

Research Article

Operating with External Arguments of Douady and Hubbard

G. Pastor, M. Romera, G. Alvarez, J. Nunez, D. Arroyo, and F. Montoya

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The external arguments of the external rays theory of Douady and Hubbard is a valuable tool in order to analyze the Mandelbrot set, a typical case of discrete dynamical system used to study nonlinear phenomena. We suggest here a general method for the calculation of the external arguments of external rays landing at the hyperbolic components root points of the Mandelbrot set. Likewise, we present a general method for the calculation of the external arguments of external rays landing at Misiurewicz points.

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1. Introduction

The most important elements constituting the Mandelbrot set are the hyperbolic components [1] and the Misiurewicz points [2–4]. A hyperbolic component is a connected set of periodic points with the same period p . These points are stable, and p is the period of the hyperbolic component. However, Misiurewicz points are preperiodic with preperiod n and eventually these points are unstable periodic with period p .

In two previous papers [5, 6], we studied how to operate with external arguments of Douady and Hubbard [7]. In [5], we only studied the external arguments of a part of the hyperbolic components, the structural components [8], and only in the Mandelbrot set antenna. In [6], we studied the external arguments of the same structural components, but this time in the chaotic bands of the Mandelbrot set. Here, we will cover all types of hyperbolic components and in all the Mandelbrot set. We will show how to calculate the external arguments of these hyperbolic components and Misiurewicz points. As in the previous papers, the results presented here are not rigorously proved theorems, but the results of a vast quantity of data based on computer experiments.

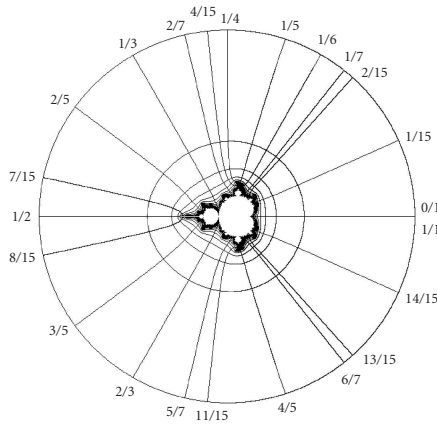


FIGURE 1.1. Mandelbrot set with some equipotential lines, external rays, and external arguments.

Let us see the concept of external argument of the external rays theory according to Douady’s words. Suppose a capacitor made of a metallic bar, whose crosssection is the Mandelbrot set M , placed along the axis of a hollow metallic cylinder of a large radius and length. Set the bar at potential zero and the cylinder at a high potential. This creates an electric field in the region between the cylinder and the bar. An electric potential function is also established in this region. We restrict our attention to the plane perpendicular to the axis of the cylinder, through its middle. In this plane, the electric potential defines equipotential lines enclosing M (which is the crosssection of the bar). Following the electric field, one gets field-lines, called the external rays of M . Each external ray starts at a point x on the boundary of M , and reaches a point y of the great circle which is the crosssection of the cylinder. The position of y is identified by an angle, called the external argument θ of x with respect to M .

A point x of the Mandelbrot set can have several external arguments. However, a specific external argument only corresponds to one point x of this set. That is why external arguments are good identifiers of the points x of the Mandelbrot set.

Figure 1.1 shows the Mandelbrot set with some of its equipotential lines and external rays. As is well known, the rational numbers in the figure are the values of the external arguments of the points where the external rays land. These external arguments are not measured in radians, but as fractions of complete turns. Using this unit, hyperbolic components and Misiurewicz points have rational external arguments, hyperbolic components with odd denominator, and Misiurewicz points with even denominator.

Each one of the Mandelbrot set discs has a root point where the disc is attached to other bigger disc or to a cardioid. Each one of the cardioids of the set, except for the main one, has a cusp point from where a filament, which joins it to the rest of the set, emerges. Root and cusp points (sometimes both points are called root points) and Misiurewicz points, normally branch points, or tips of filaments are notable points of the Mandelbrot

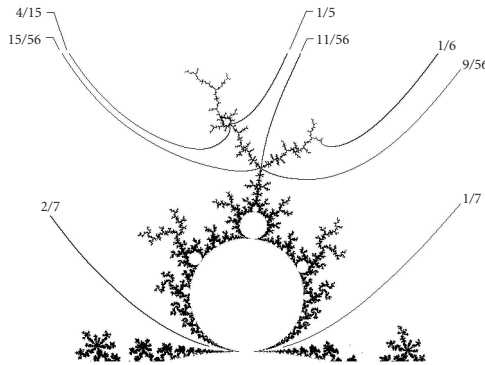


FIGURE 1.2. Four notable points of the boundary of the Mandelbrot set. Two of them, one root point and one cusp point, have external arguments with odd denominator; and the other two, one branch point and one tip point, are Misiurewicz points that have external arguments with even denominator.

set boundary. There are other notable points (as the Myrberg-Feigenbaum point which is the limit of the period doubling cascade, and the point that separate the periodic and chaotic region) which will not be considered in this paper.

In Figure 1.2, four examples of notable points are shown. Two of these points have external arguments with odd denominators. One of them is a root point with two arguments, $1/7$ and $2/7$, and the other one is a cusp point with two arguments, $3/15$ (or $1/5$) and $4/15$. The other two notable points are Misiurewicz points with external arguments with even denominator, one of them a branch point with three arguments, $9/56$, $11/56$, and $15/56$, and the other one a tip with only one argument, $1/6$. The external rays of these notable points were drawn by means of a computer program of Jung [9].

As is known, a rational external argument with odd denominator is periodic under angle-doubling modulo 1. Likewise, a rational external argument with even denominator is preperiodic under angle-doubling modulo 1. In the Mandelbrot set, root points are periodic and have rational external arguments with odd denominators, whereas Misiurewicz points are preperiodic and have rational external arguments with even denominators.

A hyperbolic component can be identified by means of each one of the two external arguments of its root point. The first and second external arguments, respectively, correspond to the external arguments with the smallest and greatest values. Likewise, a Misiurewicz point can be identified by means of each one of the external arguments of the external rays landing at such a point. If the Misiurewicz point is a branch point, there are as many arguments as branches, and if the Misiurewicz point is a tip, there is only one argument.

The binary expansion of an external argument θ , $0 < \theta < 1$, is the binary expansion of the rational number, $\theta = 0.a_1a_2a_3\dots$, where $a_k \in \{0,1\}$ and $\theta = \sum_{k=1}^{\infty} a_k 2^{-k}$. In this paper, external arguments are normally given in binary expansions.

2. External arguments calculation of the root points of the hyperbolic components

Let us succinctly see what has been published up to now about the calculation of the external arguments. Let us begin with the Schleicher algorithm which obtains the external arguments of the root point of the greatest disc placed between other two discs with known external arguments [10]. Other ways of calculating external arguments are the well-known tuning algorithm and the algorithm to calculate the external arguments of a notable point, both of them of Douady [11]. To finish, let us say that harmonics provide a compact tool to calculate the external arguments of the structural components [12, 13] of both the Mandelbrot set main antenna [5] and the shrub₀ of chaotic bands [6].

In this work, we will calculate the external arguments of both root points and Misiurewicz points, the first ones in this section and the last ones in the following section. Let us recall some concepts that can be found in [14] and will be used later. The wake of a hyperbolic component is the domain bounded by the two rays landing at its root. A hyperbolic component is narrow if it contains no component of equal or lesser period in its wake, and the external arguments of the two external rays landing at its root point then differ by $1/(2^p - 1)$, where p is the period of the component.

Let us see a general algorithm to calculate the root point external arguments of any hyperbolic component of the Mandelbrot set: the outer algorithm, which is so called because the inner wake is not used but only the outer wake.

2.1. Outer algorithm. Let $(\overline{\theta}_1, \overline{\theta}_2)$ be the unknown binary expansions, $\overline{\theta}_1 < \overline{\theta}_2$, of the rays landing at the root point of a hyperbolic component with known period p . One starts from a point of the ray $\overline{\theta}_1$ and goes in the normal direction to such a ray by going away of the hyperbolic component wake, until one finds the first ray with period $p_1 < p$. Let \overline{s} be the binary expansion of this ray. One starts again from a point of the ray $\overline{\theta}_1$ and goes in the opposite direction to the previous one, until one finds the first ray with period $p_2 = p - p_1$. Let \overline{t} be the binary expansion of this ray. Afterwards, one repeats the process by starting on a point of the ray $\overline{\theta}_2$, and one goes away of the wake until one finds the first ray with period $p_3 < p$ and binary expansion \overline{u} . One starts again from the ray $\overline{\theta}_2$ and one goes in the opposite direction, until one finds the first ray with period $p_4 = p - p_3$ and binary expansion \overline{v} . The unknown binary expansions are

$$(\overline{\theta}_1, \overline{\theta}_2) = (\overline{s\bar{t}}, \overline{u\bar{v}}). \quad (2.1)$$

In other words, each one of these binary expansions, $\overline{\theta}_1$ and $\overline{\theta}_2$, is obtained by composing other two binary expansions, that can be considered as their ancestors. To calculate the first ancestor of $\overline{\theta}_1$, that is, the first term of the composition, one starts from a point of the ray $\overline{\theta}_1$ and goes away of the wake in the normal direction. The first ray found with period $p_1 < p$, \overline{s} , is the first ancestor. In order to calculate the second ancestor of $\overline{\theta}_1$, that is, the second term of the composition, one starts again from a point of the ray $\overline{\theta}_1$ and one goes in the opposite direction. The first ray found with period $p_2 = p - p_1$, \overline{t} , is the second ancestor. Then the two ancestors of $\overline{\theta}_1$ are $\overline{s} = \overline{s_1 s_2 \cdots s_{p_1}}$ and $\overline{t} = \overline{t_1 t_2 \cdots t_{p_2}}$, where $p_1 + p_2 = p$. Therefore, $\overline{\theta}_1$ is obtained by composing in the order seen before the

two previous ancestors in accordance with the following rule of composition:

$$\overline{\theta}_1 = \overline{s} \widehat{+} \overline{t} = \overline{s_1 s_2 \cdots s_{p_1}} \widehat{+} \overline{t_1 t_2 \cdots t_{p_2}} = \overline{s_1 s_2 \cdots s_{p_1} t_1 t_2 \cdots t_{p_2}} = \overline{st}. \quad (2.2)$$

The two ancestors of $\overline{\theta}_2$ are calculated in a similar way, and they are composed with the same rule of composition.

It is not difficult to prove the outer algorithm; however, we do not do it because this is not the aim of this paper that is focused on the results based on computational data.

In Figure 2.1, several very easy application examples of the outer algorithm can be seen. Only primary discs directly attached to the main cardioid (Figure 2.1(a)), and cardioids of the antenna (Figure 2.1(b), i.e., a magnification of the antenna marked with a rectangle in Figure 2.1(a)) are treated. According to Devaney [10], the discs directly attached to the main cardioid are identified by their rotation number, which is a rational number that should not be mistaken with the rational value of an external argument. The first argument of the period-2 disc $1/2$, (Figure 2.1(a)), is $\overline{0} \widehat{+} \overline{1} = \overline{01}$, and the second one is $\overline{1} \widehat{+} \overline{0} = \overline{10}$. Likewise, the arguments of the disc $1/3$ are $\overline{0} \widehat{+} \overline{01} = \overline{001}$ and $\overline{01} \widehat{+} \overline{0} = \overline{010}$. Finally, the arguments of the disc $1/4$ are $\overline{0} \widehat{+} \overline{001} = \overline{0001}$ and $\overline{001} \widehat{+} \overline{0} = \overline{0010}$. Note that, in the discs attached to the main cardioid, our algorithm is similar to the Schleicher algorithm [10]. Figure 2.1(b) shows the external arguments of many cardioids of the antenna. All of them can easily be calculated by applying the outer algorithm. Thus, the external arguments of the period-3 cardioid are $\overline{01} \widehat{+} \overline{1} = \overline{011}$ and $\overline{10} \widehat{+} \overline{0} = \overline{100}$.

Other examples a little more complicated will be seen now with the help of Figure 2.2. Figure 2.2(a) shows the Mandelbrot set with some external arguments. Figure 2.2(b) shows a sketch of the shrub $(1/3)$. Figure 2.2(c) is a magnification of the rectangle of Figure 2.2(a) and it shows the shrub $(1/3)$ with the external arguments of some structural components and Misiurewicz points. Calculating ancestors is very easy when one has achieved some practice and certain knowledge of the Mandelbrot set. Next, let us analyze some examples of this case with the help of the three parts of the Figure 2.2.

The structural component of the branch 1 is a period-4 cardioid [12]. Let us calculate its arguments $(\overline{x_1 \cdots x_4}, \overline{y_1 \cdots y_4})$. The first ancestor of $\overline{x_1 \cdots x_4}$ is $\overline{001}$ since it is the first argument with period less than 4 that is found when one goes away to the wake. The second ancestor has to be the first argument with period 1 when one goes in the opposite direction, therefore it has to be $\overline{1}$. Hence, $\overline{x_1 \cdots x_4} = \overline{001} \widehat{+} \overline{1} = \overline{0011}$. Likewise, the first ancestor of $\overline{y_1 \cdots y_4}$ is $\overline{010}$, and the second ancestor is $\overline{0}$. Hence, $\overline{y_1 \cdots y_4} = \overline{010} \widehat{+} \overline{0} = \overline{0100}$. Let us calculate now the structural component of the branch 22, that is, a period-7 cardioid $(\overline{x_1 \cdots x_7}, \overline{y_1 \cdots y_7})$ [12]. The first ancestor of $\overline{x_1 \cdots x_7}$ is $\overline{00101}$ and the second ancestor is $\overline{01}$. Hence, $\overline{x_1 \cdots x_7} = \overline{00101} \widehat{+} \overline{01} = \overline{0010101}$. Likewise, the first and second ancestors of $\overline{y_1 \cdots y_7}$ are $\overline{001011}$ and $\overline{0}$. Hence, $\overline{y_1 \cdots y_7} = \overline{001011} \widehat{+} \overline{0} = \overline{0010110}$.

If one wants to keep calculating any of the other pairs of external arguments showed in Figure 2.2, one has to proceed in the same way as in the cases shown. Obviously, if one of the ancestors, or both, are unknown, they have to be calculated by using second

generation ancestor. If these ancestors are again unknown, they have to be calculated by using now third generation ancestor. And so on until reaching ancestors already known, since starting from them the ancestors of the previous generation can be obtained, and from them, the ancestors of the previous one, and so on as far as the pair of the original external arguments.

Up to now, only narrow hyperbolic components have been calculated. However, non-narrow hyperbolic components also can be calculated. Let us see an example. The external arguments of the period-6 disc of the period doubling cascade of the primary disc $1/3$ are $(.001010, .010001)$, as can be calculated from the tuning algorithm or from the harmonics cited at the beginning of this section. But they also can be calculated by means of the outer algorithm since the first and second ancestors of the first argument are, respectively, $.001$ and $.010$; and the first and second ancestors of the second argument are $.010$ and $.001$. Hence, the pair of external arguments is $(.001 \hat{+} .010, .010 \hat{+} .001) = (.001010, .010001)$.

Let us see more complex cases with the help of Figure 2.3. Figure 2.3(a) shows the neighborhood of the primary hyperbolic component $1/3$ with some external arguments and shrubs. Figure 2.3(b) is a sketch of the shrub $(1/3 \cdot 1/4)$. Figure 2.3(c) is a magnification of the rectangle of Figure 2.3(a), and it shows the shrub $(1/3 \cdot 1/4)$ with the external arguments of some notable structural components and Misiurewicz points.

Let us begin with the first subshrub S_1 . Let us calculate the external arguments of the period-15 cardioid of the branch 1 of S_1 , $(\overline{x_1 \cdots x_{15}}, \overline{y_1 \cdots y_{15}})$. By applying the outer algorithm, the two ancestors of $\overline{x_1 \cdots x_{15}}$ are $.001001001010$ and $.010$; and the two ancestors of $\overline{y_1 \cdots y_{15}}$ are $.001001010001$ and $.001$. Therefore, the pair of external arguments are $(.001001001010010, .001001010001001)$. Let us now calculate the external arguments of the period-21 cardioid of the branch 3 of S_1 , $(\overline{x_1 \cdots x_{21}}, \overline{y_1 \cdots y_{21}})$. The first argument has the first argument of $1/3 \cdot 1/4$ as first ancestor, and the first argument of $1/3 \cdot 1/3$ as second ancestor. Likewise, the second argument has the first argument of the period-18 cardioid of the branch 2 of S_1 as first ancestor, and the first argument of $1/3$ as second ancestor. Hence,

$$(\overline{x_1 \cdots x_{21}}, \overline{y_1 \cdots y_{21}}) = (.001001001010001001010, .001001001010001010001). \quad (2.3)$$

To end the discussion about the outer algorithm, let us see now an example in the second subshrub S_2 . Let us calculate the external arguments of the period-16 cardioid of the branch 1 of the portion that comes from $2\bar{1}$, $(\overline{x_1 \cdots x_{16}}, \overline{y_1 \cdots y_{16}})$ [12]. The argument $\overline{x_1 \cdots x_{16}}$ is obtained by composing the first argument of $1/3 \cdot 1/4$ with the first argument of the period-4 cardioid of the branch 1 of the shrub $(1/3)$; and $\overline{y_1 \cdots y_{16}}$ is obtained by composing the first argument of the period-15 cardioid of the branch 1 of S_1 with the first argument of the main cardioid. Hence,

$$(\overline{x_1 \cdots x_{16}}, \overline{y_1 \cdots y_{16}}) = (.0010010010101, .0010010010100100). \quad (2.4)$$

One still could see new examples of the outer algorithm in tertiary, quaternary, ..., shrubs, but it is not worth because the process is identical to the previous ones. However, the outer algorithm is not the only possible algorithm to calculate the arguments of a component. Let us see next other algorithms.

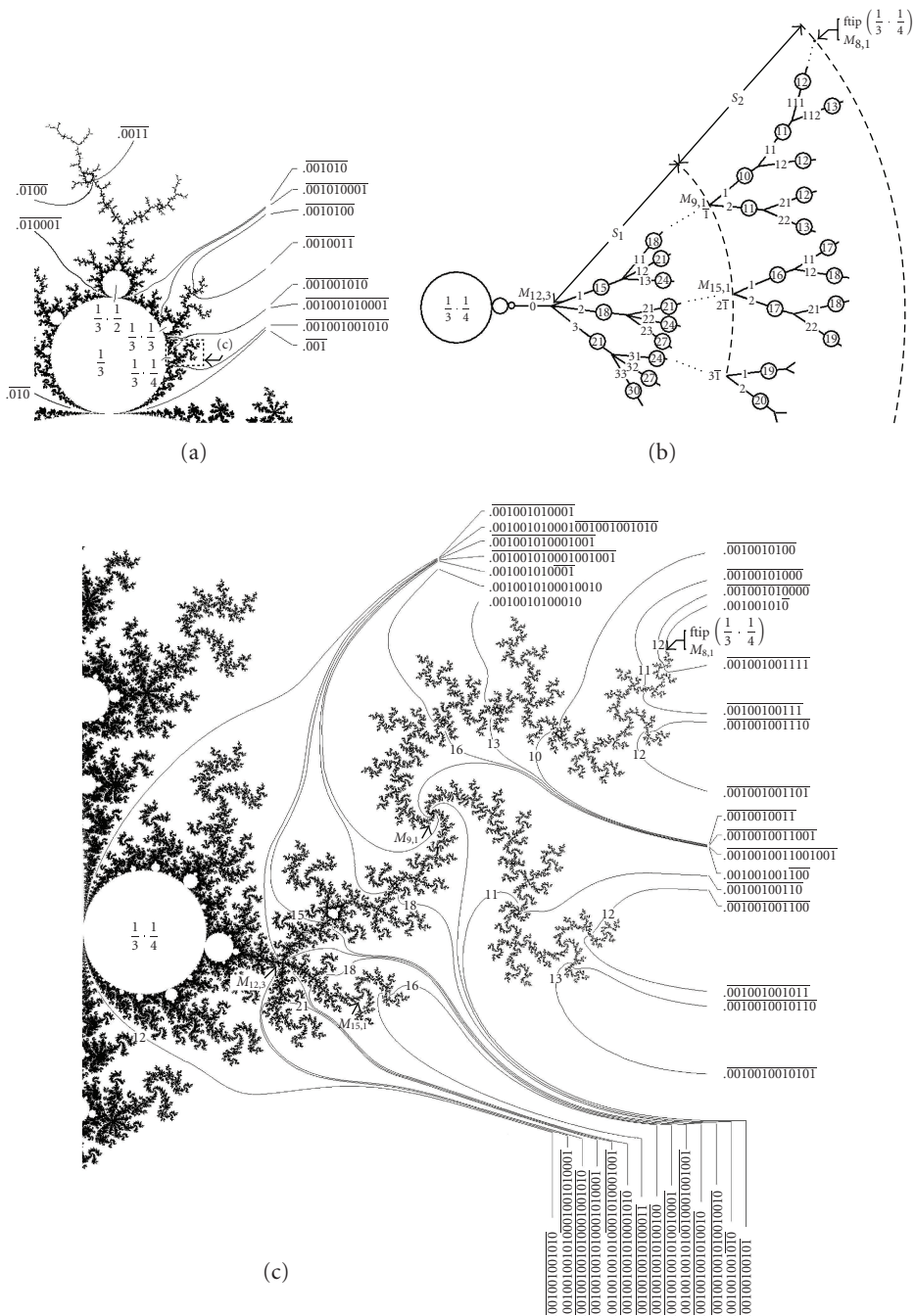


FIGURE 2.3. (a) A detail of the neighborhood of the primary hyperbolic component $1/3$, where some external arguments and shrubs are shown. (b) Sketch of a secondary shrub, shrub $(1/3 \cdot 1/4)$. (c) Magnification of the rectangle of (a) showing the shrub $(1/3 \cdot 1/4)$. The external arguments of some notable components and Misiurewicz points are shown.

2.2. Inner algorithm. Let $(\overline{\theta}_1, \overline{\theta}_2)$ be the unknown binary expansions, $\overline{\theta}_1 < \overline{\theta}_2$, of the rays landing at the root point of a hyperbolic component with known period p . Starting from a point of the ray $\overline{\theta}_1$ and going in the normal direction towards the inside of the hyperbolic component wake, one finds the first ray with period $p_1 < p$. Let \overline{s} be the binary expansion of this ray. Starting again from a point of the ray $\overline{\theta}_1$ and going in the opposite direction to the previous one, one finds the first ray with period $p_2 = p - p_1$. Let \overline{t} be the binary expansion of this ray. Afterwards, one repeats the process by starting from a point of the ray $\overline{\theta}_2$, and one goes in the normal direction towards the inside of the hyperbolic component wake until one finds the first ray with period $p_3 < p$ and binary expansion \overline{u} . One starts again from the ray $\overline{\theta}_2$ and going in the opposite direction, one finds the first ray with period $p_4 = p - p_3$ and binary expansion \overline{v} . The unknown binary expansions are

$$(\overline{\theta}_1, \overline{\theta}_2) = (\overline{st}, \overline{uv}). \tag{2.5}$$

This algorithm, which we call inner algorithm because we start by going towards the inside of the wake, can be useful in some cases, but it has important limitations since it has no sense in the narrow hyperbolic components. Obviously, by taking into account the similarity between the outer algorithm and the inner algorithm, one unique algorithm including both could easily be given.

Let us apply the inner algorithm to calculate the external arguments $(\overline{x_1 \cdots x_6}, \overline{y_1 \cdots y_6})$ of the period-6 first disc of the period doubling cascade of the primary disc 1/3 already calculated above (Figure 2.2). Since now, we first go toward the wake inside, the first ancestor of $\overline{x_1 \cdots x_6}$ is $\overline{.00101}$ and the second ancestor is $\overline{.0}$. Likewise, the first ancestor of $\overline{y_1 \cdots y_6}$ is $\overline{.0100}$ and the second ancestor is $\overline{.01}$. Hence,

$$(\overline{x_1 \cdots x_6}, \overline{y_1 \cdots y_6}) = (\overline{.001010}, \overline{.010001}). \tag{2.6}$$

3. External arguments calculation of the Misiurewicz points

Any Misiurewicz point has as many external arguments as branches meeting on it. Thus, the Misiurewicz points of the real axis have only two arguments (except for the tip which only has one, but can be considered as two arguments with the same value) whereas the main node of the shrub (q/p) has p arguments. Let us see how the external arguments of a Misiurewicz point can be calculated. The problem will be studied from two different approaches.

Let us see the first approach. The external arguments of a Misiurewicz point $M_{n,p}$ are calculated as the limit of external arguments of sequences of an infinite number of hyperbolic components (hc 's) tending to the Misiurewicz point. We call Misiurewicz route to a sequence of components converging to a Misiurewicz point. A Misiurewicz route is composed by the infinite number of the last appearance hyperbolic components ($lahc$'s) found in the interval of such a route (in practice, a few components is enough). That is to say, one starts from the first component of the sequence, hc_1 . The second component of the sequence, hc_2 , is the hyperbolic component with the smallest period between hc_1 and $M_{n,p}$. The third component of the sequence, hc_3 , is the hyperbolic component with the smallest period between hc_2 and $M_{n,p}$. In general, the N th component of the sequence,

hc_N , is the hyperbolic component with the smallest period between hc_{N-1} and $M_{n,p}$. In such a way, the sequence is indeed composed by the *lahcs* when one goes from hc_1 to $M_{n,p}$. We call Misiurewicz interval to the region between hc_1 and $M_{n,p}$.

Let us see it from a second approach. As we will see next, in the previous process only two external arguments are involved, which will be called first ancestor and second ancestor. Indeed, as it can be seen in the next examples, the binary expansion of the N th component of the sequence is obtained by adding p digits to the binary expansion of the $(N - 1)$ th component of the sequence. This occurs for any N , therefore each component is always obtained by adding the same p digits to the previous component. Hence, p is going to be the number of digits of the periodic part of the Misiurewicz point. It is going to have two associated external arguments: one with period p , that appears indefinitely, which we call second ancestor, and another one, that is, what is left after removing the repeating part, and that we call first ancestor.

When the ancestors are given as binary expansions, the algorithm of composition is very easy: one writes once the first ancestor followed by infinite times the second ancestor. That is to say, let $\bar{s} = .\bar{s}_1\bar{s}_2 \cdots \bar{s}_n$, with period n , and $\bar{t} = \bar{t}_1\bar{t}_2 \cdots \bar{t}_p$, with period p , be the first and second ancestors of one of the external arguments of a Misiurewicz point. In this case, the composition rule to calculate the first external argument of the Misiurewicz point has the very easy and intuitive expression

$$\bar{s} [\widehat{+} \bar{t}]^\infty = \overline{.s_1 s_2 \cdots s_n} [\widehat{+} \overline{.t_1 t_2 \cdots t_p}]^\infty = \overline{.st}^\infty = \bar{s}\bar{t}. \quad (3.1)$$

Obviously the result is preperiodic. As it will be seen later, if there is no simplification, the preperiod has n digits and the period p digits. If there are simplifications, the period has again p digits, but the preperiod has less than n digits.

Let us consider now the external argument $\bar{s}^* = \overline{.s_1^* s_2^* \cdots s_n^*}$, that is, the pair of the previous first ancestor \bar{s} , and the external argument $\bar{t}^* = \overline{.t_1^* t_2^* \cdots t_p^*}$, that is, the pair of the previous second ancestor \bar{t} . The arguments \bar{s}^* and \bar{t}^* are the first and second ancestors of another external argument of the same Misiurewicz point. In this case, the composition rule to calculate the other external argument is $\bar{s}^* [\widehat{+} \bar{t}^*]^\infty = \overline{.s^* t^*}^\infty = \bar{s}^* \bar{t}^*$, where again the result is preperiodic, with a preperiod of n digits and a period of p digits.

A very frequent particular case is when $\bar{s}^* = \bar{t}$ and $\bar{t}^* = \bar{s}$. In this case, the pair of arguments is $(\bar{s}\bar{t}, \bar{t}\bar{s})$.

In order to clarify what has just been said, we will give several examples to calculate Misiurewicz points where the first approach is normally used, though sometimes we will use the second approach. Let us begin with the easiest examples. Let us observe Figure 2.1(b), which is a magnification of the rectangle of the Mandelbrot set antenna shown in Figure 2.1(a). Let us notice the chaotic band B_0 and its structural components, that are the *lahcs* in such a band, with periods 3, 4, 5, 6, ... The infinite limit of these structural components is the tip placed in $c = -2$. Therefore, the external argument of the tip is the limit of the structural components arguments. The first arguments of the structural components are $\overline{.011}, \overline{.0111}, \overline{.01111}, \overline{.011111}, \dots$, whose limit is $\overline{.01}$. The second arguments of the structural components are $\overline{.100}, \overline{.1000}, \overline{.10000}, \overline{.100000}, \dots$, whose limit is $\overline{.10}$. The tip has two arguments with the same value, that is, it has a unique argument.

The limit of the structural components of the chaotic band B_1 , with periods 6, 8, 10, ..., is the band merging point m_1 (which is the separation between the chaotic bands B_0 and B_1) [8]. Therefore, the external arguments of m_1 are the limit of $\overline{.011010}$, $\overline{.01101010}$, $\overline{.0110101010}$, ..., and the limit of $\overline{.100101}$, $\overline{.10010101}$, $\overline{.1001010101}$, ..., that is, $(.01\overline{10}, .100\overline{1})$. In these two examples, the Misiurewicz routes cover the chaotic bands B_0 and B_1 , and therefore the calculated Misiurewicz points are the upper limits of these bands.

Let us use Figure 2.1(c), that is, a magnification of the rectangle of Figure 2.1(b), to see more examples where the first and second ancestors belong to different components. Let us consider the Misiurewicz route that starts from the structural component $(\overline{.0111}, \overline{.1000})$, with period 4, and reaches the Misiurewicz point $M_{4,1}$ placed in $c = -1.973932044...$ [3]. (Let us point out that in [3] any Misiurewicz point had one unit more in its preperiodic part because the iteration started from the critical point instead of the critical value. Thus $M_{4,1}$ was given as $M_{5,1}$.) In order to calculate the Misiurewicz point, one starts from the period-4 structural component $(\overline{.0111}, \overline{.1000})$, and one covers the rest of the Misiurewicz route by following the sequence of *hc*'s with periods 6, 8, 10, ..., that are the *lahcs* in this Misiurewicz interval. This sequence is

$$(\overline{.011110}, \overline{.100001}), (\overline{.01111010}, \overline{.10000101}), (\overline{.0111101010}, \overline{.1000010101}), \dots, \quad (3.2)$$

and its limit is $(.0111\overline{10}, .1000\overline{01})$, which is the $M_{4,1}$ wanted.

A pair of remarks has to be done. First, in this case, the first ancestors come from the period-4 structural component $(\overline{.0111}, \overline{.1000})$ and the second ancestors from the first disc of the period doubling cascade $(\overline{.01}, \overline{.10})$. And second, the period of $M_{4,1}$ is 1, nevertheless the number of digits of the periodic part is 2. The period can coincide or not coincide with the number of digits, but always the period divides the number of digits.

Starting from the same period-4 component of the chaotic band B_0 , other Misiurewicz routes could be followed, for example, that one which reaches the Misiurewicz point $M_{4,3}$ placed in $c = -1.955694045...$ [3]. In this route, the sequences of *lahcs* have periods 4, 7, 10, ... This sequence is

$$(\overline{.0111}, \overline{.1000}), (\overline{.0111100}, \overline{.1000011}), (\overline{.0111100100}, \overline{.1000011011}), \dots, \quad (3.3)$$

and its limit is $(.0111\overline{100}, .1000\overline{011})$, which is the $M_{4,3}$ wanted. A couple of remarks have to be done again. First, in this case, the first ancestors come, as previously, from the period-4 structural component $(\overline{.0111}, \overline{.1000})$, but the second ancestors come now from the only period-3 cardioid of the antenna $(\overline{.011}, \overline{.100})$. And second, the period of $M_{4,3}$ is 3, that now coincides with the number of digits of the periodic part.

As can be seen in Figure 2.1(c), the previously mentioned Misiurewicz point $M_{4,1}$ can be reached by following two different routes. The first route, that starts from $(\overline{.0111}, \overline{.1000})$, has already been seen above. The second route to reach $M_{4,1}$ starts from the period-5 structural component $(\overline{.01111}, \overline{.10000})$, and follows the rest of the Misiurewicz

route sequence of hc 's

$$(\overline{.0111101}, \overline{.1000010}), (\overline{.011110101}, \overline{.100001010}), \dots, \quad (3.4)$$

with periods 7, 9, ... After simplifying, $(.0111\overline{10}, .1000\overline{01})$ is obtained again.

Let us abandon the real axis and go to a primary shrub [12], for example, shrub (1/3) shown in Figure 2.2. The first tip [12], ftip (1/3) of Figures 2.2(b) and 2.2(c), is a Misiurewicz point $M_{2,1}$ (as in [3], in [12] all the Misiurewicz points are given with one unit more in their preperiods, therefore this $M_{2,1}$ is given as $M_{3,1}$). Now, the external argument can be calculated as the limit of the arguments of the structural components of the branches 1, 11, 111, ..., with periods 4, 5, 6, ..., that is, the limit of $\overline{.0011}, \overline{.00111}, \overline{.001111}, \dots$, or the limit of $\overline{.0100}, \overline{.01000}, \overline{.010000}, \dots$. Hence, $\text{ftip}(1/3) = \overline{.001} = \overline{.010}$, that are two equal values.

Likewise, it is known that the main node of a subshrub is a Misiurewicz point. In the case of shrub (1/3), the main node is an $M_{3,1}$ [12] placed in $c = -0.101096 \dots + 0.956286 \dots i$, where the branch 0 finishes and branches 1 and 2 emerges (see Figures 2.2(b) and 2.2(c)). We are going to calculate its external arguments, which are 3 because it is a branch point with 3 branches. If one goes towards $M_{3,1}$ through the branch 0, whose *lahcs* have periods 3, 6, 9, 12, ..., the first and third external arguments can be calculated. Indeed, the first external argument is the limit of $\overline{.001}, \overline{.001010}, \overline{.001010010}, \overline{.001010010010}, \dots$, therefore is $\overline{.001010}$; and the third argument is the limit of $\overline{.010}, \overline{.010001}, \overline{.010001001}, \overline{.010001001001}, \dots$, and therefore is $\overline{.010001}$. As was said before, the preperiodic part has 3 digits, one less than the 4th given in [12]. But the period is 1, which divides to 3. Likewise, if one goes towards $M_{3,1}$ through the branch 1 starting from the branch representative and following the *lahcs* with periods 4, 7, 10, ..., the second and third external arguments can be calculated as the limits of $\overline{.0011}, \overline{.0011001}, \overline{.0011001001}, \overline{.0011001001001}, \dots$, and $\overline{.0100}, \overline{.0100010}, \overline{.0100010010}, \overline{.0100010010010}, \dots$; therefore they are $\overline{.001100}$ and $\overline{.010001}$. The last argument had already been calculated through the branch 0, obviously by obtaining the same result. As can be noted in the figure, starting from the representative of the branch 2 and following the *lahcs* with periods 5, 8, 11, ..., the first and second external arguments already calculated can also be calculated again as the limits of $\overline{.00101}, \overline{.00101001}, \overline{.00101001001}, \overline{.00101001001001}, \dots$, and $\overline{.00110}, \overline{.00110010}, \overline{.00110010010}, \overline{.00110010010010}, \dots$; giving $\overline{.001010}$ and $\overline{.001100}$. As has just been seen, each argument can be calculated by following two Misiurewicz routes.

Let us calculate the three external arguments of the Misiurewicz point $M_{4,1}$, placed in $c = -0.175890 \dots + 1.086624 \dots i$, where the branch 1 finishes and branches 11 and 12 emerge. Through the branch 1, by following the *lahcs* which periods 4, 7, 10, ..., the first and third arguments can be calculated. Indeed, the first and third arguments are the limits of $\overline{.0011}, \overline{.0011010}, \overline{.0011010010}, \overline{.0011010010010}, \dots$, and $\overline{.0100}, \overline{.0100001}, \overline{.0100001001}, \overline{.0100001001001}, \dots$, that is to say, $\overline{.0011010}$ and $\overline{.0100001}$. Likewise, through the branch 11, by following the *lahcs* which periods 5, 8, 11, ..., the second and third arguments can be calculated. Since the third argument has already been calculated, we calculate now the second argument that is the limit of $\overline{.00111}, \overline{.00111001}, \overline{.00111001001}, \overline{.00111001001001}, \dots$, and therefore is $\overline{.0011100}$.

Let us see some examples in the case of a secondary shrub, for instance, the shrub $(1/3 \cdot 1/4)$ shown in Figure 2.3, that has two subshrubs S_1 and S_2 . As can be seen in Figures 2.3(b) and 2.3(c), the first tip, $\text{ftip}(1/3 \cdot 1/4)$, that is, a Misiurewicz point $M_{8,1}$, can be calculated as the limit of the representatives of the branches 1, 11, 111, ... of the portion of S_2 that emerges from $\bar{1}$, whose periods are 10, 11, 12, ..., that is to say, the limit of $\overline{.0010010011}, \overline{.00100100111}, \overline{.001001001111}, \dots$, or the limit of $\overline{.0010010100}, \overline{.00100101000}, \dots$. Hence, $\text{ftip}(1/3 \cdot 1/4) = .00100100\bar{1} = .00100101\bar{0}$, two equal values and therefore only one result.

Let us calculate the arguments of the point $\bar{1}$, where the portion S_2 that we have just seen emerges. The point $\bar{1}$ is a Misiurewicz point $M_{9,1}$ with 3 branches, therefore with 3 external arguments, and it is a boundary point between S_1 and S_2 . If one goes towards $M_{9,1}$ by following the representatives of the branches 0, 1, 11, ... of S_1 whose periods are 12, 15, 18, ..., the first and third external arguments are calculated. From the limit of

$$\overline{.001001001010}, \overline{.001001001010010}, \overline{.001001001010010010}, \dots, \quad (3.5)$$

the first external argument $.0010010010\bar{10}$ is calculated; and from the limit of

$$\overline{.001001010001}, \overline{.001001010001001}, \overline{.001001010001001001}, \dots, \quad (3.6)$$

the third external argument $.0010010100\bar{01}$ is calculated. The second external argument that has not been found yet is calculated in S_2 , for example, by starting from the period-10 representative of the branch 1 of the portion that emerges from $\bar{1}$, as the limit of the first argument of the components whose periods are 10, 13, 16, ... (see Figure 2.3(c)). Therefore, it is the limit of

$$\overline{.0010010011}, \overline{.0010010011001}, \overline{.0010010011001001}, \dots, \quad (3.7)$$

that is to say $.001001001\bar{100}$.

As a last example, let us calculate the four arguments of the main node of the shrub $(1/3 \cdot 1/4)$. This is a Misiurewicz point $M_{12,3}$ as can be seen in Figure 3.1, that is, a magnification of the neighborhood of the main node $M_{12,3}$ of Figure 2.3(c). In Figure 3.1, the arguments of $1/3 \cdot 1/4$ have not been shown due to space problems. For example, if one goes towards $M_{12,3}$ through the branch 0, the first and fourth arguments can be calculated, and if one goes towards $M_{12,3}$ through the branch 2, the second and third arguments can be calculated. Thus, in the branch 0, by following the *lahcs* with periods 12, 24, 36, ..., the limit of

$$\overline{.001001001010}, \overline{.001001001010001001010001}, \quad (3.8)$$

$$\overline{.001001001010001001010001001010001}, \dots$$

is $.00100100101000100101000\bar{1}$, that gives the first argument, and the limit of

$$\overline{.001001010001}, \overline{.001001010001001001001010}, \quad (3.9)$$

$$\overline{.001001010001001001001010001001001010}, \dots$$

is $.001001001010001010001001$, that gives the second argument, and the limit of

$$\begin{aligned} & \overline{.0010010010100010001}, \overline{.001001001010010001001001010001}, \\ & \overline{.001001001010010001001001010001001001010001}, \dots \end{aligned} \quad (3.11)$$

is $.001001001010010001001001$, that gives the third argument. These values also can be obtained starting from the two other branches.

Other examples for the cases of tertiary, quaternary, ..., shrubs could be seen, but the process is the same, and therefore it is not worth going on seeing more examples.

4. Conclusions

We present here a general method for the calculation of the external arguments of the root points of the Mandelbrot set. Two algorithms for the calculation of the external arguments have been given, one of them is a general algorithm (the outer algorithm), and the other is a useful particular algorithm (the inner algorithm).

Likewise, we present a general method, from two different approaches, for the calculation of the external arguments of Misiurewicz points.

The Schleicher algorithm [10] to calculate external arguments only can be applied in particular cases. The Douady algorithm [11] to calculate the external arguments of a notable point cannot be applied in a lot of cases. On the contrary, our method can be applied in any case and completely solve the problem of calculating the external arguments of hyperbolic components and Misiurewicz points.

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G. Pastor: Instituto de Física Aplicada, Consejo Superior de Investigaciones Científicas,
Serrano 144, 28006 Madrid, Spain
Email address: gerardo@iec.csic.es

M. Romera: Instituto de Física Aplicada, Consejo Superior de Investigaciones Científicas,
Serrano 144, 28006 Madrid, Spain
Email address: miguel@iec.csic.es

G. Alvarez: Instituto de Física Aplicada, Consejo Superior de Investigaciones Científicas,
Serrano 144, 28006 Madrid, Spain
Email address: gonzalo@iec.csic.es

J. Nunez: Instituto de Física Aplicada, Consejo Superior de Investigaciones Científicas,
Serrano 144, 28006 Madrid, Spain
Email address: juana.nunez@iec.csic.es

D. Arroyo: Instituto de Física Aplicada, Consejo Superior de Investigaciones Científicas,
Serrano 144, 28006 Madrid, Spain
Email address: david.arroyo@iec.csic.es

F. Montoya: Instituto de Física Aplicada, Consejo Superior de Investigaciones Científicas,
Serrano 144, 28006 Madrid, Spain
Email address: fausto@iec.csic.es