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Research Article

Positive Solutions for Integral Boundary Value Problem with ϕ-Laplacian Operator

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We consider the existence, multiplicity of positive solutions for the integral boundary value problem with ϕ-Laplacian $(\phi(u'(t)))' + f(t,u(t),u'(t)) = 0$, $t \in [0,1]$, $u(0) = \int_0^1 u(r)g(r)\mathrm{d}r$, $u(1) = \int_0^1 u(r)h(r)\mathrm{d}r$, where ϕ is an odd, increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$. We show that it has at least one, two, or three positive solutions under some assumptions by applying fixed point theorems. The interesting point is that the nonlinear term f is involved with the first-order derivative explicitly.

1. Introduction

We are interested in the existence of positive solutions for the integral boundary value problem

$$\begin{gathered}(\phi(u'(t)))' + f(t,u(t),u'(t)) = 0, \quad t \in [0,1], \\ u(0) = \int_0^1 u(r)g(r)\mathrm{d}r, \qquad u(1) = \int_0^1 u(r)h(r)\mathrm{d}r,\end{gathered} \tag{1.1}$$

where ϕ, f, g, and h satisfy the following conditions.

(H1) ϕ is an odd, increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$, and there exist two increasing homeomorphisms ψ_1 and ψ_2 of $(0,\infty)$ onto $(0,\infty)$ such that

$$\psi_1(u)\phi(v) \le \phi(uv) \le \psi_2(u)\phi(v) \quad \forall u, v > 0. \tag{1.2}$$

Moreover, $\phi, \phi^{-1} \in C^1(R)$, where ϕ^{-1} denotes the inverse of ϕ.

(H2) $f : [0,1] \times [0,+\infty) \times (-\infty,+\infty) \to (0,+\infty)$ is continuous. $g, h \in L^1[0,1]$ are nonnegative, and $0 < \int_0^1 g(t)\mathrm{d}t < 1$, $0 < \int_0^1 h(t)\mathrm{d}t < 1$.

The assumption (H1) on the function ϕ was first introduced by Wang [1, 2], it covers two important cases: $\phi(u) = u$ and $\phi(u) = |u|^{p-2}u$, $p > 1$. The existence of positive solutions for two above cases received wide attention (see [3–10]). For example, Ji and Ge [4] studied the multiplicity of positive solutions for the multipoint boundary value problem

$$\begin{gathered} \left(\phi_p\left(u'(t)\right)\right)' + q(t)f\left(t, u(t), u'(t)\right) = 0, \quad t \in (0,1), \\ u(0) = \sum_{i=1}^{m} \alpha_i u(\xi_i), \qquad u(1) = \sum_{i=1}^{m} \beta_i u(\xi_i), \end{gathered} \tag{1.3}$$

where $\phi_p(s) = |s|^{p-2}s$, $p > 1$. They provided sufficient conditions for the existence of at least three positive solutions by using Avery-Peterson fixed point theorem. In [5], Feng et al. researched the boundary value problem

$$\begin{gathered} \left(\phi_p\left(u'(t)\right)\right)' + q(t)f(t, u(t)) = 0, \quad t \in (0,1), \\ u(0) = \sum_{i=1}^{m-2} a_i u(\xi_i), \qquad u(1) = \sum_{i=1}^{m-2} b_i u(\xi_i), \end{gathered} \tag{1.4}$$

where the nonlinear term f does not depend on the first-order derivative and $\phi_p(s) = |s|^{p-2}s$, $p > 1$. They obtained at least one or two positive solutions under some assumptions imposed on the nonlinearity of f by applying Krasnoselskii fixed point theorem.

As for integral boundary value problem, when $\phi(u) = u$ is linear, the existence of positive solutions has been obtained (see [8–10]). In [8], the author investigated the positive solutions for the integral boundary value problem

$$\begin{gathered} u'' + f(u) = 0, \\ u(0) = \int_0^1 u(\tau)\mathrm{d}\alpha(\tau), \qquad u(1) = \int_0^1 u(\tau)\mathrm{d}\beta(\tau). \end{gathered} \tag{1.5}$$

The main tools are the priori estimate method and the Leray-Schauder fixed point theorem. However, there are few papers dealing with the existence of positive solutions when ϕ satisfies (H1) and f depends on both u and u'. This paper fills this gap in the literature. The aim of this paper is to establish some simple criteria for the existence of positive solutions of BVP(1.1). To get rid of the difficulty of f depending on u', we will define a special norm in Banach space (in Section 2).

This paper is organized as follows. In Section 2, we present some lemmas that are used to prove our main results. In Section 3, the existence of one or two positive solutions for BVP(1.1) is established by applying the Krasnoselskii fixed point theorem. In Section 4, we give the existence of three positive solutions for BVP(1.1) by using a new fixed point theorem introduced by Avery and Peterson. In Section 5, we give some examples to illustrate our main results.

2. Preliminaries

The basic space used in this paper is a real Banach space $C^1[0,1]$ with norm $\|\cdot\|_1$ defined by $\|u\|_1 = \max\{\|u\|_c, \|u'\|_c\}$, where $\|u\|_c = \max_{0\le t\le 1}|u(t)|$. Let

$$K = \left\{ u \in C^1[0,1] \mid u(t) \ge 0,\ u(1) = \int_0^1 u(t)h(t)\mathrm{d}t,\ u \text{ is concave on } [0,1] \right\}. \tag{2.1}$$

It is obvious that K is a cone in $C^1[0,1]$.

Lemma 2.1 (see [7]). *Let $u \in K, \eta \in (0,1/2)$, then $u(t) \ge \eta \max_{0\le t\le 1}|u(t)|$, $t \in [\eta, 1-\eta]$.*

Lemma 2.2. *Let $u \in K$, then there exists a constant $M > 0$ such that* $\max_{0\le t\le 1}|u(t)| \le M\max_{0\le t\le 1}|u'(t)|$.

Proof. The mean value theorem guarantees that there exists $\tau \in [0,1]$, such that

$$u(1) = u(\tau)\int_0^1 h(t)\mathrm{d}t. \tag{2.2}$$

Moreover, the mean value theorem of differential guarantees that there exists $\sigma \in [\tau, 1]$, such that

$$\left(\int_0^1 h(t)\mathrm{d}t - 1\right)u(\tau) = u(1) - u(\tau) = (1-\tau)u'(\sigma). \tag{2.3}$$

So we have

$$|u(t)| \le |u(\tau)| + \left|\int_\tau^t u'(s)\mathrm{d}s\right| \le \left(\frac{1-\tau}{1-\int_0^1 h(t)\mathrm{d}t} + 1\right)\max_{0\le t\le 1}\left|u'(t)\right| \le \frac{2-\int_0^1 h(t)\mathrm{d}t}{1-\int_0^1 h(t)\mathrm{d}t}\max_{0\le t\le 1}\left|u'(t)\right|. \tag{2.4}$$

Denote $M = (2-\int_0^1 h(t)\mathrm{d}t)/(1-\int_0^1 h(t)\mathrm{d}t)$; then the proof is complete. □

Lemma 2.3. *Assume that (H1), (H2) hold. If u is a solution of BVP(1.1), there exists a unique $\delta \in (0,1)$, such that $u'(\delta) = 0$ and $u(t) \ge 0$, $t \in [0,1]$.*

Proof. From the fact that $(\phi(u'))' = -f(t, u(t), u'(t)) < 0$, we know that $\phi(u'(t))$ is strictly decreasing. It follows that $u'(t)$ is also strictly decreasing. Thus, $u(t)$ is strictly concave on [0, 1]. Without loss of generality, we assume that $u(0) = \min\{u(0), u(1)\}$. By the concavity of u, we know that $u(t) \ge u(0)$, $t \in (0,1]$. So we get $u(0) = \int_0^1 u(t)g(t)\mathrm{d}t \ge u(0)\int_0^1 g(t)\mathrm{d}t$. By $0 < \int_0^1 g(t)\mathrm{d}t < 1$, it is obvious that $u(0) \ge 0$. Hence, $u(t) \ge 0$, $t \in [0,1]$.

On the other hand, from the concavity of u, we know that there exists a unique δ where the maximum is attained. By the boundary conditions and $u(t) \ge 0$, we know that $\delta \ne 0$ or 1, that is, $\delta \in (0,1)$ such that $u(\delta) = \max_{0\le t\le 1}u(t)$ and then $u'(\delta) = 0$. □

Lemma 2.4. *Assume that (H1), (H2) hold. Suppose u is a solution of BVP(1.1); then*

$$\begin{aligned} u(t) &= \frac{1}{1-\int_0^1 g(r)\mathrm{d}r}\int_0^1 g(r)\int_0^r \phi^{-1}\left(\int_s^\delta f(\tau,u(\tau),u'(\tau))\mathrm{d}\tau\right)\mathrm{d}s\,\mathrm{d}r \\ &\quad + \int_0^t \phi^{-1}\left(\int_s^\delta f(\tau,u(\tau),u'(\tau))\mathrm{d}\tau\right)\mathrm{d}s \end{aligned} \tag{2.5}$$

or

$$\begin{aligned} u(t) &= \frac{1}{1-\int_0^1 h(r)\mathrm{d}r}\int_0^1 h(r)\int_r^1 \phi^{-1}\left(\int_\delta^s f(\tau,u(\tau),u'(\tau))\mathrm{d}\tau\right)\mathrm{d}s\,\mathrm{d}r \\ &\quad + \int_t^1 \phi^{-1}\left(\int_\delta^s f(\tau,u(\tau),u'(\tau))d\tau\right)\mathrm{d}s. \end{aligned} \tag{2.6}$$

Proof. First, by integrating (1.1) on $[0,t]$, we have

$$\phi(u'(t)) = \phi(u'(0)) - \int_0^t f(s,u(s),u'(s))\mathrm{d}s, \tag{2.7}$$

then

$$u'(t) = \phi^{-1}\left(\phi(u'(0)) - \int_0^t f(s,u(s),u'(s))\mathrm{d}s\right). \tag{2.8}$$

Thus

$$u(t) = u(0) + \int_0^t \phi^{-1}\left(\phi(u'(0)) - \int_0^s f(\tau,u(\tau),u'(\tau))\mathrm{d}\tau\right)\mathrm{d}s \tag{2.9}$$

or

$$u(t) = u(1) - \int_t^1 \phi^{-1}\left(\phi(u'(0)) - \int_0^s f(\tau,u(\tau),u'(\tau))\mathrm{d}\tau\right)\mathrm{d}s. \tag{2.10}$$

According to the boundary condition, we have

$$\begin{aligned} u(0) &= \frac{1}{1-\int_0^1 g(r)\mathrm{d}r}\int_0^1 g(r)\int_0^r \phi^{-1}\left(\phi(u'(0)) - \int_0^s f(\tau,u(\tau),u'(\tau))\mathrm{d}\tau\right)\mathrm{d}s\,\mathrm{d}r, \\ u(1) &= -\frac{1}{1-\int_0^1 h(r)\mathrm{d}r}\int_0^1 h(r)\int_r^1 \phi^{-1}\left(\phi(u'(0)) - \int_0^s f(\tau,u(\tau),u'(\tau))\mathrm{d}\tau\right)\mathrm{d}s\,\mathrm{d}r. \end{aligned} \tag{2.11}$$

By a similar argument in [5], $\phi(u'(0)) = \int_0^\delta f(\tau, u(\tau), u'(\tau))d\tau$; then the proof is completed. □

Now we define an operator T by

$$Tu(t) = \begin{cases} \dfrac{1}{1-\int_0^1 g(r)dr}\displaystyle\int_0^1 g(r)\int_0^r \phi^{-1}\left(\int_s^\delta f(\tau, u(\tau), u'(\tau))d\tau\right)ds\,dr \\ \quad +\displaystyle\int_0^t \phi^{-1}\left(\int_s^\delta f(\tau, u(\tau), u'(\tau))d\tau\right)ds, & 0 \le t \le \delta, \\ \dfrac{1}{1-\int_0^1 h(r)dr}\displaystyle\int_0^1 h(r)\int_r^1 \phi^{-1}\left(\int_\delta^s f(\tau, u(\tau), u'(\tau))d\tau\right)ds\,dr \\ \quad +\displaystyle\int_t^1 \phi^{-1}\left(\int_\delta^s f(\tau, u(\tau), u'(\tau))d\tau\right)ds, & \delta \le t \le 1. \end{cases} \tag{2.12}$$

Lemma 2.5. *$T : K \to K$ is completely continuous.*

Proof. Let $u \in K$; then from the definition of T, we have

$$(Tu)'(t) = \begin{cases} \phi^{-1}\left(\displaystyle\int_t^\delta f(\tau, u(\tau), u'(\tau))d\tau\right) \ge 0, & 0 \le t \le \delta, \\ -\phi^{-1}\left(\displaystyle\int_\delta^t f(\tau, u(\tau), u'(\tau))d\tau\right) \le 0, & \delta \le t \le 1. \end{cases} \tag{2.13}$$

So $(Tu)'(t)$ is monotone decreasing continuous and $(Tu)'(\delta) = 0$. Hence, $(Tu)(t)$ is nonnegative and concave on $[0, 1]$. By computation, we can get $Tu(1) = \int_0^1 Tu(t)h(t)dt$. This shows that $T(\mathrm{K}) \subset K$. The continuity of T is obvious since ϕ^{-1}, f is continuous. Next, we prove that T is compact on $C^1[0,1]$.

Let D be a bounded subset of K and $m > 0$ is a constant such that $\int_0^1 f(\tau, u(\tau), u'(\tau))d\tau < m$ for $u \in D$. From the definition of T, for any $u \in D$, we get

$$|Tu(t)| < \begin{cases} \dfrac{\phi^{-1}(m)}{1-\int_0^1 g(r)dr}, & 0 \le t \le \delta, \\ \dfrac{\phi^{-1}(m)}{1-\int_0^1 h(r)dr}, & \delta \le t \le 1, \end{cases} \tag{2.14}$$

$$\left|(Tu)'(t)\right| < \phi^{-1}(m), \quad 0 \le t \le 1.$$

Hence, TD is uniformly bounded and equicontinuous. So we have that TD is compact on $C[0,1]$. From (2.13), we know for $\forall \varepsilon > 0$, $\exists \kappa > 0$, such that when $|t_1 - t_2| < \kappa$, we have

$|\phi(Tu)'(t_1) - \phi(Tu)'(t_2)| < \varepsilon$. So $\phi(TD)'$ is compact on $C[0,1]$; it follows that $(TD)'$ is compact on $C[0,1]$. Therefore, TD is compact on $C^1[0,1]$.

Thus, $T : K \to K$ is completely continuous. □

It is easy to prove that each fixed point of T is a solution for BVP(1.1).

Lemma 2.6 (see [1]). *Assume that (H1) holds. Then for $u, v \in (0,\infty)$,*

$$\psi_2^{-1}(u)v \le \phi^{-1}\big(u\phi(v)\big) \le \psi_1^{-1}(u)v. \tag{2.15}$$

To obtain positive solution for BVP(1.1), the following definitions and fixed point theorems in a cone are very useful.

Definition 2.7. The map α is said to be a nonnegative continuous concave functional on a cone of a real Banach space E provided that $\alpha : K \to [0,\infty)$ is continuous and

$$\alpha\big(tx + (1-t)y\big) \ge t\alpha(x) + (1-t)\alpha\big(y\big) \tag{2.16}$$

for all $x, y \in K$ and $0 \le t \le 1$. Similarly, we say the map γ is a nonnegative continuous convex functional on a cone of a real Banach space E provided that $\gamma : K \to [0,\infty)$ is continuous and

$$\gamma\big(tx + (1-t)y\big) \le t\gamma(x) + (1-t)\gamma\big(y\big) \tag{2.17}$$

for all $x, y \in K$ and $0 \le t \le 1$.

Let γ and θ be a nonnegative continuous convex functionals on K, α a nonnegative continuous concave functional on K, and ψ a nonnegative continuous functional on K. Then for positive real number a, b, c, and d, we define the following convex sets:

$$\begin{gathered} P\big(\gamma, d\big) = \big\{u \in K \mid \gamma(u) < d\big\}, \\ P\big(\gamma, \alpha, b, d\big) = \big\{u \in K \mid \alpha(u) \ge b,\ \gamma(u) \le d\big\}, \\ P\big(\gamma, \theta, \alpha, b, c, d\big) = \big\{u \in K \mid \alpha(u) \ge b,\ \theta(u) \le c,\ \gamma(u) \le d\big\}, \\ R\big(\gamma, \psi, a, d\big) = \big\{u \in K \mid \psi(u) \ge a,\ \gamma(u) \le d\big\}. \end{gathered} \tag{2.18}$$

Theorem 2.8 (see [11]). *Let E be a real Banach space and $K \subset E$ a cone. Assume that Ω_1 and Ω_2 are two bounded open sets in E with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$. Let $T : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$ be completely continuous. Suppose that one of following two conditions is satisfied:*

(1) $\|Tu\| \le \|u\|$, $u \in K \cap \partial\Omega_1$, *and* $\|Tu\| \ge \|u\|$, $u \in K \cap \partial\Omega_2$*;*

(2) $\|Tu\| \ge \|u\|$, $u \in K \cap \partial\Omega_1$, *and* $\|Tu\| \le \|u\|$, $u \in K \cap \partial\Omega_2$.

Then T has at least one fixed point in $\overline{\Omega}_2 \setminus \Omega_1$.

Theorem 2.9 (see [12]). *Let K be a cone in a real Banach space E. Let γ and θ be a nonnegative continuous convex functionals on K, α a nonnegative continuous concave functional on K, and ψ*

a nonnegative continuous functional on K satisfying $\psi(\lambda u) \le \lambda\psi(u)$ for $0 \le \lambda \le 1$, such that for positive number M and d,

$$\alpha(u) \le \psi(u), \quad \|u\| \le M\gamma(u) \tag{2.19}$$

for all $u \in \overline{P(\gamma,d)}$. Suppose $T : \overline{P(\gamma,d)} \to \overline{P(\gamma,d)}$ is completely continuous and there exist positive numbers a, b, and c with $a < b$ such that

(S1) *$\{u \in P(\gamma,\theta,\alpha,b,c,d) \mid \alpha(u) > b\} \ne \emptyset$ and $\alpha(Tu) > b$ for $u \in P(\gamma,\theta,\alpha,b,c,d)$;*

(S2) *$\alpha(Tu) > b$ for $u \in P(\gamma,\alpha,b,d)$ with $\theta(Tu) > c$;*

(S3) *$0 \notin R(\gamma,\psi,a,d)$ and $\psi(Tu) < a$ for $u \in R(\gamma,\psi,a,d)$ with $\psi(u) = a$.*

Then T has at least three fixed points $u_1, u_2, u_3 \in \overline{P(\gamma,d)}$, such that

$\gamma(u_i) \le d$ *for* $i = 1,2,3$,

$\alpha(u_1) > b$,

$\psi(u_2) > a$ *with* $\alpha(u_2) < b$,

$\psi(u_3) < a$.

3. The Existence of One or Two Positive Solutions

For convenience, we denote

$$L = \max\left\{\frac{\int_0^1 \varphi_1^{-1}(1-s)\mathrm{d}s}{1-\int_0^1 g(s)\mathrm{d}s}, 1\right\}, \qquad N = \min\left\{\int_0^{1/2} \varphi_2^{-1}\left(\frac{1}{2}-s\right)\mathrm{d}s, \int_{1/2}^1 \varphi_2^{-1}\left(s-\frac{1}{2}\right)\mathrm{d}s\right\},$$
$$f^{\mu} = \limsup_{\|u\|_c+\|v\|_c\to\mu} \max_{t\in[0,1]} \frac{f(t,u(t),v(t))}{\phi(\|u\|_c+\|v\|_c)}, \qquad f_{\mu} = \liminf_{\|u\|_c+\|v\|_c\to\mu} \min_{t\in[0,1]} \frac{f(t,u(t),v(t))}{\phi(\|u\|_c+\|v\|_c)}, \tag{3.1}$$

where μ denotes 0 or ∞.

Theorem 3.1. *Assume that (H1) and (H2) hold. In addition, suppose that one of following conditions is satisfied.*

(i) *There exist two constants r, R with $0 < r < (N/L)R$ such that*

(a) *$f(t,u,v) \ge \phi(r/N)$ for $(t,u,v) \in [0,1]\times[0,r]\times[-r,r]$ and*

(b) *$f(t,u,v) \le \phi(R/L)$ for $(t,u,v) \in [0,1]\times[0,R]\times[-R,R]$;*

(ii) *$f^{\infty} < \varphi_1(1/2L)$, $f_0 > \varphi_2(1/N)$;*

(iii) *$f^{0} < \varphi_1(1/2L)$, $f_{\infty} > \varphi_2(1/N)$.*

Then BVP(1.1) has at least one positive solution.

Proof. (i) Let $\Omega_1 = \{u \in K \mid \|u\|_1 < r\}$, $\Omega_2 = \{u \in K \mid \|u\|_1 < R\}$.

For $u \in \partial\Omega_1$, we obtain $u \in [0, r]$ and $u' \in [-r, r]$, which implies $f(t, u, u') \geq \phi(r/N)$. Hence, by (2.12) and Lemma 2.6,

$$\begin{aligned}
\|Tu\|_c &= \max_{0\leq t\leq 1}|Tu(t)| \\
&= \frac{1}{1-\int_0^1 g(r)\mathrm{d}r}\int_0^1 g(r)\int_0^r \phi^{-1}\left(\int_s^\delta f(\tau,u(\tau),u'(\tau))\mathrm{d}\tau\right)\mathrm{d}s\,\mathrm{d}r \\
&\quad + \int_0^\delta \phi^{-1}\left(\int_s^\delta f(\tau,u(\tau),u'(\tau))\mathrm{d}\tau\right)\mathrm{d}s \\
&= \frac{1}{1-\int_0^1 h(r)\mathrm{d}r}\int_0^1 h(r)\int_r^1 \phi^{-1}\left(\int_\delta^s f(\tau,u(\tau),u'(\tau))\mathrm{d}\tau\right)\mathrm{d}s\,\mathrm{d}r \\
&\quad + \int_\delta^1 \phi^{-1}\left(\int_\delta^s f(\tau,u(\tau),u'(\tau))\mathrm{d}\tau\right)\mathrm{d}s \\
&\geq \min\left\{\frac{1}{1-\int_0^1 g(r)\mathrm{d}r}\int_0^1 g(r)\int_0^r \phi^{-1}\left(\int_s^\delta f(\tau,u(\tau),u'(\tau))\mathrm{d}\tau\right)\mathrm{d}s\,\mathrm{d}r\right. \\
&\qquad + \int_0^{1/2} \phi^{-1}\left(\int_s^{1/2} f(\tau,u(\tau),u'(\tau))\mathrm{d}\tau\right)\mathrm{d}s, \\
&\qquad \frac{1}{1-\int_0^1 h(r)\mathrm{d}r}\int_0^1 h(r)\int_r^1 \phi^{-1}\left(\int_\delta^s f(\tau,u(\tau),u'(\tau))\mathrm{d}\tau\right)\mathrm{d}s\,\mathrm{d}r \\
&\qquad \left. + \int_{1/2}^1 \phi^{-1}\left(\int_{1/2}^s f(\tau,u(\tau),u'(\tau))\mathrm{d}\tau\right)\mathrm{d}s\right\} \\
&\geq \min\left\{\int_0^{1/2} \phi^{-1}\left(\int_s^{1/2} f(\tau,u(\tau),u'(\tau))\mathrm{d}\tau\right)\mathrm{d}s, \int_{1/2}^1 \phi^{-1}\left(\int_{1/2}^s f(\tau,u(\tau),u'(\tau))\mathrm{d}\tau\right)\mathrm{d}s\right\} \\
&\geq \min\left\{\int_0^{1/2} \phi^{-1}\left(\phi\left(\frac{r}{N}\right)\left(\frac{1}{2}-s\right)\right)\mathrm{d}s, \int_{1/2}^1 \phi^{-1}\left(\phi\left(\frac{r}{N}\right)\left(s-\frac{1}{2}\right)\right)\mathrm{d}s\right\} \\
&\geq \frac{r}{N}\min\left\{\int_0^{1/2} \varphi_2^{-1}\left(\frac{1}{2}-s\right)\mathrm{d}s, \int_{1/2}^1 \varphi_2^{-1}\left(s-\frac{1}{2}\right)\mathrm{d}s\right\} \\
&= r = \|u\|_1.
\end{aligned} \tag{3.2}$$

This implies that

$$\|Tu\|_1 \geq \|u\|_1 \quad \text{for } u \in \partial\Omega_1. \tag{3.3}$$

Next, for $u \in \partial\Omega_2$, we have $f(t,u,v) \le \phi(R/L)$. Thus, by (2.12) and Lemma 2.6,

$$\begin{aligned}
\|Tu\|_c &= \max_{0\le t\le 1}|Tu(t)| \\
&\le \frac{1}{1-\int_0^1 g(r)\mathrm{d}r}\int_0^1 g(r)\int_0^1 \phi^{-1}\left(\int_s^1 f(\tau,u(\tau),u'(\tau))\mathrm{d}\tau\right)\mathrm{d}s\,\mathrm{d}r \\
&\quad + \int_0^1 \phi^{-1}\left(\int_s^1 f(\tau,u(\tau),u'(\tau))\mathrm{d}\tau\right)\mathrm{d}s \\
&\le \frac{1}{1-\int_0^1 g(r)\mathrm{d}r}\int_0^1 \phi^{-1}\left((1-s)\phi\left(\frac{R}{L}\right)\right)\mathrm{d}s \\
&\le \frac{R}{L}\frac{\int_0^1 \varphi_1^{-1}(1-s)\mathrm{d}s}{1-\int_0^1 g(r)\mathrm{d}r} \\
&\le R = \|u\|_1.
\end{aligned} \tag{3.4}$$

From (2.13), we have

$$\begin{aligned}
\left\|(Tu)'\right\|_c &= \max\left\{\phi^{-1}\left(\int_0^\delta f(\tau,u(\tau),u'(\tau))\mathrm{d}\tau\right),\phi^{-1}\left(\int_\delta^1 f(\tau,u(\tau),u'(\tau))\mathrm{d}\tau\right)\right\} \\
&\le \phi^{-1}\left(\int_0^1 f(\tau,u(\tau),u'(\tau))\mathrm{d}\tau\right) \\
&\le \phi^{-1}\left(\phi\left(\frac{R}{L}\right)\right) \\
&\le R = \|u\|_1.
\end{aligned} \tag{3.5}$$

This implies that

$$\|Tu\|_1 \le \|u\|_1 \quad \text{for } u \in \partial\Omega_2. \tag{3.6}$$

Therefore, by Theorem 2.8, it follows that T has a fixed point in $\overline{\Omega}_2 \setminus \Omega_1$. That is BVP(1.1) has at least one positive solution such that $0 < r \le \|u\|_1 \le R$.

(ii) Considering $f^\infty < \varphi_1(1/2L)$, there exists $\rho_0 > 0$ such that

$$f(t,u,v) \le \varphi_1\left(\frac{1}{2L}\right)\phi(\|u\|_c + \|v\|_c) \quad \text{for } t \in [0,1],\ \|u\|_c + \|v\|_c \ge 2\rho_0. \tag{3.7}$$

Choosing $\overline{M} > \rho_0$ such that

$$\max\{f(t,u,v) \mid \|u\|_c + \|v\|_c \le 2\rho_0\} \le \varphi_1\left(\frac{1}{2L}\right)\phi\left(\overline{M}\right), \tag{3.8}$$

then for all $\rho > \overline{M}$, let $\Omega_3 = \{u \in K \mid \|u\|_1 < \rho\}$. For every $u \in \partial\Omega_3$, we have $\|u\|_c + \|u'\|_c \leq 2\rho$. In the following, we consider two cases.

Case 1 ($\|u\|_c + \|u'\|_c \leq 2\rho_0$). In this case,

$$f(t,u,u') \leq \varphi_1\left(\frac{1}{2L}\right)\phi\left(\overline{M}\right) \leq \phi\left(\frac{\overline{M}}{2L}\right) \leq \phi\left(\frac{\rho}{L}\right). \tag{3.9}$$

Case 2 ($2\rho_0 \leq \|u\|_c + \|u'\|_c \leq 2\rho$). In this case,

$$f(t,u,u') \leq \varphi_1\left(\frac{1}{2L}\right)\phi\left(\|u\|_c + \|u'\|_c\right) \leq \varphi_1\left(\frac{1}{2L}\right)\phi(2\rho) \leq \phi\left(\frac{\rho}{L}\right). \tag{3.10}$$

Then it is similar to the proof of (3.6); we have $\|Tu\|_1 \leq \|u\|_1$ for $u \in \partial\Omega_3$.

Next, turning to $f_0 > \varphi_2(1/N)$, there exists $0 < \xi < \rho$ such that

$$f(t,u,v) \geq \varphi_2\left(\frac{1}{N}\right)\phi(\|u\|_c + \|v\|_c) \quad \text{for } t \in [0,1],\ \|u\|_c + \|v\|_c \leq 2\xi. \tag{3.11}$$

Let $\Omega_4 = \{u \in K \mid \|u\|_1 < \xi\}$. For every $u \in \partial\Omega_4$, we have $\|u\|_c + \|u'\|_c \leq 2\xi$. So

$$f(t,u,u') \geq \varphi_2\left(\frac{1}{N}\right)\phi\left(\|u\|_c + \|u'\|_c\right) \geq \varphi_2\left(\frac{1}{N}\right)\phi(\|u\|_1) \geq \phi\left(\frac{\xi}{N}\right). \tag{3.12}$$

Then like in the proof of (3.3), we have $\|Tu\|_1 \geq \|u\|_1$ for $u \in \partial\Omega_4$. Hence, BVP(1.1) has at least one positive solution such that $0 < \xi \leq \|u\|_1 \leq \rho$.

(iii) The proof is similar to the (i) and (ii); here we omit it. □

In the following, we present a result for the existence of at least two positive solutions of BVP(1.1).

Theorem 3.2. *Assume that (H1) and (H2) hold. In addition, suppose that one of following conditions is satisfied.*

(I) $f^0 < \varphi_1(1/2L)$, $f^\infty < \varphi_1(1/2L)$, *and there exists* $m_1 > 0$ *such that*

$$f(t,u,v) \geq \phi\left(\frac{m_1}{N}\right) \quad \text{for } t \in [0,1],\ m_1 \leq \|u\|_c + \|v\|_c \leq 2m_1; \tag{3.13}$$

(II) $f_0 > \varphi_2(1/N)$, $f_\infty > \varphi_2(1/N)$, *and there exists* $m_2 > 0$ *such that*

$$f(t,u,v) \leq \phi\left(\frac{m_2}{L}\right) \quad \text{for } t \in [0,1],\ \|u\|_c + \|v\|_c \leq 2m_2. \tag{3.14}$$

Then BVP(1.1) *has at least two positive solutions.*

4. The Existence of Three Positive Solutions

In this section, we impose growth conditions on f which allow us to apply Theorem 2.9 of BVP(1.1).

Let the nonnegative continuous concave functional α, the nonnegative continuous convex functionals γ, θ, and nonnegative continuous functional ψ be defined on cone K by

$$\gamma(u) = \max_{0\le t\le 1}\left|u'(t)\right|, \qquad \psi(u) = \theta(u) = \max_{0\le t\le 1}|u(t)|, \qquad \alpha(u) = \min_{\eta\le t\le 1-\eta}|u(t)|. \tag{4.1}$$

By Lemmas 2.1 and 2.2, the functionals defined above satisfy

$$\eta\theta(u) \le \alpha(u) \le \psi(u) = \theta(u), \quad \|u\|_1 = \max\{\gamma(u),\theta(u)\} \le M\gamma(u), \tag{4.2}$$

for all $u \in K$. Therefore, the condition (2.19) of Theorem 2.9 is satisfied.

Theorem 4.1. *Assume that (H1) and (H2) hold. Let* $0 < a < b \le d\eta/(1+((1-\int_0^1 h(t)\mathrm{d}t)/(\int_0^1 h(t)(1-t)\mathrm{d}t)))$ *and suppose that f satisfies the following conditions:*

($P1$) $f(t,u,v) \le \phi(d)$ *for* $(t,u,v) \in [0,1]\times[0,Md]\times[-d,d]$;

($P2$) $f(t,u,v) > \phi(b/\eta K)$ *for* $(t,u,v) \in [\eta,1-\eta]\times[b,(b/\eta)(1+(1-\int_0^1 h(t)\mathrm{d}t)/\int_0^1 h(t)(1-t)\mathrm{d}t)]\times[-d,d]$.

($P3$) $f(t,u,v) < \phi(a/L)$ *for* $(t,u,v) \in [0,1]\times[0,a]\times[-d,d]$;

Then BVP(1.1) *has at least three positive solutions u_1, u_2, and u_3 satisfying*

$$\begin{gathered}\max_{0\le t\le 1}\left|u_i'(t)\right| \le d \quad \text{for } i = 1,2,3, \qquad \min_{\eta\le t\le 1-\eta}|u_1(t)| > b,\\ \max\nolimits_{0\le t\le 1}|u_2(t)| > a \quad \text{with } \min\nolimits_{\eta\le t\le 1-\eta}|u_2(t)| < b, \qquad \max\nolimits_{0\le t\le 1}|u_3(t)| < a,\end{gathered} \tag{4.3}$$

where L defined as (3.1), $K = \min\{\int_\eta^{1/2}\varphi_2^{-1}(1/2-s)\mathrm{d}s, \int_{1/2}^{1-\eta}\varphi_2^{-1}(s-1/2)\mathrm{d}s\}$.

Proof. We will show that all the conditions of Theorem 2.9 are satisfied.

If $u \in \overline{P(\gamma,d)}$, then $\gamma(u) = \max_{0\le t\le 1}|u'(t)| \le d$. With Lemma 2.2 implying $\max_{0\le t\le 1}|u(t)| \le Md$, so by ($P1$), we have $f(t,u(t),u'(t)) \le \phi(d)$ when $0 \le t \le 1$. Thus

$$\begin{aligned}\gamma(Tu) &= \max_{0\le t\le 1}\left|(Tu)'(t)\right|\\ &= \max\left\{\phi^{-1}\left(\int_0^\delta f(\tau,u(\tau),u'(\tau))\mathrm{d}\tau\right),\phi^{-1}\left(\int_\delta^1 f(\tau,u(\tau),u'(\tau))\mathrm{d}\tau\right)\right\}\\ &\le \phi^{-1}\left(\int_0^1 f(\tau,u(\tau),u'(\tau))\mathrm{d}\tau\right)\\ &\le \phi^{-1}(\phi(d)) = d.\end{aligned} \tag{4.4}$$

This proves that $T : \overline{P(\gamma,d)} \to \overline{P(\gamma,d)}$.

To check condition $(S1)$ of Theorem 2.9, we choose

$$u_0(t) = \frac{b}{\eta} + \frac{b\left(1 - \int_0^1 h(t)\mathrm{d}t\right)}{\eta\left(\int_0^1 h(t)(1-t)\mathrm{d}t\right)}(1-t), \quad 0 \le t \le 1. \tag{4.5}$$

Let

$$c = \frac{b}{\eta}\left(1 + \frac{1 - \int_0^1 h(t)\mathrm{d}t}{\int_0^1 h(t)(1-t)\mathrm{d}t}\right). \tag{4.6}$$

Then $u_0(t) \in P(\gamma,\theta,\alpha,b,c,d)$ and $\alpha(u_0) > b$, so $\{u \in P(\gamma,\theta,\alpha,b,c,d) \mid \alpha(u) > b\} \neq \emptyset$. Hence, for $u \in P(\gamma,\theta,\alpha,b,c,d)$, there is $b \le u(t) \le c$, $|u'(t)| \le d$ when $\eta \le t \le 1-\eta$. From assumption $(P2)$, we have

$$f(t,u(t),u'(t)) > \phi\left(\frac{b}{\eta K}\right) \quad \text{for } t \in [\eta, 1-\eta]. \tag{4.7}$$

It is similar to the proof of assumption (i) of Theorem 3.1; we can easily get that

$$\alpha(Tu) = \min_{\eta \le t \le 1-\eta} |(Tu)(t)| \ge \eta \max_{0 \le t \le 1} |(Tu)(t)| > b \quad \text{for } u \in P(\gamma,\theta,\alpha,b,c,d). \tag{4.8}$$

This shows that condition $(S1)$ of Theorem 2.9 is satisfied.

Secondly, for $u \in P(\gamma,\alpha,b,d)$ with $\theta(Tu) > c$, we have

$$\alpha(Tu) \ge \eta\theta(Tu) \ge \eta c > b. \tag{4.9}$$

Thus condition $(S2)$ of Theorem 2.9 holds.

Finally, as $\psi(0) = 0 < a$, there holds $0 \notin R(\gamma,\psi,a,d)$. Suppose that $u \in R(\gamma,\psi,a,d)$ with $\psi(u) = a$; then by the assumption $(P3)$,

$$f(t,u(t),u'(t)) < \phi\left(\frac{a}{L}\right) \quad \text{for } t \in [0,1]. \tag{4.10}$$

So like in the proof of assumption (i) of Theorem 3.1, we can get

$$\psi(Tu) = \max_{0 \le t \le 1} |(Tu)(t)| < a. \tag{4.11}$$

Hence condition $(S3)$ of Theorem 2.9 is also satisfied.

Thus BVP(1.1) has at least three positive solutions u_1, u_2, and u_3 satisfying

$$\max_{0\le t\le 1}\left|u_i'(t)\right| \le d \quad \text{for } i = 1,2,3, \qquad \min_{\eta\le t\le 1-\eta}|u_1(t)| > b,$$
$$\max_{0\le t\le 1}|u_2(t)| > a \quad \text{with } \min_{\eta\le t\le 1-\eta}|u_2(t)| < b, \qquad \max_{0\le t\le 1}|u_3(t)| < a. \tag{4.12}$$

□

5. Examples

In this section, we give three examples as applications.

Example 5.1. Let $\phi(u) = |u|u$, $g(t) = h(t) = 1/2$. Now we consider the BVP

$$(\phi(u'))' + f(t, u(t), u'(t)) = 0, \quad t \in [0,1],$$
$$u(0) = \frac{1}{2}\int_0^1 u(t)\mathrm{d}t, \qquad u(1) = \frac{1}{2}\int_0^1 u(t)\mathrm{d}t, \tag{5.1}$$

where $f(t,u,v) = (1+t)(18+u)(4+\cos v)$ for $(t,u,v) \in [0,1] \times [0,\infty) \times (-\infty,\infty)$.

Let $\psi_1(u) = \psi_2(u) = u^2$, $u > 0$. Choosing $r = 1, R = 100$. By calculations we obtain

$$L = \frac{4}{3}, \qquad N = \frac{2}{3}\left(\frac{1}{2}\right)^{3/2}, \qquad \phi\left(\frac{r}{N}\right) = 18, \qquad \phi\left(\frac{R}{L}\right) = 75^2. \tag{5.2}$$

For $(t,u,v) \in [0,1] \times [0,1] \times [-1,1]$,

$$f(t,u,v) = (1+t)(18+u)(4+\cos v)$$
$$\ge 18 \times 4 > 18, \tag{5.3}$$

for $(t,u,v) \in [0,1] \times [0,100] \times [-100,100]$,

$$f(t,u,v) = (1+t)(18+u)(4+\cos v)$$
$$\le 2 \times 118 \times 5 < 75^2. \tag{5.4}$$

Hence, by Theorem 3.1, BVP(5.1) has at least one positive solution.

Example 5.2. Let $\phi(u) = u$, $g(t) = h(t) = 1/2$. Consider the BVP

$$(\phi(u'))' + f(t, u(t), u'(t)) = 0, \quad t \in [0,1],$$
$$u(0) = \frac{1}{2}\int_0^1 u(t)\mathrm{d}t, \qquad u(1) = \frac{1}{2}\int_0^1 u(t)\mathrm{d}t, \tag{5.5}$$

where $f(t,u,v) = (1+t)(1/10+u)(1/100+v^2)[1+(\|u\|_c+\|v\|_c)^2]$ for $(t,u,v) \in [0,1] \times [0,\infty) \times (-\infty,\infty)$.

Let $\psi_1(u) = \psi_2(u) = u$, $u > 0$. Then $L = 1, N = 1/8$. It easy to see

$$f_0 = f_\infty = \infty > \psi_2\left(\frac{1}{N}\right) = 8. \tag{5.6}$$

Choosing $m_2 = 1/10$, for $t \in [0,1]$, $\|u\|_c + \|v\|_c \le 2m_2$.

$$\begin{aligned} f(t,u,v) &= (1+t)\left(\frac{1}{10}+u\right)\left(\frac{1}{100}+v^2\right)\left[1+(\|u\|_c+\|v\|_c)^2\right] \\ &\le 2\left(\frac{1}{10}+\frac{1}{5}\right)\left(\frac{1}{100}+\frac{1}{25}\right)\left(1+\frac{1}{25}\right). \\ &= \frac{39}{1250} < \frac{1}{10} = \phi\left(\frac{m_2}{L}\right). \end{aligned} \tag{5.7}$$

Hence, by Theorem 3.2, BVP(5.5) has at least two positive solutions.

Example 5.3. Let $\phi(u) = |u|u$, $g(t) = h(t) = 1/2$; consider the boundary value problem

$$\begin{gathered} \left(|u'|u'\right)' + f(t,u(t),u'(t)) = 0, \quad t \in [0,1], \\ u(0) = u(1) = \frac{1}{2}\int_0^1 u(t)\mathrm{d}t, \end{gathered} \tag{5.8}$$

where

$$f(t,u,v) = \begin{cases} \dfrac{\sin t}{10^4} + 2500u^6 + \dfrac{1}{10^4}\left(\dfrac{v}{10^5}\right)^3, & u \le 12, \\ \dfrac{\sin t}{10^4} + 2500\cdot 12^6 + \dfrac{1}{10^4}\left(\dfrac{v}{10^5}\right)^3, & u > 12. \end{cases} \tag{5.9}$$

Choosing $a = 1/10$, $b = 1$, $\eta = 1/4$, $d = 10^5$, then by calculations we obtain that

$$L = \frac{4}{3}, \qquad K = \frac{2}{3}\left(\frac{1}{4}\right)^{3/2}, \qquad \phi\left(\frac{b}{\eta K}\right) = 2304, \qquad \phi\left(\frac{a}{L}\right) = \frac{9}{1600}. \tag{5.10}$$

It is easy to check that

$$\begin{gathered} f(t,u,v) < \phi(d) = 10^{10} \quad \text{for } 0 \le t \le 1,\ 0 \le u \le 3\cdot 10^5,\ -10^5 \le v \le 10^5, \\ f(t,u,v) > 2304 \quad \text{for } \frac{1}{4} \le t \le \frac{3}{4},\ 1 \le u \le 12,\ -10^5 \le v \le 10^5, \\ f(t,u,v) < \frac{9}{1600} \quad \text{for } 0 \le t \le 1,\ 0 \le u \le \frac{1}{10},\ -10^5 \le v \le 10^5. \end{gathered} \tag{5.11}$$

Thus, according to Theorem 4.1, BVP(5.8) has at least three positive solutions u_1, u_2, and u_3 satisfying

$$\max_{0\le t\le 1}\left|u_i'(t)\right| \le 10^5 \quad \text{for } i = 1,2,3, \qquad \min_{1/4\le t\le 3/4}|u_1(t)| > 1,$$
$$\max_{0\le t\le 1}|u_2(t)| > \frac{1}{10} \quad \text{with } \min_{1/4\le t\le 3/4}|u_2(t)| < 1, \qquad \max_{0\le t\le 1}|u_3(t)| < \frac{1}{10}. \tag{5.12}$$

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