid \|x\| \leq R\}. \tag{2.2}$$

Then there exists an unbounded connected component $\mathcal{C}$ in $\mathfrak{D}$ and $z^* \in \mathcal{C}$.

Define the map $T_\lambda : Y \to E$ by

$$T_\lambda u(t) = \lambda \int_0^1 H(t,s) f(u(s)) ds, \tag{2.3}$$

where

$$H(t,s) = G(t,s) + \frac{\sum_{i=1}^{m-2} \alpha_i G(\eta_i, s)}{1 - \sum_{i=1}^{m-2} \alpha_i \eta_i} t, \quad G(t,s) = \begin{cases} (1-t)s, & 0 \le s \le t \le 1, \\ t(1-s), & 0 \le t \le s \le 1. \end{cases} \tag{2.4}$$

It is easy to verify that the following lemma holds.

Lemma 2.4. *Assume that (A1)-(A2) hold. Then $T_\lambda : Y \to E$ is completely continuous.*

For $r > 0$, let

$$\Omega_r = \{u \in Y \mid \|u\|_\infty < r\}. \tag{2.5}$$

Lemma 2.5. *Let (A1)-(A2) hold. If $u \in \partial\Omega_r,\ r > 0$, then*

$$\|T_\lambda u\|_\infty \le \lambda \widehat{M}_r \left(1 + \frac{\sum_{i=1}^{m-2} \alpha_i}{1 - \sum_{i=1}^{m-2} \alpha_i \eta_i}\right) \int_0^1 G(s,s) ds, \tag{2.6}$$

where $\widehat{M}_r = 1 + \max_{0 \le |s| \le r}\{|f(s)|\}$.

Proof. The proof is similar to that of Lemma 3.5 in [6]; we omit it. □

Lemma 2.6. *Let (A1)-(A2) hold, and $\{(\mu_l, y_l)\} \subset \Phi_k^\nu$ is a sequence of solutions of* (1.1), (1.2). *Assume that $\mu_l \le C_0$ for some constant $C_0 > 0$, and $\lim_{l\to\infty}\|y_l\| = \infty$. Then*

$$\lim_{l\to\infty} \|y_l\|_\infty = \infty. \tag{2.7}$$

Proof. From the relation $y_l(t) = \mu_l \int_0^1 H(t,s) f(y_l(s)) ds$, we conclude that $y_1'(t) = \mu_l \int_0^1 H_t(t,s) f(y_l(s)) ds$. Then

$$\|y_l'\|_\infty \le C_0 \left(1 + \frac{\sum_{i=1}^{m-2} \alpha_i}{1 - \sum_{i=1}^{m-2} \alpha_i \eta_i}\right) \int_0^1 |f(y_l(s))| ds, \tag{2.8}$$

which implies that $\{\|y_l'\|_\infty\}$ is bounded whenever $\{\|y_l\|_\infty\}$ is bounded. □

3. Proof of the Main Results

For each $n \in \mathbb{N}$, define $f^{[n]}(s) : \mathbb{R} \to \mathbb{R}$ by

$$f^{[n]}(s) = \begin{cases} f(s), & s \in \left(\frac{1}{n}, \infty\right) \cup \left(-\infty, -\frac{1}{n}\right), \\ nf\left(\frac{1}{n}\right)s, & s \in \left[-\frac{1}{n}, \frac{1}{n}\right]. \end{cases} \tag{3.1}$$

Then $f^{[n]} \in C(\mathbb{R},\mathbb{R}) \cap C^1(\mathbb{R} \setminus \{\pm 1/n\},\mathbb{R})$ with

$$f^{[n]}(s)s > 0, \quad \forall s \neq 0, \qquad \left(f^{[n]}\right)_0 = nf\left(\frac{1}{n}\right). \tag{3.2}$$

By (A3), it follows that

$$\lim_{n\to\infty} \left(f^{[n]}\right)_0 = \infty. \tag{3.3}$$

Now let us consider the auxiliary family of the equations

$$u'' + \lambda f^{[n]}(u) = 0, \quad t \in (0,1), \tag{3.4}$$

$$u(0) = 0, \qquad u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i). \tag{3.5}$$

Lemma 3.1 (see [1, Proposition 4.1]). *Let (A1), (A2) hold. If $(\mu, u) \in \mathbb{E}$ is a nontrivial solution of* (3.4), (3.5)*, then $u \in T_k^{\nu}$ for some k, ν.*

Let $\zeta^{[n]} \in C(\mathbb{R},\mathbb{R})$ be such that

$$f^{[n]}(u) = \left(f^{[n]}\right)_0 u + \zeta^{[n]}(u) = nf\left(\frac{1}{n}\right)u + \zeta^{[n]}(u). \tag{3.6}$$

Note that

$$\lim_{|s|\to 0} \frac{\zeta^{[n]}(s)}{s} = 0. \tag{3.7}$$

Let us consider

$$Lu - \lambda\left(f^{[n]}\right)_0 u = \lambda\zeta^{[n]}(u) \tag{3.8}$$

as a bifurcation problem from the trivial solution $u \equiv 0$.

Equation (3.8) can be converted to the equivalent equation

$$u(t) = \int_0^1 H(t,s)\left[\lambda\left(f^{[n]}\right)_0 u(s) + \lambda\zeta^{[n]}(u(s))\right]ds$$
$$:= \lambda L^{-1}\left[\left(f^{[n]}\right)_0 u(\cdot)\right](t) + \lambda L^{-1}\left[\zeta^{[n]}(u(\cdot))\right](t). \tag{3.9}$$

Further we note that $\|L^{-1}[\zeta^{[n]}(u)]\| = o(\|u\|)$ for u near 0 in E.

The results of Rabinowitz [8] for (3.8) can be stated as follows. For each integer $k \geq 1$, $\nu \in \{+,-\}$, there exists a continuum $\{C_k^{\nu[n]}\}$ of solutions of (3.8) joining $(\lambda_k/(f^{[n]})_0, 0)$ to infinity in $\mathbb{E}$. Moreover, $\{C_k^{\nu[n]}\} \setminus \{(\lambda_k/(f^{[n]})_0, 0)\} \subset \Phi_k^\nu$.

Proof of Theorem 1.5. Let us verify that $\{C_k^{\nu[n]}\}$ satisfies all of the conditions of Lemma 2.3.

Since

$$\lim_{n\to\infty}\frac{\lambda_k}{\left(f^{[n]}\right)_0} = \lim_{n\to\infty}\frac{\lambda_k}{nf(1/n)} = 0, \tag{3.10}$$

condition (i) in Lemma 2.3 is satisfied with $z^* = (0,0)$. Obviously

$$r_n = \sup\left\{\ \lambda + \|y\| \mid (\lambda, y) \in \mathcal{C}_k^{\nu[n]}\right\} = \infty, \tag{3.11}$$

and accordingly, (ii) holds. (iii) can be deduced directly from the Arzela-Ascoli Theorem and the definition of $f^{[n]}$. Therefore, the superior limit of $\{C_k^{\nu[n]}\}$, $\mathfrak{D}_k^\nu$, contains an unbounded connected component $\mathcal{C}_k^\nu$ with $(0,0) \in \mathcal{C}_k^\nu$.

From the condition (A2), applying Lemma 2.2 with $p = 2$ in [10], we can show that the initial value problem

$$v'' + \lambda f(v) = 0, \quad t \in (0,1),$$
$$v(t_0) = 0, \quad v(t_0) = \beta \tag{3.12}$$

has a unique solution on $[0,1]$ for every $t_0 \in [0,1]$ and $\beta \in \mathbb{R}$. Therefore, any nontrivial solution u of (1.1), (1.2) has only simple zeros in $(0,1)$ and $u'(0) \neq 0$. Meanwhile, (A1) implies that $u'(1) \neq 0$ [1, proposition 4.1]. Since $\mathcal{C}_k^{\nu[n]} \subset \Phi_k^\nu$, we conclude that $\mathcal{C}_k^\nu \subset \Phi_k^\nu$. Moreover, $\mathcal{C}_k^\nu \subset \Sigma_k^\nu$ by (1.1) and (1.2).

We divide the proof into three cases.

Case 1 ($f_\infty = 0$). In this case, we show that $\mathrm{Proj}_{\mathbb{R}}\mathcal{C}_k^\nu = [0,\infty)$.

Assume on the contrary that

$$\sup\{\lambda \mid (\lambda, u) \in \mathcal{C}_k^\nu\} < \infty, \tag{3.13}$$

then there exists a sequence $\{(\mu_l, y_l)\} \subset \mathcal{C}_k^{\nu}$ such that

$$\lim_{l\to\infty} \|y_l\| = \infty, \qquad \mu_l \le C_0, \tag{3.14}$$

for some positive constant C_0 depending not on l. From Lemma 2.6, we have

$$\lim_{l\to\infty} \|y_l\|_\infty = \infty. \tag{3.15}$$

Set $v_l(t) = y_l(t)/\|y_l\|_\infty$. Then $\|v_l\|_\infty = 1$. Now, choosing a subsequence and relabelling if necessary, it follows that there exists $(\mu_*, v_*) \in [0, C_0] \times E$ with

$$\|v_*\|_\infty = 1, \tag{3.16}$$

such that

$$\lim_{l\to\infty} (\mu_l, v_l) = (\mu_*, v_*), \quad \text{in } \mathbb{R} \times E. \tag{3.17}$$

Since $\lim_{|u|\to\infty} f(u)/u = 0$, we can show that

$$\lim_{l\to\infty} \frac{|f(y_l(t))|}{\|y_l\|_\infty} = 0. \tag{3.18}$$

The proof is similar to that of the step 1 of Theorem 1 in [7]; we omit it. So, we obtain

$$v_*''(t) + \mu_* \cdot 0 = 0, \quad t \in (0,1), \tag{3.19}$$

$$v_*(0) = 0, \qquad v_*(1) = \sum_{i=1}^{m-2} \alpha_i v_*(\eta_i), \tag{3.20}$$

and subsequently, $v_*(t) \equiv 0$ for $t \in [0,1]$. This contradicts (3.16). Therefore

$$\sup\{\lambda \mid (\lambda, y) \in \mathcal{C}_k^{\nu}\} = \infty. \tag{3.21}$$

Case 2 ($f_\infty \in (0,\infty)$). In this case, we can show easily that $\mathcal{C}$ joins $(0,0)$ with $(\lambda_k/f_\infty, \infty)$ by using the same method used to prove Theorem 5.1 in [2].

Case 3 ($f_\infty = \infty$). In this case, we show that $\mathcal{C}_k^{\nu}$ joins $(0,0)$ with $(0,\infty)$.

Let $\{(\mu_l, y_l)\} \subset \mathcal{C}_k^{\nu}$ be such that

$$\mu_l + \|y_l\| \longrightarrow \infty, \quad l \longrightarrow \infty. \tag{3.22}$$

If $\{\|y_l\|\}$ is bounded, say, $\|y_l\| \le M_1$, for some M_1 depending not on l, then we may assume that

$$\lim_{l\to\infty} \mu_l = \infty. \tag{3.23}$$

Taking subsequences again if necessary, we still denote $\{(\mu_l, y_l)\}$ such that $\{y_l\} \subset T_k^\nu \cap S_k^\nu$. If $\{y_l\} \subset T_k^\nu \cap S_{k+1}^\nu$, all the following proofs are similar.

Let

$$0 = \tau_l^0 < \tau_l^1 < \cdots < \tau_l^{k-1} \tag{3.24}$$

denote the zeros of y_l in $[0,1]$. Then, after taking a subsequence if necessary, $\lim_{l\to\infty}\tau_l^j := \tau_\infty^j$, $j \in \{0,1,\ldots,k-1\}$. Clearly, $\tau_\infty^0 = 0$. Set $\tau_\infty^k = 1$. We can choose at least one subinterval $(\tau_\infty^j, \tau_\infty^{j+1}) \triangleq I_\infty^j$ which is of length at least $1/k$ for some $j \in \{0,1,\ldots,k-1\}$. Then, for this j, $\tau_l^{j+1} - \tau_l^j > 3/4k$ if l is large enough. Put $(\tau_l^j, \tau_l^{j+1}) \triangleq I_l^j$.

Obviously, for the above given k, ν and j, $y_l(t)$ have the same sign on I_l^j for all l. Without loss of generality, we assume that

$$y_l(t) > 0, \quad t \in I_l^j. \tag{3.25}$$

Moreover, we have

$$\max_{t\in I_l^j} |u_l(t)| \le M_1. \tag{3.26}$$

Combining this with the fact

$$\frac{f(y_l(t))}{y_l(t)} \ge \inf\left\{ \frac{f(s)}{s} \mid 0 < s \le M_1 \right\} > 0, \quad t \in \left(\tau_l^j, \tau_l^{j+1}\right), \tag{3.27}$$

and using the relation

$$y_l''(t) + \mu_l \frac{f(y_l(t))}{y_l(t)} y_l(t) = 0, \quad t \in \left(\tau_l^j, \tau_l^{j+1}\right), \tag{3.28}$$

we deduce that y_l must change its sign on (τ_l^j, τ_l^{j+1}) if l is large enough. This is a contradiction. Hence $\{\|y_l\|\}$ is unbounded. From Lemma 2.6, we have that

$$\lim_{l\to\infty} \|y_l\|_\infty = \infty. \tag{3.29}$$

Note that $\{(\mu_l, y_l)\}$ satisfies the autonomous equation

$$y_l'' + \mu_l f(y_l) = 0, \quad t \in (0,1). \tag{3.30}$$

We see that y_l consists of a sequence of positive and negative bumps, together with a truncated bump at the right end of the interval $[0,1]$, with the following properties (ignoring the truncated bump) (see, [1]):

(i) all the positive (resp., negative) bumps have the same shape (the shapes of the positive and negative bumps may be different);

(ii) each bump contains a single zero of y_l', and there is exactly one zero of y_l between consecutive zeros of y_l';

(iii) all the positive (negative) bumps attain the same maximum (minimum) value.

Armed with this information on the shape of y_l, it is easy to show that for the above given I_l^j, $\{\|y_l\|_{I_l^j,\infty} := \max_{I_l^j} y_l(t)\}_{l=1}^{\infty}$ is an unbounded sequence. That is

$$\lim_{l\to\infty} \|y_l\|_{I_l^j,\infty} = \infty. \tag{3.31}$$

Since y_l is concave on I_l^j, for any $\sigma > 0$ small enough,

$$y_l(t) \geq \sigma \|y_l\|_{I_l^j,\infty}, \quad \forall t \in \left[\tau_l^j + \sigma, \tau_l^{j+1} - \sigma\right]. \tag{3.32}$$

This together with (3.31) implies that there exist constants α, β with $[\alpha,\beta] \subset I_\infty^j$, such that

$$\lim_{l\to\infty} y_l(t) = \infty, \quad \text{uniformly for } t \in [\alpha,\beta]. \tag{3.33}$$

Hence, we have

$$\lim_{l\to\infty} \frac{f(y_l(t))}{y_l(t)} = \infty, \quad \text{uniformly for } t \in [\alpha,\beta]. \tag{3.34}$$

Now, we show that $\lim_{l\to\infty}\mu_l = 0$.

Suppose on the contrary that, choosing a subsequence and relabeling if necessary, $\mu_l \geq b_0$ for some constant $b_0 > 0$. This implies that

$$\lim_{l\to\infty} \mu_l \frac{f(y_l(t))}{y_l(t)} = \infty, \quad \text{uniformly for } t \in [\alpha,\beta]. \tag{3.35}$$

From (3.28) we obtain that y_l must change its sign on $[\alpha,\beta]$ if l is large enough. This is a contradiction. Therefore $\lim_{l\to\infty}\mu_l = 0$. □

Proof of Theorem 1.6. (a) and (b) are immediate consequence of Theorem 1.5(a) and (b), respectively.

To prove (c), we rewrite (1.1), (1.2) to

$$u = \lambda \int_0^1 H(t,s) f(u(s))\,ds = T_\lambda u(t). \tag{3.36}$$

By Lemma 2.5, for every $r > 0$ and $u \in \partial\Omega_r$,

$$\|T_\lambda u\|_\infty \leq \lambda \widehat{M}_r \left(1 + \frac{\sum_{i=1}^{m-2} \alpha_i}{1 - \sum_{i=1}^{m-2} \alpha_i \eta_i}\right) \int_0^1 G(s,s)ds, \tag{3.37}$$

where $\widehat{M}_r = 1 + \max_{0 \leq |s| \leq r}\{|f(s)|\}$.

Let $\lambda_r > 0$ be such that

$$\lambda_r \widehat{M}_r \left(1 + \frac{\sum_{i=1}^{m-2} \alpha_i}{1 - \sum_{i=1}^{m-2} \alpha_i \eta_i}\right) \int_0^1 G(s,s)ds = r. \tag{3.38}$$

Then for $\lambda \in (0, \lambda_r)$ and $u \in \partial\Omega_r$,

$$\|T_\lambda u\|_\infty < \|u\|_\infty. \tag{3.39}$$

This means that

$$\Sigma_k^\nu \cap \{(\lambda, u) \in (0,\infty) \times E \mid 0 < \lambda < \lambda_r, u \in E : \|u\|_\infty = r\} = \emptyset. \tag{3.40}$$

By Lemma 2.6 and Theorem 1.5, it follows that $\mathcal{C}_k^\nu$ is also an unbounded component joining $(0,0)$ and $(0,\infty)$ in $[0,\infty) \times Y$. Thus, (3.40) implies that for $\lambda \in (0, \lambda_r)$, (1.1), (1.2) has at least two solutions in T_k^ν. □

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